Coloring Games Group Formation

… why it's hard to keep your friends when you have enemies

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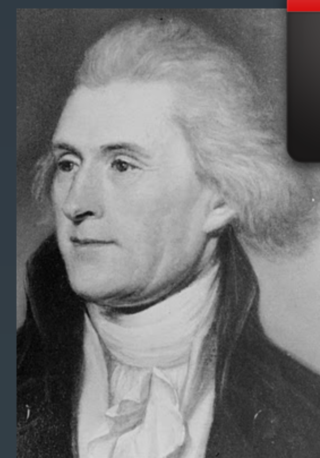
joint work with Guillaume, Dorian and Juba

Why do we have social networks?

American idealism or French Hedonism

"He who receives an idea from me,
receives instruction himself
without lessening mine"



"Nul plaisir n'a goust pour moi
sans communication,
mais il vaut mieux encore estre seul
qu'en compagnie ennuyeuse et inepte."

These principles in practice today

- Your network position can be an advantage
  - To find about jobs [Gr74], innovation [CKM57]

  Information sharing = mutual benefits

- But sharing information can be harmful
  - Tension, self-censor, "privacy paradox" [B06]

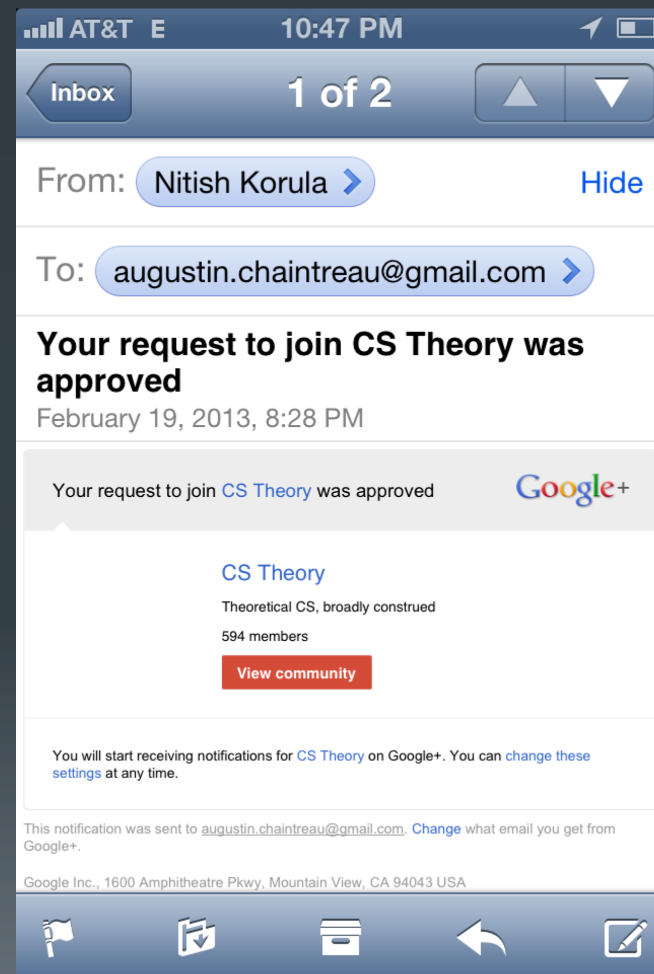
  Privacy = right to protect against this risk

- This tradeoff explains links and communities

Communities = context to share

- Communities create boundaries needed to control information
  - Maximize mutual benefit
  - Avoid negative exposure

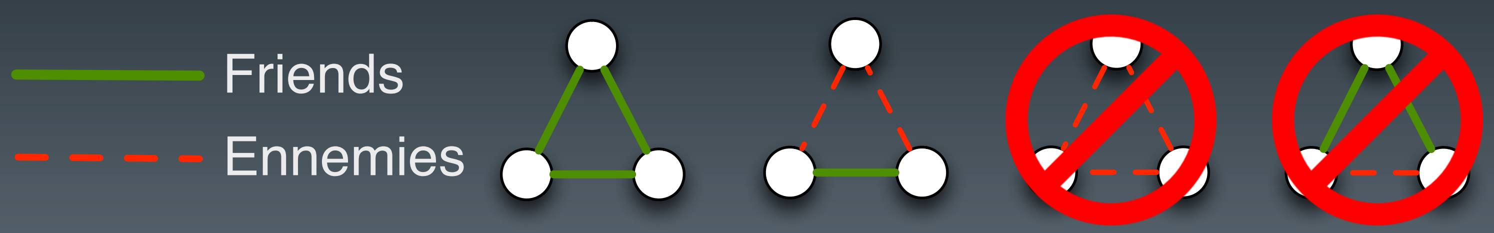
- A simple principle
  - Surprisingly unexplored
  - Need to handle +/- links



Background: Communities

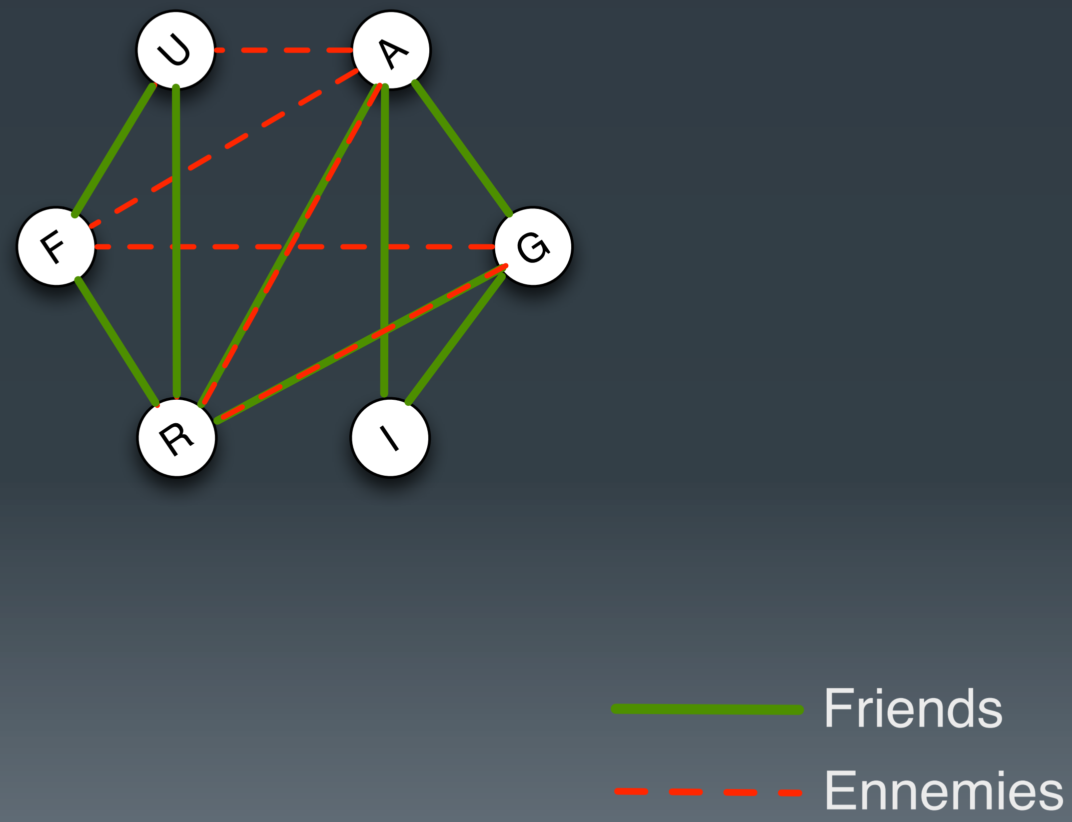
Communities: what we know so far

- Whenever the graphs contains positive links
  - Partition (most) vs. Overlapping (some)
  - In practice, fast clustering methods maximizing a score: modularity [N06] conductance [L08]
  - In theory, positive results focus with bisection
- Whenever the graphs contains signed links



  - Premise: evolve toward structural balance [C56]

Structural balance at work



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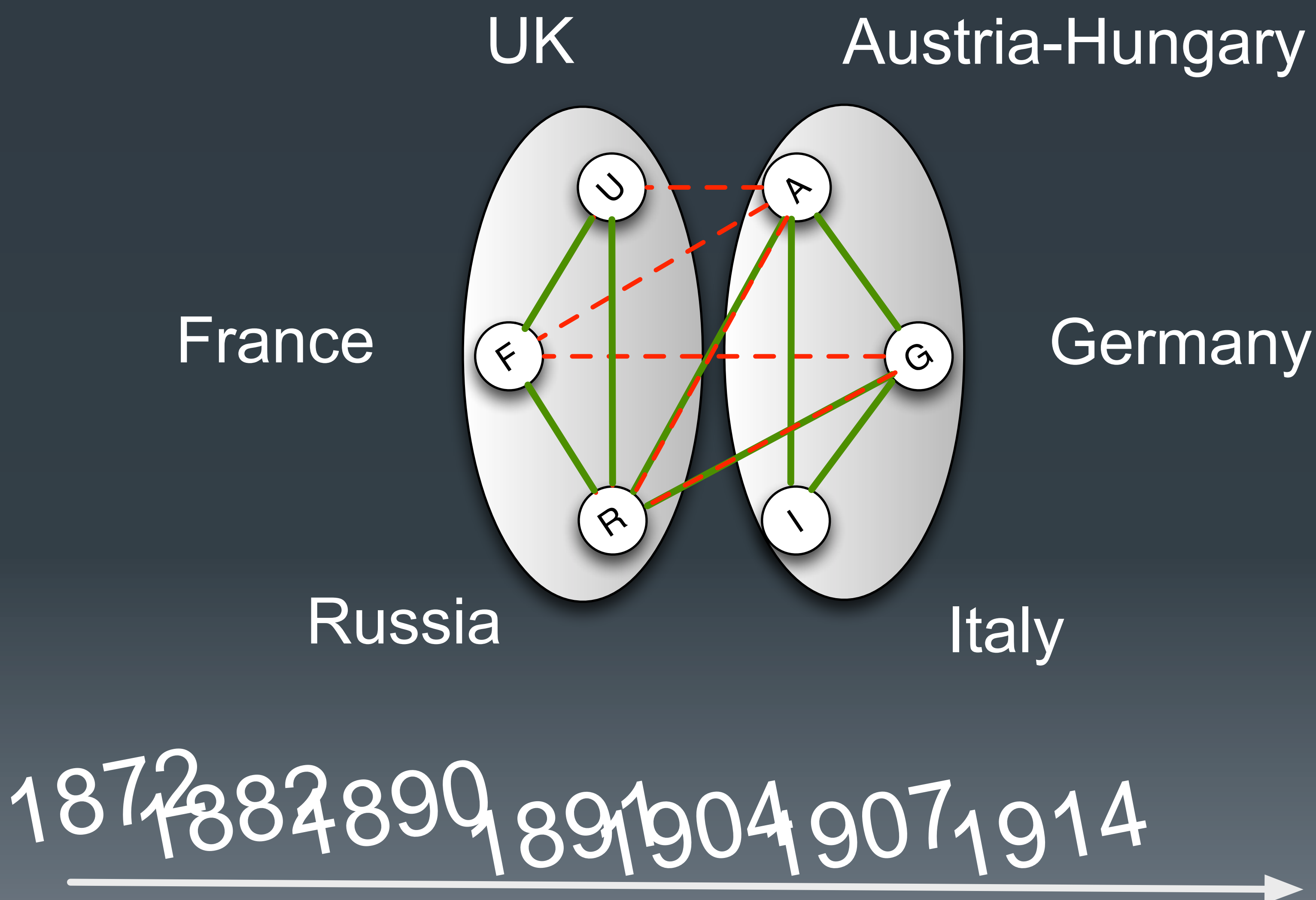
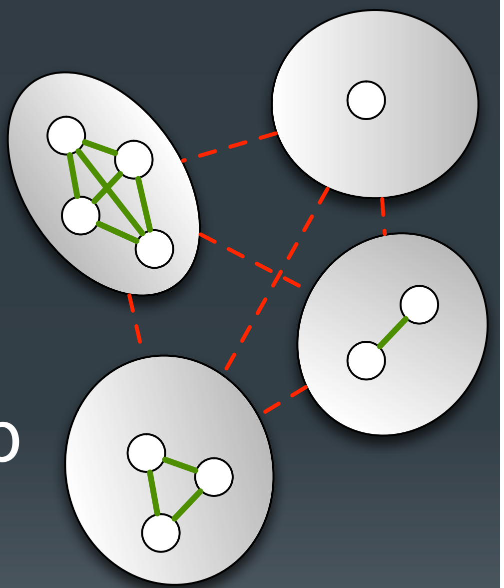
Structural balance at work



Structural Balance form group

- THM: Graphs that are strongly (resp. weakly) structurally balanced form 2 (resp. a few) antagonistic communities.
  - Graphs evolve to form self-reinforcing cliques



- But does not offer a model of group formation for information sharing
  - Graph usually fixed and not balanced [LHK10]
  - This dynamics does not represent node's utility

We need to revisit group formation

A different group formation dynamics

… capturing the benefit/risk tradeoff of sharing

- Utility representing how a group benefits a user depending on who she can reach

- Just like structural balance, we would like communities to be self-enforced
  - But with no global assumption on the graph

- General: Overlapping communities, …

Structure of this talk

- Coloring games

- Uniform case: friends + enemies
- Non-uniform case: friends + enemies + boring
- Extensions

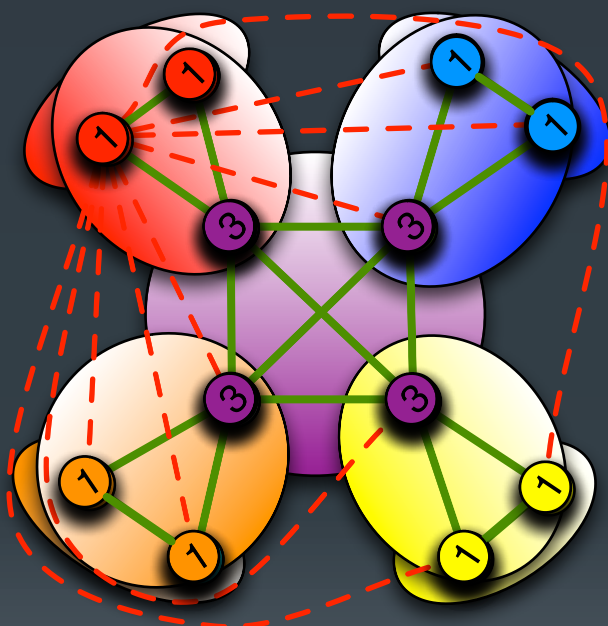
- Open problems

Utility = sum of weights for edges to neighbors in the same color!

Friends

Ennemies

Simplification 1:
friends weight +1
enemies weight -∞
No other weight exists



Has the game stabilized?

No node can benefit from changing color

Nor can subsets up to 3 nodes can

Simplification 2:
A node choose 1 color and can change at anytime

But a 4-deviation exists!

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Challenges of Group formation

1. A coloring (or partition) of the graph is k-stable if no k-deviation exists.
   Do such coloring exist? for k=1,2,3,4,…, n?
2. How many steps required to converge?
3. Are groups formed efficient to exchange?

- What if weights not in {-∞,+1}? w=0?
- What about overlapping groups?
- What about non-linear utility?

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Uniform case: friends + enemies

- All weights are in {-∞,+1}
  - "Clique with enemies", small clubs
  - First analyzed by K. Ligett-J. Kleinberg 2010
- Thm: a k-stable configuration exists for all k
  - Construction requires to solve a NP-hard pb
  - But the following terminates for any k
  - `while(`a k-deviation exists`) {` compute configuration after deviation`}`
- L(k,n)=worst case iteration for n nodes

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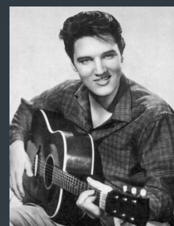
L(k,n) = # iteration in a coloring game before k stability

		Best prior bound	Our results
$k = 1$		$O(n^2)$ Potential	$\sim \frac{2}{3} n\sqrt{n}$ exact
$k = 2$		$O(n^2)$ Potential	$\sim \frac{2}{3} n\sqrt{n}$ exact
$k = 3$		$O(n^3)$ Different Potential	$\Theta(n^2)$ order tight
$k = 4$		$O(2^n)$ No Potential exists Exponential?	$\Omega(n^{a \ln(n)})$ $O(e^{\sqrt{n}})$ Not polynomial Nor exponential!

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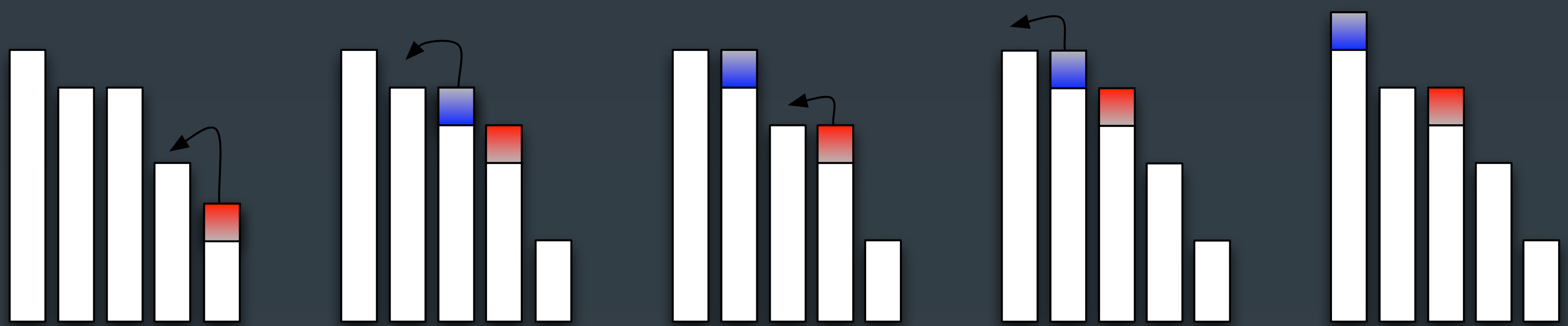
Proof:

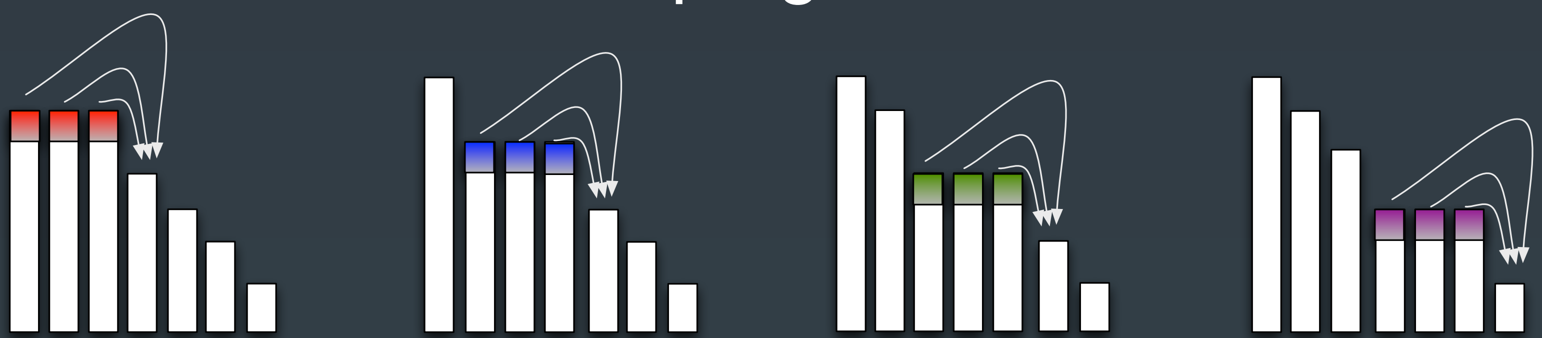
- Let order group by size and draw them



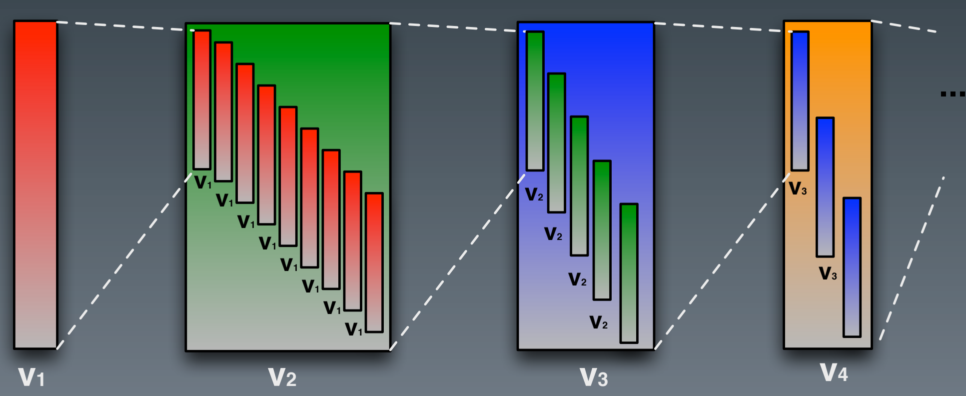
  - Evolution of groups follow integer partitions
  - Decomposition of a sand pile reversed in time
  - For k=1 this analysis is exact (extends to k=2)
  - For k=3 a special ordering offers $O(n^2)$ bounds

Two objects used in lower bounds

- k=3: “cascade” slows progress



- k=4: “recursive cascade” breaks any polynom



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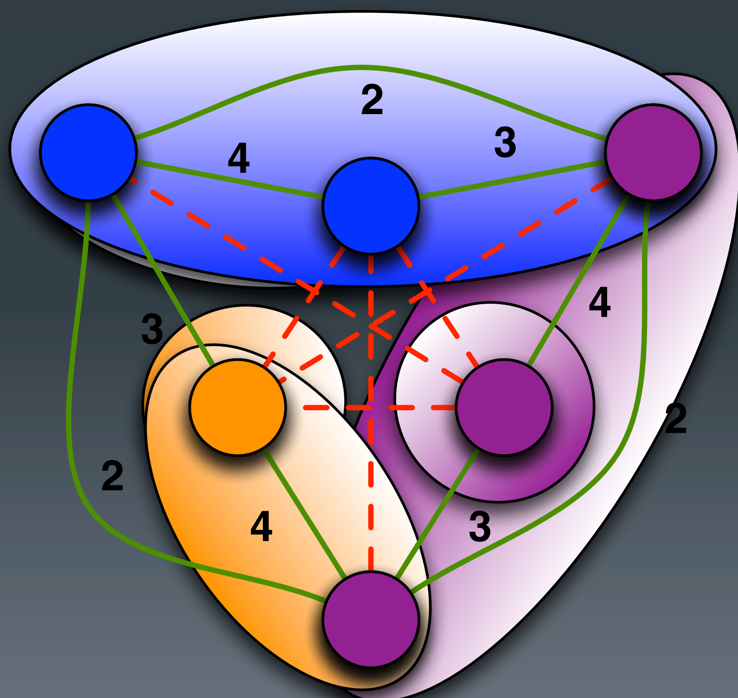
The non uniform case

- General weights:
  - Some nodes are simply boring: edge w=0
  - Not all friends are equal: edges with varying w



Characterizing when stability fails!

- No results known for general weights
- Prop: There always exist a 1-stable coloring
  - And we have just seen it's not true for k=2
  - But can we do better if we fix weights? $k_{max}(W)$
  - For instance, we have $k_{max}(\{-\infty,+1\})=\infty$ and also $k_{max}(\{-\infty,+2,+3,+4\})=1$

- Ideally, prove how set of weights impacts k_{max}
  - And also which graph cause pathologies

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Good and bad news

- We can exactly compute k_{max} for any weights
  - $k_{max}(\{-\infty,+1\})=k_{max}(\mathbb{N}^{+})=k_{max}(\mathbb{N}^{-})=k_{max}(\{-1,+n\})=\infty$
  - $k_{max}(\{-\infty,0,+1\}) = 2$ (challenging)
  - For others, if not trivially equivalent, $k_{max} = 1$

- Thm: whenever $k>k_{max}$ it is NP hard to decide whether a graph admits a k-stable coloring
  - You might find sufficient conditions, but they will necessarily be conservative

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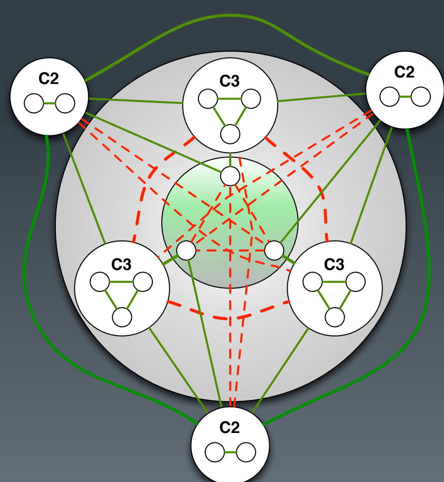
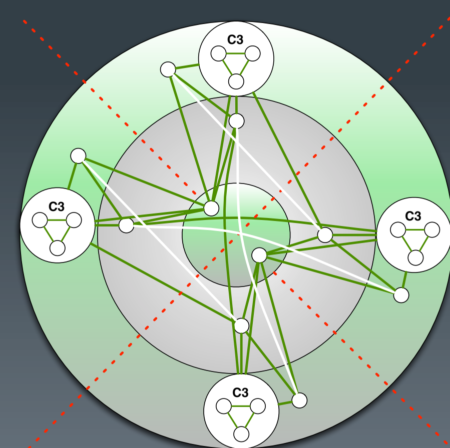
Proof: $k_{max}(\{-\infty,0,+1\}) = 2$

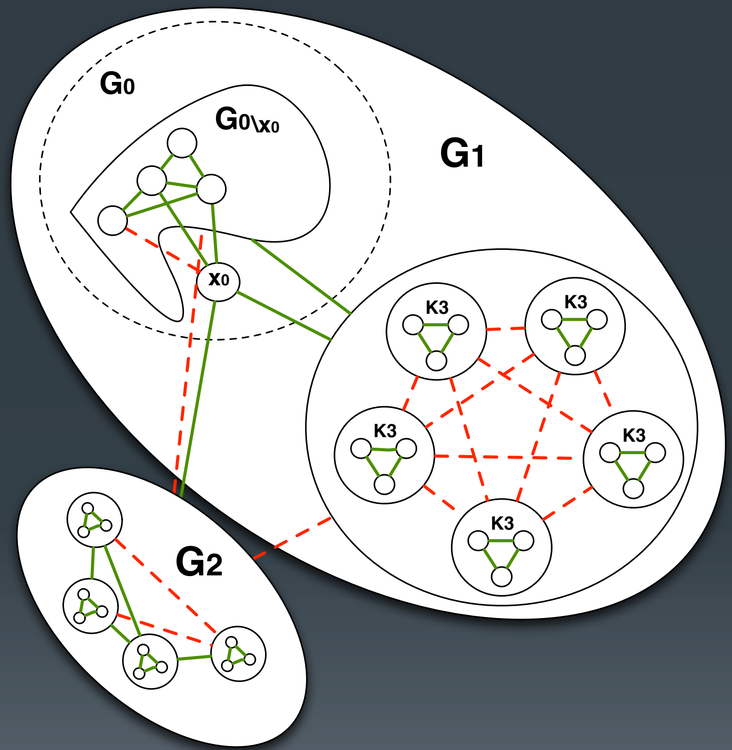
- Upper bound use a sufficient condition
  - If friendship graph has girth l, there exists a k=l-1 stable coloring found by best response
  - And all graph have l≥3 hence, k≥2
- Lower bound: one counter-example suffices





Proof: If $k>k_{max}$, stability is NP hard

- Starting from an unstable graph G_0
  - Constructs a mechanism input any graph G
  - With polynomial steps build a compound G1UG2
  - G's max. independent set contains c nodes iff the compound graph is stable



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Extension 1: Efficiency

- Thm: Approximating optimal is NP Hard.
- Is best/worst stable configuration optimal?
  - For k=1 (Nash eq.) Price of Stability/Anarchy PoS is small (optimal is Nash, but hard to find), PoA is large O(n)
  - As k increases, the gap between them narrows price of 2-anarchy is O(positive degree)
  - But also, stability sometimes ceases

- Exhibits tension between stability & efficiency

Extension 2: Overlapping Groups

- Nodes choose q colors
  - Positive results k=1,2 hold

Simplification 2: A node choose 1 color and can change at anytime

- But stability beyond k=2 can be complex!
  - Most counter examples become stable
  - Thm: Some graphs with weight {$-\infty$,+1} are not 3-stable as soon as 2 colors are available!
  - Intuitively because nodes have more choices
- Positive consequence on efficiency

Extension 3: Non linear utility

- What if nodes receive utility from group effect … not just a sum of pairwise interactions.
- A formulation using hypergraph generalizes most positive results in k=1 and k=2 cases

- A very rich model that can handle:
  - Higher order incompatibility (forbidden subsets)
  - Network effect, diminishing return

Conclusion

- The benefit/risk tradeoff of information can be analyzed in a group formation game
  - The principles are simple, the form general
  - Remarkable stability & combinatorial properties

- More work on group formation to establish
  - Practical sufficient condition on stability
  - Convergence to stability with general algorithm
  - Alliance in market with substitute/complements

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Thank you!