Specialized Macdonald polynomials, quantum K-theory, and Kirillov-Reshetikhin crystals

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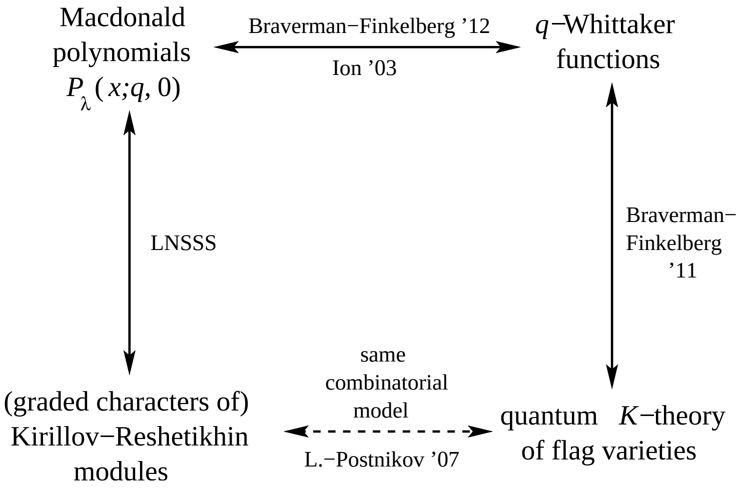
Macdonald polynomials $P_\lambda(x;q,0)$
Braverman−Finkelberg '12
Ion '03
q−Whittaker functions
LNSSS
Braverman−Finkelberg '11
(graded characters of) Kirillov−Reshetikhin modules
same combinatorial model
L.−Postnikov '07
quantum K−theory of flag varieties

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Recursive construction procedure (for the non-symmetric ones $E_\mu(x; q, t)$), based on Cherednik's intertwiners I_i.

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Let

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Theorem (Braverman-Finkelberg, Ion)

We have

$$P_\lambda(x;q,t=0) = \widehat{\Psi}_\lambda(x;q)\,.$$

Schubert calculus

Flag variety G/B, Schubert variety $X_w = \overline{B^- wB/B}$, for $w \in W$.

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A k-point GW invariant (of degree d) counts curves of degree d passing through k given Schubert varieties.

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Theorem (Braverman-Finkelberg)

In simply-laced types, the q-Whittaker function $\Psi_\lambda(x; q)$ (viewed as a function of λ) coincides with the K-theoretic J-function.

Kirillov-Reshetikhin (KR) modules

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Main Theorem (L.-Naito-Sagaki-Schilling-Shimozono)

For all untwisted affine root systems $A_{n-1}^{(1)} - G_2^{(1)}$, we have

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(2) In simply-laced types, certain affine Demazure characters were identified with $P_\lambda(x;q,0)$ (Ion), and $X_\lambda(x;q)$ (Fourier-Littelmann).

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- the tensor products of one-column KR modules (LNSSS).

The model is uniform for all Lie types $A_{n-1} - G_2$.

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The quantum Bruhat graph $\mathrm{QBG}(W)$ on W is the directed graph with labeled edges

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where

$\ell(ws_\alpha) = \ell(w) + 1$ (covers of the Bruhat order), or

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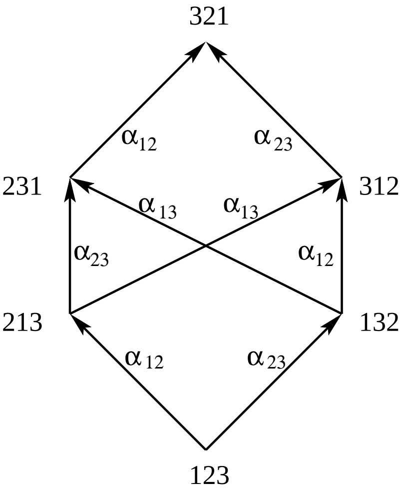
Comes from the multiplication of Schubert classes in the quantum cohomology of flag varieties $QH^*(G/B)$ (Fulton and Woodward).

Bruhat graph for S_3:



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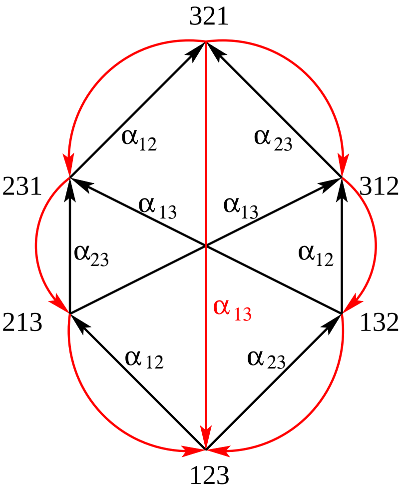

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Given a dominant weight λ, we associate with it a sequence of roots, called a λ-chain:

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Fact. The construction of a λ-chain is based on a reduced decomposition of the affine Weyl group element corresponding to $A_\circ - \lambda$. This gives a sequence of alcoves from $A_\circ$ to $A_\circ - \lambda$.

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For $w \in W$ and J, construct the chain $\pi(w, J)$ of elements in W:

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Important structures:

$$\mathcal{A}_q(\Gamma, w) := \{J \,:\, \pi(w, J) \text{ path in QBG}(W)\},$$
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Remark. For $q = 0$, we retrieve the alcove model (L. and Postnikov, cf. Gaussent and Littelmann, Littelmann):

$$P_\lambda(X; 0, 0) = \mathrm{ch}(V_\lambda) = \sum_{J \in \mathcal{A}_{\lessdot}(\Gamma)} x^{\mathrm{weight}(J)} \, .$$

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Recall: $K(G/B)$ and $QK(G/B)$ have bases of Schubert classes $[\mathcal{O}_{X_w}] = [\mathcal{O}_w]$, $w \in W$.

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In $K(G/B)$ (finite-type or Kac-Moody), we have

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Remark. Restricting the RHS, we retrieve the Chevalley formula in $QH^*(G/B)$ (Fulton-Woodward).

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  - They multiply as in the conjectured Chevalley formula.
  - They are conjectured to represent Schubert classes $[\mathcal{O}_w]$ in $QK(SL_n/B)$.

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Fact. The crystal structure on $B^{\otimes \mathbf{p}}$ is defined by a tensor product rule: $B^{\otimes \mathbf{p}} = B^{p_1,1} \otimes B^{p_2,1} \otimes \dots .$

Models for KR crystals

Fact. In the classical types $A - D$ there are tableau models (the usual column-strict fillings in type $A_{n-1}^{(1)}$, but more involved in the other types, particularly for $B_n^{(1)}$ and $D_n^{(1)}$).

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Goal. Uniform model for all types $A_{n-1}^{(1)} - G_2^{(1)}$, based on the quantum alcove model.

The quantum alcove model for KR crystals

Given $\mathbf{p} = (p_1, p_2, \ldots)$ and an arbitrary Lie type, let

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Main Theorem (L.-Naito-Sagaki-Schilling-Shimozono)

The (combinatorial) crystal $\mathcal{A}_q(\Gamma)$ is isomorphic to the tensor product of KR crystals $B^{\otimes \mathbf{p}}$.

The energy function

It originates in the theory of exactly solvable lattice models.

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The energy via the quantum alcove model

Consider $J = \{j_1 < j_2 < \ldots < j_s\}$ in $\mathcal{A}_q(\Gamma)$ for $\Gamma = (\beta_1, \ldots, \beta_m)$, i.e., we have a path in the quantum Bruhat graph

$$1_W = w_0 \xrightarrow{\beta_{j_1}} w_1 \xrightarrow{\beta_{j_2}} \ldots \xrightarrow{\beta_{j_s}} w_s \,.$$

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Theorem (L.-Naito-Sagaki-Schilling-Shimozono)

Given $J \in \mathcal{A}_q(\Gamma)$, which is identified with $B^{\otimes \mathbf{p}}$, we have

$$D_B(J) = -\mathrm{height}(J)\,.$$

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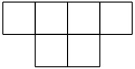
We use combinatorial moves based on certain operators on W defined by $\mathrm{QBG}(W)$, which satisfy the Yang-Baxter equation (Brenti-Fomin-Postnikov).

Example in type A_2.

$$\mathbf{p} = (1,2,2,1) =$$  $$; \quad \lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1 = (4,2,0).$$

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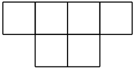
$\mathbf{p} = (1,2,2,1) =$  ; $\quad \lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1 = (4,2,0)$.

A λ-chain as a concatenation of ω_1-, ω_2-, ω_2-, and ω_1-chains:

$$\Gamma = (\ \ (1,2),\ \ (1,3)\ \mid\ (2,3),\ \ (1,3)\ \mid\ (2,3),\ \ (1,3)\ \mid\ (1,2),\ \ (1,3)\ \ )\,.$$

Example. Let $J = \{1, 2, 3, 6, 7, 8\}$.

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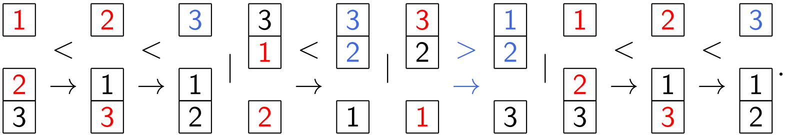
Claim: J is admissible. Indeed, the corresponding path in the quantum Bruhat graph is



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The corresponding element in $B^{\otimes \mathbf{p}} = B^{1,1} \otimes B^{2,1} \otimes B^{2,1} \otimes B^{1,1}$ represented via column-strict fillings:

$$\boxed{3} \otimes \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} \otimes \boxed{3}\,.$$

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We have

$$\mathrm{height}(J) = 2 \, .$$