Partial differential equations in the social and life science: emergent challenges in modeling, analysis, and computations

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1 Overview of the Field

In recents years the study of behavior of interacting agents has gained increasing interest in several research fields. In economics, the evolution of social phenomena like financial markets, crime hot-spots, traffic behaviors and opinion formations have been studied quite extensively. In biological programs, the analysis of collective behavior of fish, bacteria and birds and the study of self-organizing systems have been fundamental research topics.

Natural phenomena driven by interactions of agents are present in various real life applications. Dependent on the application, such interactions occur at all length scales, and they can be understood and successfully described by different mathematical tools. One of the most common tools are differential equations of mean field type. Mean field description originates from statistical physics ideas, where interactions between agents (particles, players or bacteria) are studied using a external force field: each agent contributes to the creation of the mean field, and the mean field, in return, influences the behavior of the agents themselves. Another tool to describe interactions is via network or agent-based models. One fundamental question is to understand if such models can be rewritten via certain appropriate upscaling (homogenization) processes as effective systems involving differential equations. Several applications can be also modeled by discrete, ill-posed diffusion equations; such equations are quite common in granular flow, as well as in image processing and chemotaxis. They also arise in population dynamics and opinion dynamics, where they describe interaction pairs and compromised-based opinion formations.

2 Recent Developments and Open Problems

There is an extensive applied mathematics literature developed for problems in the biological, and physical sciences. Our understanding of social science problems from a mathematical standpoint is less developed, but also presents some very interesting and intriguing questions.

Mathematical models of partial differential equations which describe the behavior of interacting agents usually present analytical and numerical challenges: the difficulties are caused on one side by the nonlinearity and high (differential) order, and consequent lack of classical theory tools. On the other side we often deal with complex systems involving multi-scales. For discrete, ill-posed differential equations, the analysis of the stabilizing effects coming from the discrete scheme is essential to the study of long time asymptotics.

3 Presentation Highlights

Presentations were grouped according to

1. Mean field equations
2. Partial Differential Equations on graph and networks
3. Non-dominating sorting
4. Numerics
5. Collective behavior and self-organized dynamics
6. Theoretical talks

3.1 Mean Field equations

There were four talks concerning mean field type equations that arise as mathematical models in various applications, such as economics and social dynamics. We devoted particular attention to the novel theory of Mean Field Games. This theory is a new branch of game theory that has been recently formulated by Lasry and Lions and has remarkable applications that extend from economics to finance and social dynamics. The game involves a large number of players. The interactions between the players (particles, agents, bacteria, etc) are modeled by an external field in the following way: each agent contributes to the creation of the mean field, and the mean field, in return, influences the behavior of the agents themselves. A mean field game model can be described in two ways: the first one is via a single partial differential equation in the space of probability measures. The second way is via a system that couples a Hamilton- Jacobi equation with a backward Fokker-Planck equation. The mathematical structure of both models is very rich, and many interesting and important mathematical questions are still open.

- *Time dependent mean-field games.* Diogo Gomes (Instituto Superior Tecnico, Lisbon).

  The speaker and collaborators have developed various techniques to establish existence of regular solutions to mean-field games with local dependence on the measures. The methods are based upon various a-priori estimates coupled with continuation methods.

- *Information aggregation and mean-field games.* Ravi Srinivasan (The University of Texas Austin).

  In the past years, information flows in large populations have been studied in literature in a variety of contexts. We focus here on Boltzmann-like models of information aggregation and corresponding mean-field games, in which agents adopt strategies to optimize their individual welfare. The spearker related to works by economists on the role of innovation in economic growth.

- *Flocking dynamics and mean-field limit in the Cucker-Smale-type model with topological interactions.* Jan Haskovec (King Abdullah University of Science and Technology).

  The speaker presented a Cucker-Smale-type model for flocking, where the strength of interaction between agents depends on their topological distance, measured in units of agents' separation (number of intermediate individuals separating the agents), which makes the model scale-free. This was motivated by recent extensive observations of starling flocks, which suggested that interaction ruling animal collective behavior depends on topological rather than metric distance, as was the case in the classical Cucker-Smale and the vast majority of other flocking models. In the topological model the conditions leading to asymptotic flocking has been studied, defined as the convergence of the agents' velocities to a common vector. The shift from metric to topological interactions requires development of new analytical methods, taking into account the graph-topological nature of the problem. Moreover, it has been shown how to pass to the mean-field limit as the number of individuals tends to infinity, recovering kinetic and hydrodynamic descriptions. In particular, the speaker introduced the novel concept of topological distance in continuum descriptions, which is applicable to a broad variety of models of collective behavior.

- *Explicit derivation and stability of coherent patterns of motion in kinetic swarming models.* Stephan Martin (Imperial College London)

  There are several modeling concepts describing the behavior of individuals in an animal swarm of for example fish or birds which focus on a model of self-propelled interacting particles. It is a well-known fact that even minimalistic interaction rules allow for the emergence of coherent macroscopic patterns observed in nature, when applied to all members of a swarm. In the mean-field limit approach, a kinetic PDE is used to model the evolution of a particle density rather than tracing individuals separately. Its macroscopic closure allows for a compact description of some coherent patterns, such as flocks or mills. The speaker discussed the possibility to explicitly compute the stationary density profile of such states using a particular type of interaction potential called Quasi-Morse. Flock and mill profiles can be predicted with a cheap numerical procedure that does not necessitate particle simulations. The speaker also presented a result on the stability of flock solutions.

3.2 Partial differential equations on graph and networks

Nowadays the analysis and understanding of network structure is a prominent topic in several applications. In social media, the analysis of the network helps in understanding the large scale structure of the connectivities between users. In neuroscience the study of neuron networks is essential to understand the specific functional reaction in an organ or tissue. Graphs are considered as mutable objects, subjects to discrete and continuous changes. We are particularly interested in partial differential equations defined on graphs. They have the advantages of keeping the original domain of interest. However it requires the development of a new framework to understand diffusion, oscillation, and other phenomena that can occur on a graph.

- *Partial differential equations on networks.* Yves van Gennip (University of California Los Angeles).

  In the continuum, close connections exist between mean curvature flow, the Allen-Cahn (AC) partial differential equation, and the Merriman-Bence-Osher (MBO) threshold dynamics scheme. Graph analogues of these processes have recently seen a rise in popularity as relaxations of NP-complete combinatorial problems, which demands deeper theoretical underpinnings of the graph processes. The aim of this talk was to introduce graph processes in the light of their continuum counterparts, provide some background, prove the first results connecting them, illustrate these processes with examples and identify open questions for future study. The speaker derived a graph curvature from the graph cut function, the natural graph counterpart of total variation (perimeter). This derivation and the resulting curvature definition differ from those in earlier literature, where the continuum mean curvature is simply discretized, and bears many similarities to the continuum nonlocal curvature or nonlocal means formulation. This new graph curvature is not only relevant for graph MBO dynamics, but also appears in the variational formulation of a discrete time graph mean curvature flow. Estimates proving that the dynamics are trivial for both MBO and AC evolutions provided the parameters (the time-step and diffuse interface scale, respectively) are sufficiently small (a phenomenon known as "freezing" or "pinning"). It was also shown that the dynamics for MBO are nontrivial if the time step is large enough. These bounds are in terms of graph quantities such as the spectrum of the graph Laplacian and the graph curvature. Adapting a Lyapunov functional for the continuum MBO scheme to graphs, we prove that the graph MBO scheme converges to a stationary state in a finite number of iterations. Variations on this scheme have recently become popular in the literature as ways to minimize (continuum) nonlocal total variation.

- *Mathematics of Crime.* Andrea Bertozzi (UCLA)

This lecture used crime as a case study for using applied mathematical techniques in a social science application and covers a variety of mathematical methods that are applicable to such problems. Recent work on agent based models, methods in linear and nonlinear partial differential equations, variational methods for inverse problems and statistical point process models were reviewed. Applications are residential burglaries and gang crimes. Examples consider both ”bottom up” and ”top down” approaches to understanding the mathematics of crime, and how the two approaches could converge to a unifying theory.

- *Epidemiology on random graphs.* Reinhard Illner (University of Victoria)

  Classical epidemiology models assume that the population is well mixed, clearly a simplistic and unrealistic assumption. More recently, population structures have been modeled as random graphs. The speaker discussed the propagation of a disease that occured on such a structure and what the basic reproduction number looks like. An ODE model first suggested by Miller and Volz was presented and generalized to heterogeneous populations, where the graph structure reflects different groups. The behavior of the basic reproduction number in such contexts was discussed. The speaker noted that the basic reproduction number always decreases if connections are severed,showing that there are subtleties in the process.

3.3 Non-dominating sorting

In the workshop we had a speaker talking about non-dominated sorting. Non-dominated sorting is a fundamental problem in multi-objective optimization, and is equivalent to several important combinatorial problems. It can be used to, for instance, combine results from multiple search engines, or retrieve images from a database that are similar to multiple queries.

- *Hamilton-Jacobi equation for the continuum limit of non-dominated sorting.* Jeff Calder (University of Michigan).

  In one of the speaker's recent work, it is shown that non-dominated sorting of random points in Euclidean space has a continuum limit that corresponds to solving a Hamilton-Jacobi partial differential equation. In this talk the speaker describes this result, sketch the proof, and give some theoretical and practical applications.

3.4 Numerics

- *Inverse point source identification problems and applications.* Richard Tsai (The University of Texas Austin)

  The design of algorithms for the problem of identification or discovery of point sources in a complicated domain present several challenges. With robotic applications in mind, in which robots can be sent into the complicated domain to gather data, the speakers presented algorithms that determine in a sequential manner, where in the domain should additional data collected in order to improve the estimate of the source locations. The governing PDEs under consideration model respectively wave and diffusion phenomena as well as nonlinear reaction-diffusion equations.

- *Kinetic modeling and simulation algorithms of self-organized systems interacting with a few individuals.* Lorenzo Pareschi (Universita di Ferrara)

  In nature, self-organized systems such as flocks of birds, schools of fish and herds of sheep have to deal with the presence of external agents such as predators or leaders that modify their internal dynamics. Such situations involve a large number of individuals with their own social behavior interacting with a small number of other individuals acting as external point-source forces. Starting from a microscopic description, the speaker presented the derivation of different kinetic models and introduced a macroscopic model via a suitable hydrodynamic approximation. The efficient numerical solution of such systems was also discussed and several numerical results presented.

- *Evolution of Sets and Solving PDEs with Discontinuous Solutions.* Jean-Christophe Nave (McGill University).

  In this talk the speaker presented methods to (i) evolve arbitrary surfaces and curves and (ii) solve problems with jumps on co-dim 1 interfaces.

- *Volume-constrained nonlocal diffusion problems and their numerical approximations.* Qiang Du (Pennsylvania State University).

  This talk focused on the study a class of volume-constrained nonlocal diffusion problems on bounded domains. The speaker showed how nonlocal vector calculus provides analogies between the nonlocal model and classical models for diffusion as well as the notion of local and nonlocal fluxes. The analytical framework also allows for finite-dimensional approximations using both discontinuous or continuous Galerkin methods. Results on convergence, error analysis and condition number estimates were presented in both nonlocal setting and in local limit.

- *Centroidal Voronoi Tessellations of Rigid Bodies.* Lisa Powers (McGill University).

  The Centroidal Voronoi Tessellation (CVT) is an optimal configuration of points in convex domains. These tessellations are used in many areas from facility location problems to mesh generation.

  The speaker presented an extention of the notion of a CVT from points to rigid bodies in two and three dimensions. Given a finite set of shapes, their location can be optimized via translation and rotation by minimizing a suitable cost function. The CVT optimization problem for points is typically tackled using quasi-Newton methods and an iterative algorithm called Lloyd's method. The optimization problem for rigid bodies is challenging in part because integrals over generalized Voronoi regions must be calculated. The novelty of the presented algorithm is that the generalized Voronoi diagram is never explicitly calculated.

3.5 More analytic talks

- Pierre-Emmanuel Jabin (University of Maryland) Title: New regularity estimates for compressible transport phenomena. This talk focused on the critical regularity estimates for the advection of particles or biological organisms. Those are transported by a rough velocity field with unbounded divergence. This creates two major challenges in order to control the oscillations of the density: The lack of smoothness of the velocity and possible compressions or dilatations.

- Alexis Vasseur (Texas, Austin) Title: Relative entropy applied to stability of shocks in fluid mechanics, and asymptotic limits.

3.6 Collective behavior and self-organized dynamics

Self-organized dynamics is driven by the interaction of agents with their neighbors. Examples range from consensus of voters and traffic flows to the formation of flocks of birds and tumor growth. When the interaction consists of global self-alignment, the large time behavior leads into consensus or flocking. When the self-alignment is purely local, the dynamics evolves into one or more clusters and the open questions regarding the emergence of consensus are related to the connectivity of the underlying graph.

- *Clustering, consensus and critical thresholds in self-organized dynamics.* Eitan Tadmor (University of Maryland)

  At the hydrodynamic level, the large time behavior is dictated by the balance between nonlinear convection and convolution-based interaction based on non-local means. Finite time breakdown depends on whether the initial configuration crosses intrinsic, O(1) critical thresholds (CT). The speaker demonstrated critical threshold phenomena with several n-dimensional prototype models. These include prolonged life-span of sub-critical 2D shallow-water solutions, 3D restricted Euler and Euler-Poisson equations, and the hydrodynamic descriptions of self-organized dynamics

- *Quasi–static evolution and congested crowd motion.* Yao Yao (University of Wisconsin Madison).

  In this talk the speaker presented a model , given by partial differential equations, that describes in a simplified setting the congested crowd motion with a density constraint. When the drift potential is convex, the crowd density is likely to aggregate, and thus if the initial density starts as a patch (i.e. if it is a characteristic function of some set), then the density evolves like a patch. The equation is a transport equation with a drift potential, where a constraint on the L^∞ norm is imposed on the density. The speaker showed that the patch evolves according to a quasi-static evolution equation, which is a free boundary problem connected to the Hele-Shaw equation. Methods used come from both viscosity solutions theory as well as from the gradient flow structure of the problem. The paper on this talk was published after the meeting. The paper can be found as D. Alexander, I. Kim and Y. Yao, Quasi-static evolution and congested crowd transport, Nonlinearity, 27: 1-36, 2014.

- *A coupled system of gradient flows for chemotaxis.* Daniel Matthes (TU Munich).

  The speaker presented a system of two coupled non-linear drift-diffusion equations modeling the behavior of bacteria: one equation governs the production and degradation of a chemical signal substance, the other describes the according movement of the bacteria. Unlike in the parabolic–parabolic Keller-Segel model, the chemical substance has a fixed saturation level. The equation system can be cast in the form of a gradient flow with respect to a joint Wasserstein-L2-metric. The potential, however, is not geodesically lambda-convex due to the coupling between the components. U using variational methods, existence and exponential equilibration, were shown. The work was joint with Jonathan Zinsl (TU Munich).

- *Stable stationary states for repulsive-attractive potentials.* Jose Antonio Carrillo (Imperial College).

  This talk considered local minimizers (in the topology of transport distances) of the interaction energy associated to a repulsive-attractive potential were considered. The link between the dimensionality of the support of local minimizers and the repulsive strength of the potential at the origin was shown to be fundamental to understand pattern formation in collective behavior. Some real life applications' examples were given.

- *Nonlocal transport vs nonlinear diffusion: from particle description to large time asymptotics.* Marco Di Francesco (University of Bath).

  Aggregation phenomena in microbiology and animal biology can be often described by PDEs of "transport" type, with a "nonlocal" velocity field. Formal derivation of non-local partial differential equations from particle-based ordinary differential equations was the focus of this talk. The speaker highlighted how variational structure often leads to well-posedness in a probability-measure sense. A major issue is providing a mathematical description of the emergence (or not) of collective behavior, or "multiple" behavior in the large-time asymptotics, depending on the choice of the initial conditions or other parameters. This issue has been partly investigated in the recent literature (cf. chemotaxis with two species). Recent results on the existence and uniqueness of non trivial steady states for a model with quadratic diffusion (joint work of the speaker with M. Burger), and on the finite time blowup and "multiple collapse" for a "purely nonlocal" model with two species of agents (joint work of the speaker with S. Fagioli) were presented.

- *Dynamics and steady states of nonlinear diffusion equations with long-range attractions.* Yanghong Huang (Imperial College London).

  The study of dynamics and steady states of a nonlocal aggregation equation with nonlinear diffusion arising in many contexts of biology and population dynamics was presented. The nonlinear diffusion is a power law type (like the porous medium equation) and the aggregation kernel is radial, attractive and integrable, with smoothness. The critical power for the nonlinear diffusion was shown to be quadratic. The steady states and their bifurcation diagrams are qualitatively different when the exponent of the nonlinear diffusion is larger or smaller than two. It was also shown that the dynamics of general random non-negative initial data also exhibits some coarse-graining behavior.

- *A population model with small density cut-off.* Alexander Lorz (Universite Pierre et Marie Curie).

  The question of 'cutting the tail' of the solution of an elliptic equation arises naturally in several contexts and leads to a singular perturbation problem with a strong cut-off. The speaker considered both the PDE with a drift and the symmetric case where a variational problem could be stated. It is known that, in both cases, the same critical scale arises for the size of the singular perturbation. More interesting is that in both cases another critical parameter (of order one) arises that decides when the limiting behavior is non-degenerate. The values of this critical parameter has been studied, both theoretically and numerically. In the symmetric case, the speaker discussed if the variational solution leads to the same value as for the maximal solution of the PDE. A weak formulation of the limiting Bernoulli problem has been proposed, which incorporates both Dirichlet and Neumann boundary condition. The work presented was in collaboration with Benoit Perthame, UPMC Univ Paris 06 and Peter Markowich, King Abdullah University of Science and Technology (KAUST), Saudi Arabia.

- *Kinetic Models for Opinion Formation.* Bertram Düring (University of Sussex).

  The speaker discusses some kinetic models for opinion formation. The time evolution of the opinion distribution was described by (systems of) Boltzmann-like equations. The speaker showed that at suitably large times, in presence of a large number of interactions in each of which individuals change a little their opinions/positions, the Boltzmann-type equations are well-approximated by Fokker-Planck type equations, which admit different, non-trivial steady states. Int his talk on a kinetic model for opinion formation at the Banff meeting the speaker mentioned an extension to inhomogeneous problems and Prof Tsai suggested a real-world example for such a model. This application will now be included in a forthcoming paper.

- *Non-local effects in Social Phenomenon.* Nancy Rodriguez (Stanford University).

  In a world with every day new connections, it is has become essential to include non-local effect when modeling any type of social phenomenon. The speaker (i) introduced some evidence of the need to include non-local effects in modeling, and (ii) discussed some progress made on the idea of "blocking" wave propagation for a non-local equation, motivated by an application to criminal activity. The results presented has consequences in other areas like ecology, pest control, and nerve-pulse propagation.

- *Fokker-Planck equations in Neuroscience.* Mara Jose Caceres (Universidad de Granada).

  A Fokker-Planck model describing the behavior of neuronal networks was presented. Specifically, the speaker analyzed the Nonlinear Noisy Leaky Integrate and Fire (NNLIF) model for neurons networks, where the main parameters in the model are the connectivity of the network and the noise. NNLIF describes the neuronal membrane potential considering as variable only the voltage.

4 Scientific Progress Made

- The talks in the workshop motivated Diogo Gomes towards a new line of research in numerics of which the following paper was a direct consequence:

  D. Gomes, R. Velho and M. T. Wolfram. "Socio-economic applications of finite state mean field games". Submitted for publication.

  D. Gomes, E. Pimentel, and H. Sanchez-Morgado. "Time dependent mean-field games in the sub quadratic case". To appear in Communications in Partial Differential Equations.

  P-E. Jabin started working on a new project with A. Vasseur on some minimization criterion for hyperbolic systems inspired by mean field games.

- Andrea Bertozzi, Braxton Osting and Yves van Gennip did some informal presentations on new work related to curvature flows on graphs. Since that time they have published a number of papers on the latter topic:

  Yves van Gennip, Nestor Guillen, Braxton Osting, and Andrea L. Bertozzi, Mean curvature, threshold dynamics, and phase field theory on finite graphs, accepted in the Milan Journal of Mathematics, 2014.

  Cristina Garcia-Cardona, Ekaterina Merkurjev, Andrea L. Bertozzi, Arjuna Flenner and Allon G. Percus, Multiclass Data Segmentation Using Diffuse Interface Methods on Graphs (prepublication version), IEEE Trans. Pattern Anal. Mach. Int., 2014.

  E. Merkurjev, C. Garcia, A. L. Bertozzi, A. Flenner, A. Percus Diffuse interface methods for multiclass segmentation of high-dimensional data Applied Math. Letters, online first, March 4, 2014.

  Huiyi Hu, Thomas Laurent, Mason A. Porter, Andrea L. Bertozzi, A Method Based on Total Variation for Network Modularity Optimization using the MBO Scheme, SIAM J. Appl. Math., 73(6), pp. 2224-2246, 2013. E. Merkurjev, T. Kostic, and A. L. Bertozzi, An MBO Scheme on Graphs for Segmentation and Image Processing, SIAM J. Imaging Sci. 6(4), 1903-1930, 2013.

  In the general area of swarming:

  Alan Mackey, Theodore Kolokolnikov, and Andrea L. Bertozzi, Two-species particle aggregation and stability of codimension one solutions accepted in DCDS-B, 2014.

  J. Zipkin, M. B. Short, A. L. Bertozzi, Cops on the dots in a mathematical model for urban crime and police response, accepted in DCDS-B 2014.

  Andrea L. Bertozzi, James von Brecht, Hui Sun, Theodore Kolokolnikov, and David Uminsky, Ring Patterns and their Bifurcations in a Nonlocal Model of Biological Swarms, accepted in Comm. Math. Sci., special issue in honor of George Papanicolau's birthday, 2013.

  James von Brecht, Benjamin Sudakov, and A. L. Bertozzi, Swarming on Random Graphs II, J. Stat. Phys., online first, 2014.

- Yao Yao in collaboration with other workshop's participants has published the following papers:

  Yao Yao and Andrea L. Bertozzi, Blow-up dynamics for the aggregation equation with degenerate diffusion, Physica D, 260, pp. 77-89, 2013, special issue on Emergent Behaviour in Multi-particle Systems with Non-local Interactions

  D. Alexander, I. Kim and Y. Yao, Quasi-static evolution and congested crowd transport, Nonlinearity, 27: 1-36, 2014.

- During the meeting we advanced in the analysis of the stability of flock solutions for 2nd order models of swarming. Their spatial profiles are determined as minimizers of repulsive-attractive potential energies. Their stability as solutions of the 1st order model determined their stability properties as solutions of the 2nd order model. This has led to two publications recently of the participants J.A. Carrillo and S. Martin:

  J. A. Carrillo, Y. Huang, S. Martin, Nonlinear stability of flock solutions in second-order swarming models, Nonlinear Analysis: Real World Applications 17, 332?343, 2014.

  G. Albi, D. Balagu, J. A. Carrillo, J. von Brecht, Stability Analysis of Flock and Mill rings for 2nd Order Models in Swarming, to appear in SIAM J. Appl. Math.

- Alexander Lorz has published this paper:

  A. Lorz, P. Markowich, and B. Perthame. Bernoulli variational problem and beyond. Arch. Ration. Mech. Anal., pages 1-29, 2013.

- Numerics Lisa Powers's paper from her talk is now accepted to SISC: "An iterative algorithm for computing measures of generalized Voronoi regions."

  Together with Rustum Choksi and Jean-Christophe Nave she would like to work on two projects that were discussed at Banff, 1) Modeling crystal growth for general crystallization nuclei (suggested by Prof. Illner), and 2) Developing a flocking model based on centroidal Voronoi tessellation ideas.

- Maria J. Caceres, Benoit Perthame, Beyond blow-up in excitatory integrate and fire neuronal networks: Refractory period and spontaneous activity, Journal of Theoretical Biology, 350 (2014), 81-89.

5 Outcome of the Meeting

We have brought together applied scientists and leading experts in mathematical modeling and theory. The analysts presented the most recents advances in the modeling areas and applied mathematicians were exposed to the new and rigorous mathematical tools which will help in the understanding of the related phenomena. The workshop was beneficiary for both groups of people; participants had direct interactions and targeted new interesting problems and applications.