Resilience of distributed averaging in large-scale networks

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BIRS workshop on Asymptotics of Large-Scale Interacting Networks
Banff, February 26, 2013

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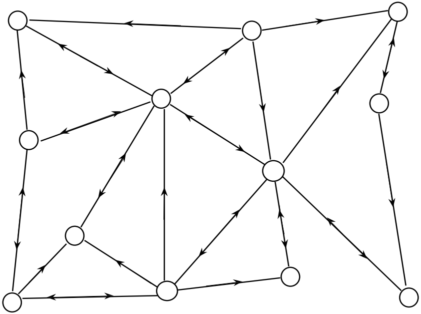
Distributed averaging

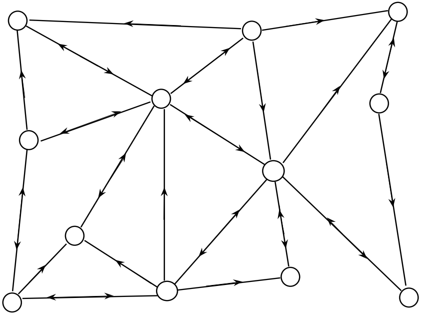


Stochastic irreducible P invariant measure $\pi = \pi P$

Averaging dynamics: $x(0) = y\,,\ x(t+1) = Px(t)$

$$(P \text{ aperiodic}) \quad \Longrightarrow \quad x_v(t) \stackrel{t\to\infty}{\longrightarrow} \pi y, \quad \forall v \in \mathcal{V}$$

Perturbation



Stochastic irreducible P $\qquad \pi = \pi P$

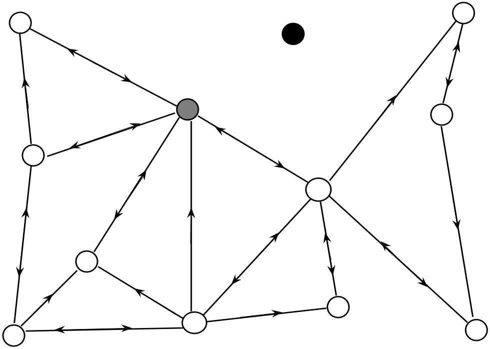
Stochastic (possibly not irreducible) $\tilde{P}$ $\qquad \tilde{\pi} = \tilde{\pi}\tilde{P}$

When is $||\tilde{\pi} - \pi||$ small?

Interested in $\mathcal{W} := \{w : P_{w\cdot} \neq \tilde{P}_{w\cdot}\}$ 'small' as $n := |\mathcal{V}| \to \infty$

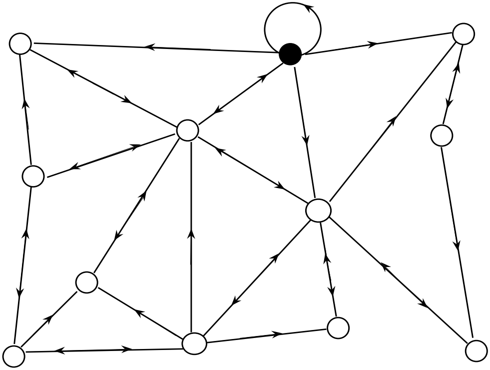
($||\tilde{P} - P||$ not necessarily small)

Example 1: node failure in sensor network



$\mathcal{U} := \{\text{failed nodes}\}$ $\qquad$ $\mathcal{W} = \mathcal{U} \cup \{v : \exists u \in \mathcal{U}, P_{vu} \neq 0\}$

Example 2: influential agents in social network



P doubly stochastic $\implies$ $\pi =$ uniform

$\mathcal{W} = \{$influential agents$\}$ [Acemoglu *et al*, '10]

$\tilde{P}_{wv} = \alpha_w P_{wv}, \quad \tilde{P}_{ww} = \alpha_w P_{ww} + 1 - \alpha_w \qquad \alpha_w \in (0,1) \quad \forall w \in \mathcal{W}$

State of the art

1 [Mitropanov, 05] $$||\tilde{\pi} - \pi||_1 \leq e\tau ||\tilde{P} - P||_\infty$$

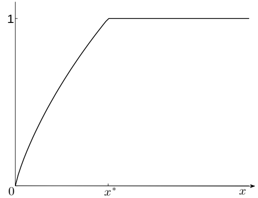
$$\tau := \inf\{t \geq 0 : \ ||\mu P^t - \pi||_1 \leq 1/e \ \forall \mu\} \qquad \text{mixing time}$$

2 [Acemoglu *et al*,'10] $$||\tilde{\pi} - \pi||_2 \leq \frac{||\tilde{P} - P||_2}{1 - \lambda_2(P)}$$

$$\lambda_2(P) := \text{ second largest eigenvaue (in module)}$$

- proof based on $\tilde{\pi} - \pi = \tilde{\pi}(\tilde{P} - P) \sum_{k \geq 0} P^k$
- fast mixing $\Longrightarrow$ more robustness
- need $||\tilde{P} - P|| = o(1)$ to prove $||\tilde{\pi} - \pi|| = o(1)$

Theorem [C., Fagnani]



P irreducible, $\pi = \pi P$, $\quad \tilde{\pi} = \tilde{\pi}\tilde{P}$.

$$\theta(x) := \begin{cases} x \ln\left(e^2/x\right) & 0 \leq x \leq x^* \\ 1 & x \geq x^* \end{cases}$$

Then,

$$||\tilde{\pi} - \pi||_{TV} \leq \theta\left(\frac{1}{\tilde{\gamma}_{\mathcal{W}}}\ \frac{\tau}{\tau^*_{\mathcal{W}}}\right)$$

- τ mixing time of P
- $\tau^*_{\mathcal{W}} := \min\left\{\mathbb{E}[T_{\mathcal{W}} | V_0 = v] :\ v \in \mathcal{V} \setminus \mathcal{W}\right\}$

where $V_0, V_1, \ldots$ Markov chain (P), $T_{\mathcal{W}} := \inf\{t :\ V_t \in \mathcal{W}\}$

- $\tilde{\gamma}_{\mathcal{W}} := \sup\limits_{t \geq 1} \min\limits_{\substack{w \in \mathcal{W}: \\ \pi_w > 0}} \frac{1}{t}\mathbb{P}(\tilde{T}_{\mathcal{V}\setminus\mathcal{W}} \leq t |\ V_0 = w)$ escapability
- proof: based on coupling, Kac's formula

Locally tree-like networks

$$\text{network locally tree-like around } \mathcal{W} \quad \Longrightarrow \quad \tau^*_{\mathcal{W}} \asymp \frac{1}{\pi(\mathcal{W})}$$

- connected Erdos-Renyi: $\mathcal{G}(n,p)$ with $p = \beta n^{-1} \log n$, $\beta > 1$
- configuration model: degree distrib. $\{p_k\}_{k\geq 3}$, $\sum_k p_k k^2 < \infty$

$$\text{random } \mathcal{W} \subseteq \mathcal{V} \qquad |\mathcal{W}| = O(n^{1-\varepsilon}) \quad \varepsilon > 0$$

$$\Longrightarrow \qquad ||\tilde{\pi} - \pi||_{TV} = o(1) \qquad \text{w.h.p. as } n \to \infty$$

d-dimensional grids

$$d \geq 3, \qquad |\mathcal{W}| \leq C \qquad \text{(possibly not localized)}$$

$$\Longrightarrow \qquad ||\tilde{\pi} - \pi||_{TV} = o(1) \qquad \text{as } n \to \infty$$