Evolutionary Stability Against Multiple Mutations

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Overview

- Definition of ESS and multiple ESS
- Some Observations
- Equivalence between Pure ESS and multiple ESS for 2×2 games
- Conclusions

We consider symmetric games with payoff function $u : \Delta \times \Delta \to \mathbb{R}$, where Δ is the probability simplex in $\mathbb{R}^k$ and u is given by the bilinear function

$$u(p, q) = \sum_{i,j=1}^{k} p_i q_j u(e^i, e^j).$$

Here e^i; $i = 1, 2, \cdots, k$, are pure strategies, and
$p = (p_1, p_2, \cdots, p_k)$, $q = (q_1, q_2, \cdots, q_k)$ are mixed strategies of the players.

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Definition (ESS)

A strategy $p \in \Delta$ is called an ESS, if for any mutant strategy $r \neq p$, there is an invasion barrier $\bar{\epsilon} = \bar{\epsilon}(r) \in (0, 1)$ such that

$$u(p, \epsilon r + (1 - \epsilon)p) > u(r, \epsilon r + (1 - \epsilon)p) \text{ for all } 0 < \epsilon \leq \bar{\epsilon}. \tag{1}$$

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Definition

Let m be a positive integer. A strategy $p \in \Delta$ is said to be evolutionarily stable (or robust) against 'm' mutations if, for every $r^1, \cdots, r^m \neq p$, there exists $\bar{\epsilon} = \bar{\epsilon}(r^1, \cdots, r^m) \in (0, 1)$ such that

$$
\begin{aligned}
&u(p, \epsilon_1 r^1 + \cdots + \epsilon_m r^m + (1 - \epsilon_1 - \cdots - \epsilon_m)p) \\
&\qquad > \max_{1 \le i \le m} u(r^i, \epsilon_1 r^1 + \cdots + \epsilon_m r^m + (1 - \epsilon_1 - \cdots - \epsilon_m)p),
\end{aligned}
$$

for all $\epsilon_1, \ldots, \epsilon_m \in (0, \bar{\epsilon}]$.
A strategy $p \in \Delta$ is said to be evolutionarily stable against multiple mutations if it is evolutionary stable against 'm' mutations for each $m = 1, 2, \cdots$.

First Observation

In Weibull, an example of a mixed strategy ESS in a 2×2 symmetric game is given and this ESS is shown not to be evolutionarily stable against two simultaneous mutations. In fact, we can prove that an evolutionarily stable strategy against multiple mutations is necessarily pure.

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Theorem

An evolutionarily stable strategy against multiple mutations is necessarily a pure strategy.

Proof:

Let p be evolutionarily stable against multiple mutations. If possible, let p be a mixed strategy. Without loss of generality let $p = (p_1, p_2, \cdots, p_l, 0, \cdots, 0)$, with $p_i > 0$, $i = 1, 2, \cdots, l$. Let $\bar{\epsilon} = \bar{\epsilon}(e^1, e^2, \cdots, e^k)$ be the invasion barrier corresponding to all the k pure mutations. Let $r = \alpha_1 e^1 + \alpha_2 e^2 + \cdots + \alpha_l e^l + (1 - \alpha_1 - \alpha_2 - \cdots - \alpha_l)p$, where $0 < \alpha_1, \alpha_2, \cdots, \alpha_l < \bar{\epsilon}$. Then, we have

$$\sum_{i=1}^{l} p_i u(e^i, r) = u(p, r) > \max\{u(e^1, r), u(e^2, r), \cdots, u(e^l, r)\}, \tag{2}$$

which is a contradiction. Thus p must be pure.

Theorem

For $p \in \Delta$, the following are equivalent:

(a) *p is robust against two mutations;*

(b) *$p \in \Delta^{NE}$, and, for every $q \in BR(p) \setminus \{p\}$ and $r \in \Delta$,*

$$u(p,q) > u(q,q) \quad \textit{and} \quad u(p,r) \geq u(q,r).$$

Note that the first condition in (b) characterizes ESS.

Second Observation

Our second observation says that the evolutionary stable strategy against two mutations and evolutionary stable strategy against m mutations are equivalent.

Theorem

A strategy is evolutionarily stable against two mutations if and only if it is evolutionarily stable against m mutations, where $m > 2$.

Let p be evolutionarily stable against two mutations. Let $r^1, r^2, \cdots, r^m$ be m mutations that appear with proportions $\epsilon_1, \epsilon_2, \cdots, \epsilon_m$, respectively. For $i = 1, 2, \cdots, m$, let

$$\begin{aligned} h_i(\epsilon_1, \epsilon_2, \cdots, \epsilon_m) := u(p, \epsilon_1 r^1 + \epsilon_2 r^2 + \cdots + \epsilon_m r^m + (1 - \epsilon_1 - \epsilon_2 - \cdots - \epsilon_m)p) \\ - u(r_i, \epsilon_1 r^1 + \epsilon_2 r^2 + \cdots + \epsilon_m r^m + (1 - \epsilon_1 - \epsilon_2 - \cdots - \epsilon_m)p) \end{aligned}$$

We need to show that for sufficiently small $\epsilon_1, \epsilon_2, \cdots, \epsilon_m$, $h_i(\epsilon_1, \epsilon_2, \cdots, \epsilon_m) > 0$ for each $i = 1, 2, \cdots, m$. Note that

$$\begin{aligned} h_i(\epsilon_1, \epsilon_2, \cdots, \epsilon_m) = \epsilon_1[u(p, r^1) - u(r^i, r^1)] + \epsilon_2[u(p, r^2) - u(r^i, r^2)] \\ + \cdots + \epsilon_m[u(p, r^m) - u(r^i, r^m)] \\ + (1 - \epsilon_1 - \epsilon_2 - \cdots - \epsilon_m)[u(p, p) - u(r^i, p)]. \end{aligned} \tag{3}$$

Second Observation: Proof

Fix i. If $r^i \in BR(p)$, then $u(r^i, p) - u(p, p) = 0$.

From Theorem 4, we have

$$u(r^i, r^i) < u(p, r^i) \text{ and } u(r^i, r^j) \leq u(p, r^j)$$

for all $j \neq i$. As a result, we have $h_i(\epsilon_1, \epsilon_2, \cdots, \epsilon_m) > 0$ for $\epsilon_1, \epsilon_2, \cdots, \epsilon_m > 0$, whenever $r^i \in BR(p)$.

Now let $r^i \notin BR(p)$. Then $u(p, p) - u(r^i, p) > 0$. Thus for sufficiently small $\epsilon_1, \epsilon_2, \cdots, \epsilon_m > 0$, we must have $h(\epsilon_1, \epsilon_2, \cdots, \epsilon_m) > 0$. And hence p is evolutionarily stable against m mutations.

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Proof:

Without loss of generality, let the pure strategy e^1 be an ESS and let $\bar{\epsilon}$ be the corresponding uniform invasion barrier.

Note that, for any $\alpha, \beta \geq 0$ with $\alpha + \beta \leq 1$, we have

$$
\begin{aligned}
& u(e^1, \alpha e^1 + \beta e^2 + (1 - \alpha - \beta) e^1) \\
& = u(e^1, \beta e^2 + (1 - \beta) e^1) > u(e^2, \beta e^2 + (1 - \beta) e^1) \\
& \qquad\qquad = u(e^2, \alpha e^1 + \beta e^2 + (1 - \alpha - \beta) e^1), \quad (4)
\end{aligned}
$$

by the definition of ESS, whenever $\beta \leq \bar{\epsilon}$ and for any $\alpha < 1$.

Let r^1, r^2 be any two mixed (or pure) strategies different from e^1. Let $0 < \epsilon_1, \epsilon_2 \leq \frac{\bar{\epsilon}}{2}$ and consider $w = \epsilon_1 r^1 + \epsilon_2 r^2 + (1 - \epsilon_1 - \epsilon_2)e^1$. We want to show that for sufficient small ϵ_1 and ϵ_2,

$$u(e^1, w) > \max\{u(r^1, w), u(r^2, w)\}.$$

Choose $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that

$$r^1 = \alpha_1 e^1 + \beta_1 e^2 \text{ and } r^2 = \alpha_2 e^1 + \beta_2 e^2.$$

Note that $\alpha_1 + \beta_1 = \alpha_2 + \beta_2 = 1$ and α_1, α_2, both are different from 1. Let

$$\hat{\alpha} = \alpha_1 \epsilon_1 + \alpha_2 \epsilon_2 \text{ and } \hat{\beta} = \beta_1 \epsilon_1 + \beta_2 \epsilon_2.$$

Clearly $\hat{\alpha}, \hat{\beta} \leq \bar{\epsilon}$. Using these notations, we have

$$w = \hat{\alpha} e^1 + \hat{\beta} e^2 - (1 - \hat{\alpha} - \hat{\beta}) e^1.$$

Applying (4), we obtain

$$u(e^1, w) > u(e^2, w).$$

Now

$$u(e^1, w) = \alpha_2 u(e^1, w) + \beta_2 u(e^1, w) > \alpha_2 u(e^1, w) + \beta_2 u(e^2, w) = u(r^2, w)$$

Similarly we can show that

$$u(e^1, w) > u(r^1, w),$$

which completes the proof of the theorem.

Thus we have the following characterization for evolutionary stability against multiple mutations in 2×2 games.

Theorem

In symmetric games with exactly two pure strategies, the pure strategy e^1 is an ESS if and only if it is either a strict symmetric Nash equilibrium or

$$u(e^1, e^1) \geq u(e^2, e^1) \text{ and } u(e^1, e^2) > u(e^2, e^2) \tag{5}$$

UNIQUE NE + PURE $\rightarrow$ MULTIESS

Now we ask the following question. Can the above result be true for general games? The answer is no in general. However, if the game has unique NE and it is pure, then we can show that it is necessarily stable against multiple mutations.

Note that a unique NE is not neessarily ESS.

Unique NE + Pure → MultiESS

Definition

A Nash equilibrium is said to be regular (or quasi-strict) provided no player has a pure best reply to the opponent's optimal strategy outside the carrier of his own strategy.

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A Nash equilibrium is said to be regular (or quasi-strict) provided no player has a pure best reply to the opponent's optimal strategy outside the carrier of his own strategy.

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Proof:

From the results of Jansen, Norde, every bimatrix game admits at least one quasi-strict Nash equilibrium. Thus the unique pure Nash equilibrium must be quasi-strict. Now by the definition of quasi-strict equilibrium, we infer that the a pure Nash equilibrium must be strict. Consequently it is evolutionarily stable against multiple mutations.

Conclusions

- The evolutionary stability against multiple mutations provides a refinement of strict Nash equilibrium.
- Some applications to economic problems is currently under exploration.
- Evolutionary games with continuous action spaces is an active area with lots of interesting mathematical questions. The difficulty is due to several topologies on the space of probability measures.
- (Evolutionary) games on graphs.

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Thank You