Testing for skew-symmetric models

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Introduction

- Let f and F be the pdf and the cdf, respectively, of a symmetric law on the real line,
$$f(x) = f(-x), \quad \forall x \in \mathbb{R}.$$
Let p be a pdf on [0,1]. According to Abtahi et al (2011), a rv X with pdf
$$g(x) = f(x)p\{F(x)\} \tag{1}$$
is said to have a unified skewed distribution with functional parameters f and p, $X \sim USD(f, p)$.

- Every skewed pdf can be expressed in the form (1).

- In fact, from Wang, Boyer and Genton (2004), any continuous pdf can be uniquely expressed as (1) for certain pdfs
$$f \in \mathcal{S} = \{f : \mathbb{R} \to \mathbb{R},\ f \text{ is a symmetric pdf}\} \quad \text{and}$$
$$p \in \mathcal{P} = \{p : [0, 1] \to \mathbb{R},\ p \text{ is a pdf}\}$$

Introduction

$$g(x) = f(x)p\{F(x)\}$$

If

- X is a continuous rv with pdf g,
- Y is a continuous rv with pdf f,
- τ is an even function

then

$$\tau(Y) \stackrel{d}{=} \tau(X)$$

In particular: $Y^2 \stackrel{d}{=} X^2$, $E(Y^{2k}) = E(X^{2k})$, $\forall k, \ldots$

Introduction

Consequence: for certain inferential objectives, it is not necessary to know the law of X, but only that of the symmetric component of its pdf.

The purpose of this work is to propose a test for testing gof for the symmetric component:

H_0 : the symmetric part of g is f $\Leftrightarrow$ $g \in \mathbb{F}_f = \{g(x) = f(x)p\{F(x)\},\ p \in \mathcal{P}\}$,
H_1 : $g \notin \mathbb{F}_f$.

The test statistic

Proposition 1 (Abtahi et al, 2011)

Let $f \in \mathcal{S}$ and $p \in \mathcal{P}$.

(a) If $X \sim USD(f, p)$, then $F(X)$ has pdf p.

(b) If X has pdf $p \in \mathcal{P}$, then $F^{-1}(X) \sim USD(f, p)$.

Let $X_1, \ldots, X_n$ from $X \sim USD(f, p)$. Assume f known. The above result led Abtahi et al (2011) to propose the following estimator

$$\hat{g}_1(x) = f(x)\hat{p}\{F(x)\},$$

where $\hat{p}$ is a kernel-based estimator of p,

$$\hat{p}(x) = \frac{1}{nh}\sum_{i=1}^{n} K_1\left(\frac{Y_i - x}{h}\right),$$

$Y_i = F(X_i)$, $1 \leq i \leq n$, h is the bandwidth and K_1 is a kernel.

Note that to build $\hat{g}_1(x)$ we only need to know the symmetric part of g

The test statistic

$$\hat{g}_1(x) = f(x)\hat{p}\{F(x)\}$$

Another consistent kernel-based estimator (could take different bandwidths)

$$\hat{g}_2(x) = \frac{1}{nh}\sum_{i=1}^{n} K_2\left(\frac{X_i - x}{h}\right)$$

To test the null hypothesis

H_0 : the symmetric part of g is f $\Leftrightarrow$ $g \in \mathbb{F}_f = \{g(x) = f(x)p\{F(x)\},\ p \in \mathcal{P}\}$,
H_1 : $g \notin \mathbb{F}_f$.

a reasonable test statistic is

$$T = \int \{\hat{g}_1(x) - \hat{g}_2(x)\}^2 \omega(x)dx,$$

where $\omega(x) \geq 0$ is a weight function.

The test statistic

$$T = \int \{\hat{g}_1(x) - \hat{g}_2(x)\}^2 \omega(x)dx,$$

T can be considered as an analogue of the test statistic studied in

- Hall (1984) for testing gof to a totally specified pdf: $\hat{g}_1(x) = g_0(x)$
- Fan (1994) for testing gof to a parametric family: $\hat{g}_1(x) = g(x; \hat{\theta})$

The test statistic

Theorem 1

Suppose that K_1 and K_2 satisfy Assumptions a–d, that X has pdf ϱ and $Y = F(X)$ has pdf υ, ϱ and υ are uniformly continuous. Suppose $h \to 0$ and $(nh)^{-1} \log n \to 0$. Suppose $\omega : \mathbb{R} \to [0, \infty)$ satisfies

$$\int \omega(x)dx < \infty, \int f^2(x)\omega(x)dx < \infty, \int f(x)\upsilon\{F(t)\}\omega(x)dx < \infty, \int \varrho(x)\omega(x)dx <$$

Then

$$T \xrightarrow{as} I = \int [f(x)\upsilon\{F(x)\} - \varrho(x)]^2 w(t)dt.$$

- Note that $I \geq 0$. If H_0 is true then $I = 0$. In fact, if $\omega(t) > 0$, $\forall t \in \mathbb{R}$, then if follows that $I = 0$ iff H_0 is true.
- The result in Theorem 1 is also true for $\omega(x) = 1$
- The statement in Theorem 1 is also true if we take different bandwidths, say h_1 and h_2, whenever $h_i \to 0$ and $(nh_i)^{-1} \log n \to 0$, $i = 1, 2$.

Asymptotic null distribution

Theorem 2

Suppose that K_1 satisfies Assumption 2, K_2 satisfies Assumption 2 (a), Assumptions 1,3,4 hold and that H_0 is true. If $nh^5 \to \delta$, for some $\delta \in \mathbb{R}$, $\delta \geq 0$, then

$$nh^{1/2}(T - \mu_{03} - \mu_{04}) \xrightarrow{\mathcal{L}} \sqrt{2}\sigma_1 N_1,$$

where $\mu_{03} = \mu_3/(nh)$,

$$\mu_3 = \int \kappa^2(t,u)\omega(t)g(t)dudt,$$

$\mu_{04} = h^4\mu_4$,

$$\mu_4 = \int \left[\tau_1 f(t)p''\{F(t)\} - \tau_2 g''(t)\right]^2 \omega(t)dt,$$

$\tau_i = \int x^2 K_i(t)dt$, $i = 1,2$, $N_1 \sim N(0,1)$,

$$\sigma_1^2 = \int \left\{ \int \kappa(t,u)\kappa(t,u+v)du \right\}^2 \omega^2(t)g^2(t)dvdt$$

and $\kappa(t,u) = f(t)K_1\{uf(t)\} - K_2(u)$.

Asymptotic null distribution

Theorem 2

$$nh^{1/2}(T - \mu_{03} - \mu_{04}) \xrightarrow{\mathcal{L}} \sqrt{2}\sigma_1 N_1$$

$$T = \frac{1}{n^2}\sum_{i,j=1}^{n} U_n(X_i, X_j) \quad \text{with} \quad U_n(x,y) = \int v_n(x;t)v_n(y;t)\omega(t)dt$$

$$\mu_n(t) = E_0\{v_n(X;t)\}, \quad w_n(x;t) = v_n(x;t) - \mu_n(t).$$

$$T = T_1 + T_2 + T_3 + T_4,$$

with

$$T_1 = \frac{1}{n^2}\sum_{i \neq j}\int w_n(X_i;t)w_n(X_j;t)\omega(t)dt, \quad T_2 = \frac{2}{n}\sum_{i=1}^{n}\int w_n(X_i;t)\mu_n(t)\omega(t)dt,$$

$$T_3 = \frac{1}{n^2}\sum_{i=1}^{n}\int w_n^2(X_i;t)\omega(t)dt, \quad T_4 = \int \mu_n^2(t)\omega(t)dt.$$

Asymptotic null distribution

$$T^{red} = \frac{1}{n(n-1)} \sum_{i \neq j} U_n(X_i, X_j).$$

Theorem 3

Suppose that assumptions in Theorem 1 hold. Suppose also that $K_i \geq 0$, $i = 1, 2$, and that

$$\int f^2(x) \upsilon\{F(t)\} \omega(x) dx < \infty.$$

Then

$$T^{red} \xrightarrow{as} I = \int \left[f(x) \upsilon\{F(x)\} - \varrho(x) \right]^2 w(t) dt.$$

Theorem 4

Suppose that assumptions in Theorem 2 hold. Then

$$nh^{1/2}(T^{red} - \mu_{04}) \xrightarrow{\mathcal{L}} \sqrt{2}\sigma_1 N_1.$$

Note that

$$\mu_{04} \approx \int E_0^2 \{\hat{g}_1(t) - \hat{g}_2(t)\} \omega(t) dt.$$

Thus, the term μ_{04} accounts for the integrated squared bias of $\hat{g}_1(t) - \hat{g}_2(t)$ as an estimator of $g(t) - g(t) = 0$.

We will restrict our study to T^{red}, since the practical use of T requires to estimate more parameters than that of T^{red}.

Estimating σ_1^2

Note that

$$\sigma_1^2 = \int R(t) g^2(t) dt,$$

with $R(t) = R_1(t)\omega^2(t)$,

$$R_1(t) = \int \left\{ \int \kappa(t,u)\kappa(t,u+v)du \right\}^2 dv,$$

$\kappa(t,u) = f(t)K_1\{uf(t)\} - K_2(u)$. The only unknown is the pdf of the data g:

$$\tilde{g}_1(x) = f(x)\tilde{p}\{F(x)\},$$

where

$$\tilde{p}(x) = \frac{1}{nh_3} \sum_{i=1}^{n} K_3\left(\frac{Y_i - x}{h_3}\right),$$

$Y_i = F(X_i)$, $1 \leq i \leq n$, h_3 is the bandwidth and K_3 is a kernel, that may differ from h and K_1 in the definition of $\hat{g}_1$, respectively.

Estimating σ_1^2

Let

$$\hat{\sigma}_1^2 = \int R(t)\tilde{g}_1^2(t)dt$$

The consistency of $\hat{\sigma}_1^2$ as an estimator of σ_1^2 follows from the next lemma.

Lemma 1

Suppose that K_3 satisfy Assumptions a–d, that $Y = F(X)$ has pdf υ, υ is uniformly continuous. Suppose $h_3 \to 0$ and $(nh_3)^{-1} \log n \to 0$. Let $R : \mathbb{R} \to \mathbb{R}$ be such that $\int |R(t)| f^2(t)dt < \infty$ and $\int |R(t)| f(t) l_1(t)dt < \infty$, where $l_1(t) = f(t)\upsilon\{F(t)\}$. Then,

$$\int R(t)\tilde{g}_1^2(t)dt \xrightarrow{as} \int R(t) l_1^2(t)dt.$$

Estimating σ_1^2

Another estimator of σ_1^2 can be derived by taking into account that

$$\sigma_1^2 = \int R(t)g^2(t)dt = \int R(t)g(t)dG(t),$$

where G is the cdf of X, which suggests

$$\tilde{\sigma}_1^2 = \int R(t)\tilde{g}_1(t)dG_n(t) = \frac{1}{n}\sum_{i=1}^{n} R(X_i)\tilde{g}_1(X_i).$$

The consistency of $\tilde{\sigma}_1^2$ as an estimator of σ_1^2 follows from the next lemma.

Lemma 2

Suppose that K_3 satisfy Assumptions a–d, that X has pdf ϱ and $Y = F(X)$ has pdf υ, υ is uniformly continuous. Suppose $h_3 \to 0$ and $(nh_3)^{-1}\log n \to 0$. Let $R : \mathbb{R} \to \mathbb{R}$ be such that $\int |R(t)|f(t)\varrho(t)dt < \infty$ and $\int |R(t)|l_1(t)\varrho(t)dt < \infty$, where l_1 is as defined in Lemma 1. Then,

$$\frac{1}{n}\sum_{i=1}^{n} R(X_i)\tilde{g}_1(X_i) \xrightarrow{as} \int R(t)l_1(t)\varrho(t)dt.$$

Estimating μ_4

- Under H_0

$$\mu_4 = \int [\{\tau_1 f(t) - \tau_2 f^3(t)\} p''\{F(t)\} - 3\tau_2 f(t) f'(t) p'\{F(t)\} - \tau_2 f''(t) p\{F(t)\}]^2 w(t) dt.$$

Thus the problem of estimating μ_4 is equivalent to that of estimating

$$\int R(t) p^{(a)}\{F(t)\} p^{(b)}\{F(t)\} dt,$$

where $R(t)$ is a known function, $p^{(a)}(u) = \frac{\partial^a}{\partial u^a} p(u)$.

- $p^{(a)}(u)$ can be estimated by

$$\tilde{p}^{(a)}(u) = \frac{\partial^a}{\partial u^a} \tilde{p}(u) = \frac{1}{n h_3^{a+1}} \sum_{i=1}^{n} K_3^{(a)} \left(\frac{Y_i - u}{h_3} \right),$$

$Y_i = F(X_i)$, $1 \leq i \leq n$, and $K_3^{(a)}(u) = \frac{\partial^a}{\partial u^a} K_3(u)$, $a = 0, 1, 2$.

- Analogously, we estimate $p^{(b)}(u)$ through

$$\tilde{\tilde{p}}^{(b)}(u) = \frac{\partial^b}{\partial u^b} \tilde{\tilde{p}}(u) = \frac{1}{n h_4^{b+1}} \sum_{i=1}^{n} K_4^{(b)} \left(\frac{Y_i - u}{h_4} \right),$$

h_4 and K_4 may differ from h_3 and K_3.

Estimating μ_4

Lemma 3

Suppose that K_3 is a times differentiable, K_4 is b times differentiable, and satisfy certain further assumptions. Suppose that $Y = F(X)$ has pdf υ, υ has uniformly continuous a and b derivatives. Suppose that $h_3 \to 0$, $n^{-1}h_3^{-2a-1}\log(1/h_3) \to 0$, $h_4 \to 0$ and $n^{-1}h_4^{-2b-1}\log(1/h_4) \to 0$. Let $R : \mathbb{R} \to \mathbb{R}$ be such that

$$\int |R(t)|dt < \infty, \quad \int |R(t)\upsilon^{(a)}\{F(t)\}|dt < \infty, \quad \int |R(t)\upsilon^{(b)}\{F(t)\}|dt < \infty.$$

Then,

$$\int R(t)\tilde{p}^{(a)}\{F(t)\}\tilde{\tilde{p}}^{(b)}\{F(t)\}dt \xrightarrow{as} \int R(t)\upsilon^{(a)}\{F(t)\}\upsilon^{(b)}\{F(t)\}dt.$$

Power: fixed alternatives

Let $\alpha \in (0,1)$. As an immediate consequence of the stated results, the test

$$\Psi_\alpha = \Psi_\alpha(X_1, \dots, X_n) = \begin{cases} 1 & \text{if } \; nh^{1/2} \dfrac{|T^{red} - h^4 \hat{\mu}_4|}{\sqrt{2}\hat{\sigma}_1} \geq Z_{1-\alpha/2}, \\ 0 & \text{otherwise,} \end{cases}$$

is consistent against any fixed alternative, that is to say, if the data have pdf $g \notin \mathbb{F}_f$, then

$$\lim_{n \to \infty} P(\Psi_\alpha = 1) = 1,$$

whenever $\omega(t) > 0$, $\forall t \in \mathbb{R}$. The result is also true if $\hat{\sigma}_1$ is replaced by $\tilde{\sigma}_1$.

Power: local alternatives

- The first problem is that of defining these alternatives. Here we consider the following:

$$H_{1,n}: \quad \text{the pdf of the data is } g_n(x) = g(x) + a_n d_1(x),$$

where $g \in \mathbb{F}_f$, that is, $g(x) = f(x)p\{F(x)\}$, for some $p \in \mathcal{P}$, $a_n \to 0$ and $\int d_1(x)dx = 0$.

- Under $H_{1,n}$, g_n can be uniquely expressed as

$$g_n(x) = f_n(x)p_n\{F_n(x)\},$$

where $f_n \in \mathcal{S}$, $F_n(x) = \int_{-\infty}^{x} f_n(u)du$ is the cdf associated with the pdf f_n,

$$f_n(x) = f(x) + a_n d(x),$$

with

$$d(x) = \{d_1(x) + d_1(-x)\}/2 \in \mathcal{S}$$

Power: local alternatives

Theorem 5

Suppose assumption in Theorem 2 hold. Assume also that d_1 and $c : [0, 1] \to \mathbb{R}$, defined as $c(u) = d_1\{F^{-1}(u)\}/f\{F^{-1}(u)\}$, are bounded, two times differentiable, with second derivative bounded and uniformly continuous. If $H_{1,n}$ holds, then

$$nh^{1/2}(T^{red} - \mu_{04} - 2a_n\mu_{05} - a_n^2\mu_{06}) \xrightarrow{\mathcal{L}} \sqrt{2}\sigma_1 N_1,$$

where μ_{04} and σ_1 are as defined in Theorem 2, $\mu_{05} = h^4\mu_5$, $\mu_{06} = h^4\mu_6$,

$$\mu_5 = \int \left[\tau_1 f(t)p''\{F(t)\} - \tau_2 g''(t)\right] \left[\tau_1 f(t)c''\{F(t)\} - \tau_2 d_1''(t)\right] \omega(t)dt,$$

$$\mu_6 = \int \left[\tau_1 f(t)c''\{F(t)\} - \tau_2 d_1''(t)\right]^2 \omega(t)dt.$$

Power: local alternatives

- The test Ψ_α is able to detect local alternatives such that

$$\mu_5 \neq 0 \quad \text{and} \quad nh^{1/2+4} a_n \nrightarrow 0$$

or

$$\mu_6 \neq 0 \quad \text{and} \quad nh^{1/2+4} a_n^2 \nrightarrow 0.$$

- Suppose that $\mu_5 \neq 0$. Since we are assuming that $nh^5 \longrightarrow \delta$, for some $\delta \geq 0$, this implies that the test Ψ_α is able to detect local alternatives converging to the null hypothesis at a rate greater than or equal to $n^{-1/10}$.
- This shortcoming persist if instead of T^{red} we consider a test based on the initially proposed test statistic T.
- The best choice for h is

$$h = cn^{-1/5}, \text{ for some } c > 0.$$

Some numerical results

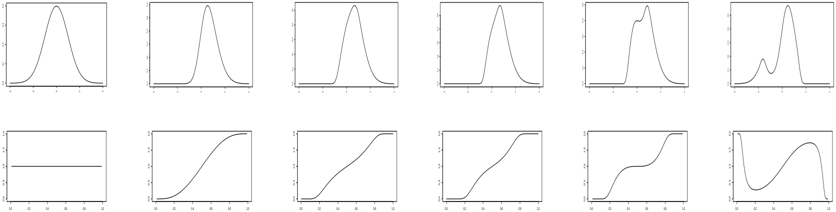
$$H_{0N}: \ f \text{ is the pfd of a } N(0,1),$$

$$K_1, \, K_2, \, h, \, w, \, \hat{\sigma}_1, \, \tilde{\sigma}_1, \, \mu_4,$$

Some numerical results

To investigate the goodness of the asymptotic approximation to the null distribution, we generated samples from generalized skew-normal distribution with pdf

$$g(x) = 2\phi(x)\Phi(\alpha_1 x + \alpha_3 x^3).$$



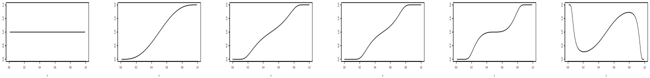
(0,0) (2,0) (1,1) (1,2) (0,2) (2,-1)

Figure 1. Graphs of the pdf g, in the top row, and of the associated pdf p, in the bottom row, for the selected values of (α_1, α_3), under each graph.

Some numerical results

Table 1. Estimated type I probability errors.

		(0,0)		(2,0)		(1,1)		(1,2)		(0,2)		(2,-1)	
n	*h*	f05	f10	f05	f10	f05	f10	f05	f10	f05	f10	f05	f10
50	.08	.038	.062	.045	.069	.040	.079	.054	.076	.046	.082	.080	.106
	.10	.042	.064	.060	.085	.049	.081	.071	.096	.074	.114	.113	.161
	.12	.048	.062	.075	.095	.072	.104	.102	.138	.128	.172	.174	.237
	.14	.056	.087	.105	.142	.100	.150	.147	.191	.198	.243	.286	.372
100	.06	.037	.067	.036	.076	.035	.064	.045	.078	.042	.063	.069	.101
	.08	.050	.073	.052	.081	.041	.072	.060	.085	.062	.101	.111	.165
	.10	.050	.074	.064	.105	.065	.109	.083	.109	.135	.192	.204	.292
	.12	.059	.089	.108	.161	.125	.191	.126	.174	.256	.326	.435	.540
200	.06	.037	.070	.068	.094	.049	.074	.051	.088	.054	.088	.096	.140
	.08	.044	.080	.080	.107	.064	.103	.082	.122	.122	.172	.159	.221
	.10	.068	.102	.102	.139	.133	.182	.150	.216	.297	.385	.316	.384
300	.05	.052	.082	.060	.088	.042	.084	.042	.064	.048	.088	.096	.162
	.06	.056	.086	.062	.090	.042	.078	.042	.060	.056	.084	.196	.274
	.07	.050	.078	.070	.102	.046	.070	.048	.066	.068	.104	.294	.390



Some numerical results

Table 2. Bootstrap estimated type I probability errors vs asymptotic approx. for $n = 50$ and $h = 0.10$.

	(0,0)		(2,0)		(1,1)		(1,2)		(0,2)		(2,-1)	
	f05	f10	f05	f10	f05	f10	f05	f10	f05	f10	f05	f10
Boot	.042	.092	.066	.114	.062	.130	.078	.150	.058	.156	.132	.230
Asym	.042	.064	.060	.085	.049	.081	.071	.096	.074	.114	.113	.161

Some numerical results

Table 3. Estimated type I probability errors with bootstrap selection of the bandwidth.

	(0,0)		(2,0)		(1,1)		(1,2)		(0,2)		(2,-1)	
n	f05	f10	f05	f10	f05	f10	f05	f10	f05	f10	f05	f10
50	.038	.062	.048	.076	.042	.088	.050	.082	.042	.074	.074	.100
100	.052	.086	.052	.090	.044	.078	.048	.090	.042	.075	.068	.104

Some numerical results

Table 4. Estimated powers with bootstrap selection of the bandwidth.

	t_5		U		χ^2_3		$M1$		$M2$	
n	f05	f10	f05	f10	f05	f10	f05	f10	f05	f10
50	0.048	0.070	1.000	1.000	0.500	0.594	0.532	0.630	0.050	0.088
100	0.048	0.082	1.000	1.000	0.820	0.872	0.860	0.904	0.513	0.613

M1=0.5 N(0,1)+0.5 N(4,1), M2=0.5 N(0,1)+0.5 N(4,σ^2), $\sigma = 2$

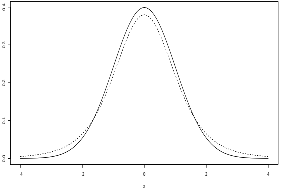


Figure 2. Graphs of the pdf of the t_5 (dashed line) and the pdf of the law $N(0, 1)$ (solid line).

Further research

The results in this talk could be extended in several directions:

- we could let the pdf in the the null hypothesis depend on unknown parameters, such a location and scale parameters;
- the results could be extended to the d-dimensional case, for any fixed $d \geq 1$;
- instead of letting the window parameter go to 0 as the sample size increases, we could keep it fixed in order to get better results for the detection of local alternatives;
- ...

Further research

For the second extension one can follow the steps in this work, by taking into account that Abtahi and Towhidi (2013) have shown that any continuous d-variate pdf g can be expressed as

$$g(x) = f(x)p\{F(x_1), F(x_2|x_1), \dots, F(x_d|x_1, x_2, \dots, x_{d-1})\}, \qquad (2)$$

where f is a symmetric pdf on on $\mathbb{R}^d$.

In addition, these authors have proven the following, which is a multivariate analogue of Proposition 1

Proposition 2 (Abtahi and Towhidi, 2013)

Let f be pdf of a symmetric random vector and let p be a pdf defined on $[0, 1]^d$.

(a) If the pdf of X is as in (2), then $\mathcal{F}(X) = (F(X_1), F(X_2|X_1), \dots, F(X_d|X_1, \dots, X_{d-1})$ has pdf p.

(b) If X has pdf $p \in \mathcal{P}$, then $(F^{-1}(X_1), F^{-1}(X_2|X_1), \dots, F^{-1}(X_d|X_1, \dots, X_{d-1})$ has pdf (2).

Thank you for your attention !!

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Assumptions for Theorem 1

ASSUMPTION a The kernel K is of bounded variation and uniformly continuous, with modulus of continuity m_K.

ASSUMPTION b $\int |K(x)|\, dx < \infty$ and $K(x) \to 0$ as $|x| \to \infty$.

ASSUMPTION c $\int K(x) dx = 1$.

ASSUMPTION d $\int |x \log |x||^{1/2}\, dK(x) dx < \infty$.

ASSUMPTION e $\int_0^1 \{\log(1/u)\}^{1/2} d\gamma(u) < \infty$, where $\gamma(u) = \{m_k(u)\}^{1/2}$.

Assumptions for Theorem 2

ASSUMPTION 1 $h = h_n \to 0$, $nh \to \infty$.

ASSUMPTION 2 (a) $K : \mathbb{R} \to [0, \infty)$ is bounded and satisfy

$$\int K(x)dx = 1, \quad \int xK(x)dx = 0, \quad \int x^2 K(x)dx < \infty.$$

(b) K is continuous.

ASSUMPTION 3 The functions f and p are bounded, two times differentiable, their second derivatives are bounded and uniformly continuous.

ASSUMPTION 4 (a) $\omega : \mathbb{R} \to [0, \infty)$ is bounded and satisfies

$$\int \omega(x)dx < \infty.$$

(b) ω is continuous.