Weak crystal operators and flag Gromov-Witten invariants

Anne Schilling (UC Davis)
joint with Jennifer Morse (Drexel)
Banff, October 16, 2013

- Littlewood-Richardson template
- Variations
- k-Schur functions
- Crystal operators on affine factorizations

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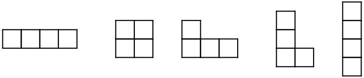
Variation 1: Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$

Indexed by partitions: 

- Tensor product multiplicities

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu} c^{\nu}_{\lambda\mu} \, V(\nu)$$

- Symmetric function coefficients

$$s_{\lambda} \, s_{\mu} \; = \; \sum_{\nu} c^{\nu}_{\lambda\mu} \, s_{\nu} \qquad \text{and} \qquad s_{\nu/\lambda} = \sum_{\mu} c^{\nu}_{\lambda\mu} \, s_{\mu}$$

- Intersections in the Grassmannian

$$c^{\nu}_{\lambda\mu} = X_{\lambda} \cap X_{\mu} \cap X_{\hat{\nu}}$$

- Cohomology of the Grassmannian structure constants

$$\sigma_{\lambda} \cup \sigma_{\mu} = \sum_{\nu \subset rect} c^{\nu}_{\lambda\mu} \, \sigma_{\nu}$$

Combinatorial description

Littlewood–Richardson rule

$c^{\nu}_{\lambda\mu}$ = # skew tableaux t of shape ν/λ and weight μ such that $\mathrm{row}(t)$ is a reverse lattice word.

Example

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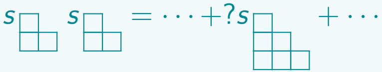

$\Rightarrow c^{321}_{21,21} = 2$

Combinatorial description

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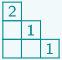
Example

$$s_{(2,1)}\, s_{(2,1)} = \cdots + ?\, s_{(3,2,1)} + \cdots$$



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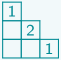

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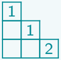

112

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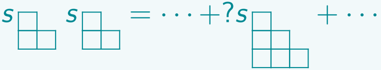

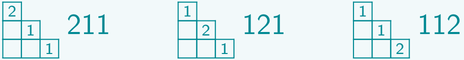

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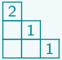
Example

$$s_{21}\, s_{21} = \cdots + ?\, s_{321} + \cdots$$



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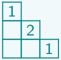

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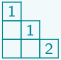

112

$\Rightarrow c^{321}_{21,21} = 2$

Gordon James (1987) on the Littlewood-Richardson rule:

> *"Unfortunately the Littlewood-Richardson rule is much harder to prove than was at first suspected. The author was once told that the Littlewood-Richardson rule helped to get men on the moon but was not proved until after they got there."*

Crystal graph

Action of crystal operators e_i, f_i, s_i on tableaux:

1. Consider letters i and $i+1$ in row reading word of the tableau
2. Successively "bracket" pairs of the form $(i+1, i)$
3. Left with word of the form $i^r(i+1)^s$

$$e_i(i^r(i+1)^s) = \begin{cases} i^{r+1}(i+1)^{s-1} & \text{if } s > 0 \\ 0 & \text{else} \end{cases}$$

$$f_i(i^r(i+1)^s) = \begin{cases} i^{r-1}(i+1)^{s+1} & \text{if } r > 0 \\ & \text{else} \end{cases}$$

$$s_i(i^r(i+1)^s) = i^s(i+1)^r$$

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Crystal reformulation

3							
	1	2	2	3			
		1	1	2	3	3	3

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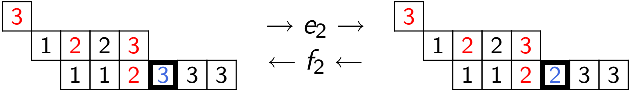
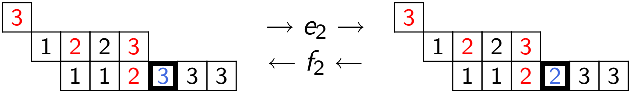


e_2: change leftmost unpaired 3 into 2
f_2: change rightmost unpaired 2 into 3

Crystal reformulation



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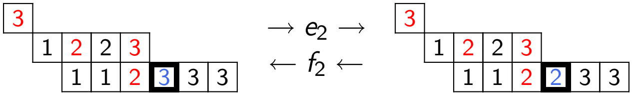
Theorem

b where all $e_i(b) = 0$ (highest weight)

$\leftrightarrow$ *connected component*

$\leftrightarrow$ *irreducible*

Crystal reformulation



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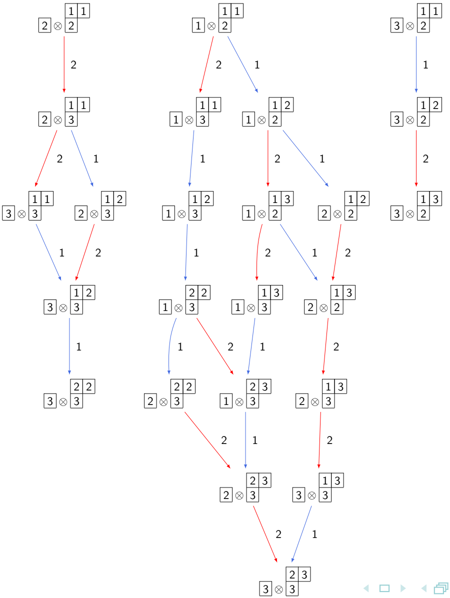
$\leftrightarrow$ connected component

$\leftrightarrow$ irreducible

Reformulation of LR rule

$c^{\nu}_{\lambda\mu}$ counts tableaux of shape ν/λ and weight μ which are highest weight.

Decomposition



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Variation 2: c_{uv}^{w}

The set $\mathbb{F}_n$ of complete flags:

$$0 = W_0 \subset W_1 \subset \cdots \subset W_n = \mathbb{C}^n$$

subvarieties indexed by permutations of S_n

Intersections in the flag variety

Count points in the intersection $c_{uv}^{w} = X_u \cap X_v \cap X_{w_0 w}$

Structure constants in cohomology of the flag variety

$$\sigma_u \cup \sigma_v = \sum_{w \in S_n} c_{uv}^{w} \sigma_w$$

Schubert polynomial coefficients

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_{w} c_{uv}^{w} \mathfrak{S}_w$$

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Variations 1 and 2 quantized

Grassmannian | Flags

Gromov-Witten invariants

Quantum cohomology

count rational curves of degree d that meet $X_\lambda, X_\mu, X_{\hat{\nu}}$

count equivalence classes of rational curves of multidegree d in $\mathbb{F}_n$

$$\sigma_\lambda *_q \sigma_\mu = \sum_{\nu \subset rect} q^d \, N^\nu_{\lambda\mu} \, \sigma_\nu$$

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Polynomial coefficients modulo an ideal

Ring of symmetric functions

$\mathbb{Z}[x_1, \ldots, x_n; q_1, \ldots, q_{n-1}]$

Schur functions

quantum Schubert polynomials

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Modulo an ideal is non-trivial

$$s_\lambda\, s_\mu \;=\; \sum_{\nu \subset rect} c^{\nu}_{\lambda\mu}\, s_\nu \;+\; \sum_{\nu \not\subset rect} c^{\nu}_{\lambda\mu}\, s_\nu$$

$$\Lambda \otimes \mathbb{Z}[q] \twoheadrightarrow QH^*(Gr_{a,n})$$

$$s_\lambda \;\mapsto\; \begin{cases} \sigma_\lambda & \text{when} \quad \lambda \subset \mathit{rectangle} \\ \pm q^* \sigma_{\tilde{\lambda}} & \text{when} \quad \lambda \not\subset \mathit{rectangle} \end{cases}$$

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k-Schur functions

- Originally an empirical study [Lascoux, Lapointe, Morse], for $\lambda_1 \leq k$,

$$H_\lambda(x;q,t) = \sum_{\mu_1 \leq k} K_{\lambda\mu}(q,t)\, A_\mu^{(k)}(x;t)\,,$$

where $K_{\lambda\mu}(q,t) \in \mathbb{N}[t]$.

- Crazy difficulty led to family of functions $\{s_\mu^{(k)}\}_{\mu_1 \leq k}$ defined in terms of a k-Pieri rule where it was conjectured that $A_\mu^{(k)}(x;1) = s_\mu^{(k)}$
- $\{s_\mu^{(k)}\}_{\mu_1 \leq k}$ basis for $\Lambda = \mathbb{Z}[h_1, \ldots, h_k]$
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Variation 1q: quantized $c^{\nu}_{\lambda\mu}$

Wess-Zumino-Witten model of Verlinde algebra

Gromov-Witten invariants of the Grassmannian

$$\sigma_\lambda *_q \sigma_\mu = \sum_{\substack{\nu \subset rect \\ |\nu|=|\lambda|+|\mu|-dn}} q^d \, N^{\nu}_{\lambda\mu} \, \sigma_\nu$$

Symmetric function coefficients

- Schur coefficients in product of Schur functions modulo an ideal
- k-Schur coefficients in a product of k-Schur functions

$$s^{(k)}_\lambda \, s^{(k)}_\mu = \sum_{\hat\nu=(a^*,\nu\subset rect)} N^{\nu}_{\lambda\mu} \, s^{(k)}_{\hat\nu} + \sum_{\hat\nu\neq(a^*,\nu\subset rect)} c^{\hat\nu}_{\lambda\mu} \, s^{(k)}_{\hat\nu}$$

Computation in Λ

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Computation in Λ

Variation 2q: Flag Gromov–Witten invariants

Affine Grassmannian

$\tilde{G}r = SL(n, \mathbb{C}((t)))/SL(n, \mathbb{C}[[t]])$ $\quad n = k + 1$

homology of affine Grassmannian $\twoheadrightarrow$ quantum cohomology of Grassm.

$\downarrow$

quantum cohomology of flags

Product of k-Schurs | **Flag Gromov-Wittens**

$s^{(k)}_\lambda \, s^{(k)}_\mu = \sum_\nu C_{\lambda\mu\nu} \, s^{(k)}_\nu$ $\qquad$ $\sigma_u *_q \sigma_v = \sum_w \sum_d q^d \langle u, v, w \rangle_d \, \sigma_{w_0 w}$

k-bounded partitions $\qquad$ permutations of S_{k+1}

Theorem (Morse-Lapointe)

Precise relation between $C_{\lambda\mu\nu}$ and $\langle u, v, w \rangle_d$ (up to relabeling).

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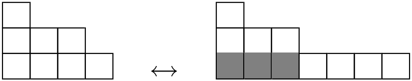
Indexing sets

k-bounded partition $\qquad$ $k+1$-core $\qquad$ $(k=4)$

$\lambda \;=$  $=$

```
3
4 0 1
0 1 2 3 4 0 1
```

Action of affine symmetric group on cores:

$$s_i\tau = \tau + \begin{cases} \text{all boxes of residue } i \text{ added} \\ \text{all boxes of residue } i \text{ removed} \\ \text{nothing} \end{cases}$$

$\emptyset \xrightarrow{s_0}$ `0` $\xrightarrow{s_4 s_3 s_2 s_1}$

```
4
0 1 2 3 4
```

$\xrightarrow{s_1 s_0}$

```
4 0 1
0 1 2 3 4 0 1
```

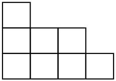
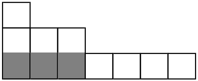
$\xrightarrow{s_3}$

```
3
4 0 1
0 1 2 3 4 0 1
```

Affine Grassmannian element in $\tilde{S}_{k+1}/S_{k+1}$: $\quad \tilde{w}_\lambda = s_3 s_1 s_0 s_4 s_3 s_2 s_1 s_0$

Indexing sets

k-bounded partition $\qquad$ $k+1$-core $\qquad$ $(k=4)$

$\lambda \quad =$  $\leftrightarrow$  $=$

3
4 0 1
0 1 2 3 4 0 1

Action of affine symmetric group on cores:

$$s_i\tau = \tau + \begin{cases} \text{all boxes of residue } i \text{ added} \\ \text{all boxes of residue } i \text{ removed} \\ \text{nothing} \end{cases}$$

$\emptyset \xrightarrow{s_0}$ 0 $\xrightarrow{s_4 s_3 s_2 s_1}$ 4 / 0 1 2 3 4 $\xrightarrow{s_1 s_0}$ 4 0 1 / 0 1 2 3 4 0 1 $\xrightarrow{s_3}$ 3 / 4 0 1 / 0 1 2 3 4 0 1

Affine Grassmannian element in $\tilde{S}_{k+1}/S_{k+1}$: $\quad \tilde{w}_\lambda = s_3 s_1 s_0 s_4 s_3 s_2 s_1 s_0$

Indexing sets

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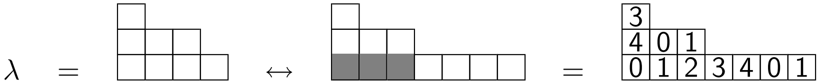

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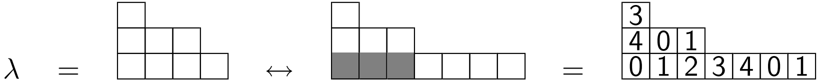

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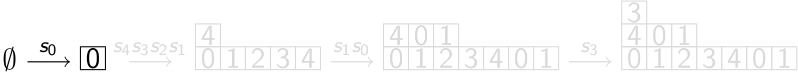

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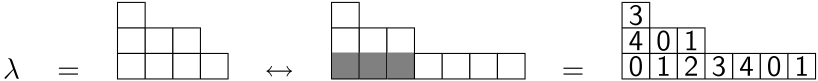

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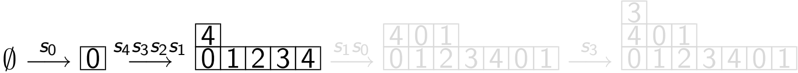

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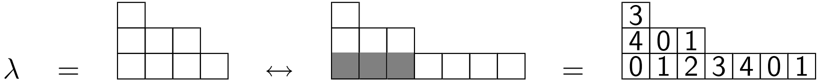

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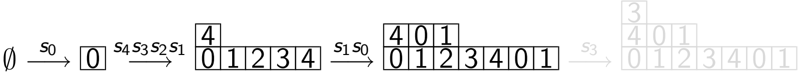

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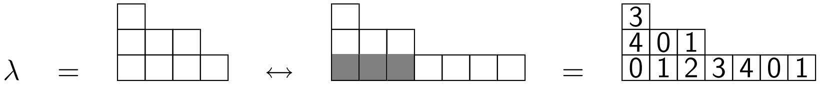

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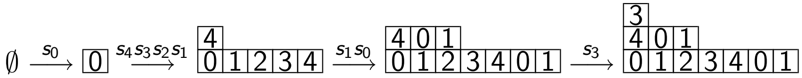

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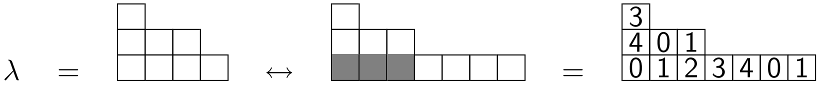

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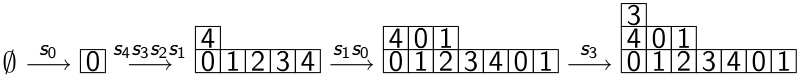

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Affine symmetric group $\tilde{S}_n$

$\langle s_0, s_1, \ldots, s_{n-1} \rangle$ where $s_i s_j = s_j s_i$

$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ (all indices mod n)

$s_i^2 = 1$

Example

For $n = 3$, $s_1 s_2 s_1 s_0 = s_2 s_1 s_2 s_0$

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Affine Grassmannian permutations

All reduced words end in s_0

Affine symmetric group

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Affine horizontal strips and Pieri rule

Schur function Pieri rule

$$h_r s_\lambda = \sum_{\substack{\nu \\ \nu/\lambda \text{ horizontal } r\text{-strip}}} s_\nu$$

k-Schur function Pieri rule

$$h_r s_\lambda^{(k)} = \sum_{\substack{\nu \\ \nu/\lambda \text{ weak horizontal } r\text{-strip}}} s_\nu^{(k)}$$

ν/λ is weak horizontal r-strip if $\tilde{w}_\nu \tilde{w}_\lambda^{-1}$ is cyclically decreasing of length r.

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Cyclically decreasing permutation

Definition

$\tilde{w} \in \tilde{S}_n$ is cyclically decreasing if every reduced word has no $j-1$ preceeding $j \pmod{n}$.

Remark

In particular, every letter in the reduced word appears at most once.

Example

For $n = 4$, cyclically decreasing: $\tilde{w} = s_1 s_0 s_3$ and $\tilde{w} = s_3 s_1$
not cyclically decreasing $\tilde{w} = s_3 s_1 s_0$

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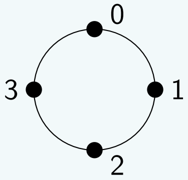
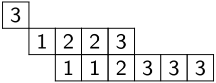


k-tableaux or affine factorizations

Schur case

tableau $\leftrightarrow$ sequence of horizontal strips



$\leftrightarrow$

k-Schur case

horizontal strip $\leftrightarrow$ cyclically decreasing element

Definition

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- $\ell(\tilde{w}_\lambda) = |\alpha|$
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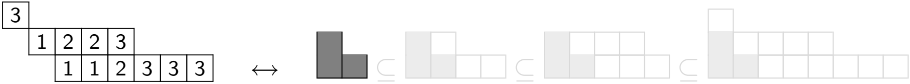

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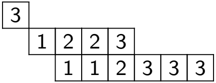
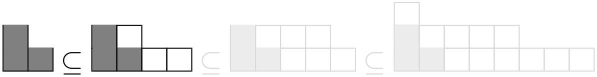


$\leftrightarrow$



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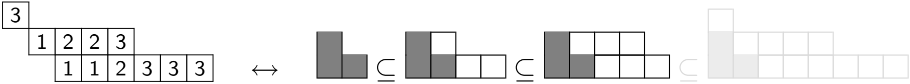

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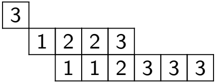

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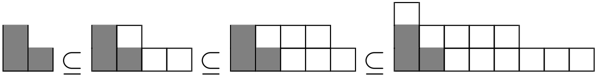

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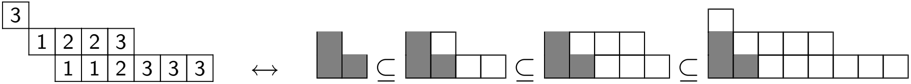

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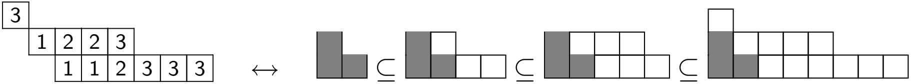

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Example

Affine factorizations of $\tilde{w}_\lambda = s_3 s_2 s_3 s_1 s_0 = s_2 s_3 s_2 s_1 s_0 \in \tilde{S}_4$

with weight $\alpha = (21^3)$ $\{(s_3)(s_2)(s_3)(s_1 s_0), (s_2)(s_3)(s_2)(s_1 s_0)\}$

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- **Littlewood-Richardson template**
- **Variations**
- ***k*-Schur functions**
- **Crystal operators on affine factorizations**

Weak crystal operators and flag Gromov-Witten invariants

- **Littlewood-Richardson template**
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Schur times k-Schur

k-Schur coefficients in $s_\mu \, s_{\tilde{\nu}}^{(k)}$ include

- all fusion coefficients
- coefficients in Schur times a Schubert polynomial
- Gromov-Witten invariants for flags $\langle u, v, w \rangle_d$ where u has one descent

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Can use Giambelli formula:

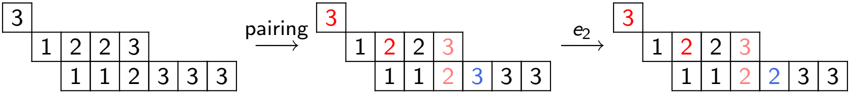
$$s_\mu\; s_{\tilde{v}}^{(k)} = det\,(h_{\mu_i+j-1})_{ij}\;\; s_{\tilde{v}}^{(k)}$$

$$= \sum_\sigma \mathrm{sgn}(\sigma)\, \underbrace{h_{\alpha_1}\cdots h_{\alpha_\ell}\, s_{\tilde{v}}^{(k)}}_{\sum_{\tilde{w}} s_{\tilde{w}\tilde{v}}^{(k)}}$$

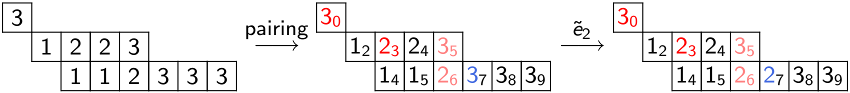
where $\tilde{w}$ is an affine factorization of weight α.

Crystal operators on affine factorizations

Recall e_i pairing and action:



Label cells diagonally



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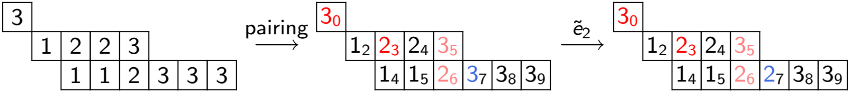
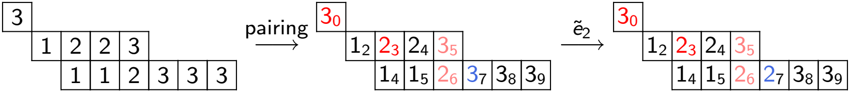


$$\underbrace{(9\,8\,7\,5\,0)}_{\text{label of 3's}}\ \underbrace{(6\,4\,3)}_{\text{label of 2's}}$$

Crystal operators on affine factorizations

Label cells diagonally



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from left to right:
pair $x \in$ 3's with smallest $y \in$ 2's that is bigger than x

Label cells diagonally

$$\begin{array}{cccccccc} 3 & & & & & & & \\ & 1 & 2 & 2 & 3 & & & \\ & & 1 & 1 & 2 & 3 & 3 & 3 \end{array} \xrightarrow{\text{pairing}} \begin{array}{cccccccc} 3_0 & & & & & & & \\ & 1_2 & 2_3 & 2_4 & 3_5 & & & \\ & & 1_4 & 1_5 & 2_6 & 3_7 & 3_8 & 3_9 \end{array} \xrightarrow{\tilde{e}_2} \begin{array}{cccccccc} 3_0 & & & & & & & \\ & 1_2 & 2_3 & 2_4 & 3_5 & & & \\ & & 1_4 & 1_5 & 2_6 & 2_7 & 3_8 & 3_9 \end{array}$$

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from left to right:
pair $x \in$ 3's with smallest $y \in$ 2's that is bigger than x
delete rightmost unpaired $z \in$ 3's and add $z - t$ to 2's

Definition

The above defines $\tilde{e}_i$ and $\tilde{f}_i$ on factorizations $\tilde{w} = v^r \cdots v^1 \in \langle s_0, \ldots, s_{\hat{x}}, \ldots, s_{n-1} \rangle$ where v^i is cyclically decreasing.

Theorem

For partition $\mu \subseteq (a^{n-a})$ and affine Grassmannian $\tilde{v}$, let

$$s_\mu \, s^{(k)}_{\tilde{v}} = \sum_{\tilde{w}} c^{\tilde{w}}_{\mu \tilde{v}} \, s^{(k)}_{\tilde{w}} \, .$$

If $\tilde{w}\tilde{v}^{-1} \in \langle s_0, \ldots, s_{\hat{x}}, \ldots, s_{n-1} \rangle$,
$c^{\tilde{w}}_{\mu,\tilde{v}} = \#$ of affine factorizations of $\tilde{w}\tilde{v}^{-1}$ with weight μ killed by all $\tilde{e}_i$.

Proof

Via sign-reversing involution using $\tilde{s}_i \tilde{e}_i$ following Remmel-Shimozono.
All terms cancel in Giambelli formula except highest weight elements..

Main Results (with Morse)

Theorem

For partition $\mu \subseteq (a^{n-a})$ and affine Grassmannian $\tilde{v}$, let

$$s_\mu \, s_{\tilde{v}}^{(k)} = \sum_{\tilde{w}} c_{\mu \tilde{v}}^{\tilde{w}} \, s_{\tilde{w}}^{(k)} \, .$$

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Corollaries

Corollary

Schubert polynomial expansion of $s_\lambda \mathfrak{S}_w$ for any $w \in S_n$ and partition λ where $|\lambda^c| < n$.

Corollary

Fusion rules $N_{\lambda\mu}^{\nu}$ for any λ, μ and ν such that

- ν/μ has a cut-point
- or λ satisfies $|\lambda^c| < n$.

Corollary

Gromov-Witten invariants for flags $\langle u, v, w\rangle_d$ when u has one descent and $v_r w_{R_r} w^{-1} \in S_{\hat{x}}$
(v_r is v shifted by r; w_{R_r} element obtained from rth k-rectangle)

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Related work

Quantum cohomology of Grassmannian

- Buch, Kresch, Tamvakis 2003
- Knutson, Tao puzzles 2003
- Coskun recursive algorithm 2009
- Buch et al. forthcoming

Quantum Flag

- Fomin, Gelfand, Postnikov quantum Monk 1997
- Postnikov quantum Pieri 1999
- Berg, Saliola, Serrano k-Schur indexed by rectangle minus a box, quantum Monk 2012

Fusion

- Tudose two row and two column case 2000
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Related work (continued)

Schur times Schubert

- Lenart growth diagrams, plactic approach 2009
- Benedetti, Bergeron relation to dual k-Schur coefficients 2012
- Meszaros, Panova, Postnikov Fomin-Kirillov algebra, hook and two-row case in quantum case 2012

Future Work

- Gromov-Witten invariants
  Closer study of crystal structure on affine factorizations and crystal operators on dual k-tableaux
- t-analogue of k-Schur functions and relation to energy on KR crystals (charge plus offset)
- Schur expansion for LLT polynomials

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Thank you !