Localization for multi-particle Anderson Hamiltonians & unique continuation principle for spectral projections

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Disordered quantum many-body systems
BIRS
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Localization for multi-particle Anderson Hamiltonians

Joint work with Son Nguyen:

- AK and Son T. Nguyen: *The bootstrap multiscale analysis for the multi-particle Anderson model*. J. Stat. Phys. **151**, 983-973 (2013).
- AK and Son T. Nguyen: *Bootstrap multiscale analysis and localization for multi-particle continuous Anderson Hamiltonians*. Preprint (to be posted soon in the arXiv).

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The n-particle Anderson Hamiltonian is the random Schrödinger operator

$$H_{\omega}^{(n)} := H_{0,\omega}^{(n)} + U \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^{nd}), \quad \text{where} \quad H_{0,\omega}^{(n)} := -\Delta^{(n)} + V_{\omega}^{(n)}.$$

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   1. $\omega = \{\omega_k\}_{k \in \mathbb{Z}^d}$ is a family of independent identically distributed random variables whose common probability distribution μ has a bounded density ρ and satisfies $\{0, M_+\} \subset \operatorname{supp} \mu \subseteq [0, M_+]$ for some $M_+ > 0$;

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$0 \le \widetilde{U}(y) \le \widetilde{U}_\infty < \infty$, $\widetilde{U}(y) = \widetilde{U}(-y)$, $\widetilde{U}(y) = 0$ for $\|y\|_\infty > r_0 \in (0,\infty)$.

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- $H_\omega^{(1)} = H_{0,\omega}^{(1)}$, so $\Sigma^{(1)} = [0, \infty)$. Letting $\Sigma_0^{(n)}$ denote the almost sure spectrum of $H_{0,\omega}^{(n)}$, we have

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5. $H_\omega^{(n)}$ will denote a fixed n-particle Anderson Hamiltonian.

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- *(Finite multiplicity of eigenvalues)* The eigenvalues of H_ω^N in $[0, E^{(N)}]$ have finite multiplicity:

$$\operatorname{tr} \chi_{\{E\}}(H_\omega^N) < \infty \quad \text{for all} \quad E \leq E^{(N)}.$$

Theorem-cont.

- *(Summable Uniform Decay of Eigenfunction Correlations (SUDEC))* . For every $\zeta \in (0,1)$ there exists a constant $C_{\omega,\zeta}$ such that for every $E \leq E^{(N)}$ and $\phi, \psi \in \operatorname{Ran} \chi_{\{E\}}(H_\omega^N)$ we have

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the supremum being taken over all Borel functions g on $\mathbb{R}$ with $\|g\|_\infty = \sup_{t\in\mathbb{R}} |g(t)| \leq 1$. In particular, we have

$$\mathbb{E}\left\{ \sup_{t\in\mathbb{R}} \left\| \chi_{\mathbf{x}} \chi_I(H_\omega^N) e^{itH_\omega^N} \chi_{\mathbf{y}} \right\| \right\} \leq C_\zeta(\mathbf{y}) e^{-(d_H(\mathbf{x},\mathbf{y}))^\zeta} \text{ for all } \mathbf{x} \in \mathbb{R}^{Nd}.$$

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- Son Nguyen will describe this extension of bootstrap multiscale analysis in his talk.

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- Localization was proved for the (discrete) multi-particle Anderson model by Chulaevsky and Suhov, using a multiscale analysis, and by Aizenman and Warzel, using the fractional moment method.
- Chulaevsky, Boutet de Monvel and Suhov extended the results of Chulaevsky and Suhov to the multi-particle Anderson Hamiltonian, obtaining localization at the bottom of the spectrum.
- Our localization results are derived from a bootstrap multiscale analysis, an enhanced multiscale analysis developed in the one-particle case by Germinet and K.
- Son Nguyen will describe this extension of bootstrap multiscale analysis in his talk.
- We extend the bootstrap multiscale analysis (and its consequences) to the multi-particle Anderson Hamiltonian without requiring a covering condition. This requires Wegner estimates without a covering condition, which will be described by Peter Hislop in his talk.

Unique continuation principle for spectral projections

Wegner estimates without a covering condition use a unique continuation principle for spectral projections, which we will now describe.

- AK, *Unique continuation principle for spectral projections of Schrödinger operators and optimal Wegner estimates for non-ergodic random Schrödinger operators*. Comm. Math Phys. **323**, 1229-1246 (2013)
- Appendix to : AK and Son T. Nguyen, *Bootstrap multiscale analysis and localization for multi-particle continuous Anderson Hamiltonians*. Preprint (to be posted soon in the arXiv).

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$$B(x,\delta) := \left\{ y \in \mathbb{R}^d ; |y - x| < \delta \right\}, \text{ with } |x| := |x|_2 = \left(\sum_{j=1}^{d} |x_j|^2 \right)^{\frac{1}{2}};$$

$$\Lambda = \Lambda_{\mathbf{L}}(a) := a + \prod_{j=1}^{d} \left(-\tfrac{L_j}{2}, \tfrac{L_j}{2}\right) = \prod_{j=1}^{d} \left(a_j - \tfrac{L_j}{2}, a_j + \tfrac{L_j}{2}\right),$$

where $a \in \mathbb{R}^d$ and $\mathbf{L} = (L_1, \ldots, L_d) \in (0,\infty)^d$.

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  where $a \in \mathbb{R}^d$ and $\mathbf{L} = (L_1, \ldots, L_d) \in (0,\infty)^d$.
- H_Λ denotes the restriction of H to the the rectangle $\Lambda \subset \mathbb{R}^d$:

$$H_\Lambda = -\Delta_\Lambda + V_\Lambda \quad \text{on} \quad \mathrm{L}^2(\Lambda).$$

  - Δ_Λ is the Laplacian on Λ with either Dirichlet or periodic bc.
  - V_Λ is the restriction of V to Λ..

Unique continuation principle for spectral projections

A UCPSP on a rectangle Λ is an estimate of the form

$$\chi_I(H_\Lambda) W_\Lambda \chi_I(H_\Lambda) \geq \kappa \chi_I(H_\Lambda) \quad \text{on} \quad \mathrm{L}^2(\Lambda),$$

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- Germinet and Klein (2013) proved a modified version of the CHK UCPSP, using Bourgain and Kenig's quantitative unique continuation principle and (some) Floquet theory, obtaining control of the constant κ in terms of the relevant parameters.

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- Rojas-Molina and Veselić (2013) proved, under the hypotheses of the Theorem, that for boxes $\Lambda = \Lambda_L(x_0)$ with $x_0 \in \mathbb{Z}^d$ and $L \in \mathbb{N}_{\mathrm{odd}}$, if ψ is an eigenfunction of H_Λ with eigenvalue $E \in]-\infty, E_0]$, then

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- Our Theorem is derived from the quantitative unique continuation principle as in Bourgain and Klein using the "dominant boxes" introduced by Rojas-Molina and Veselić.
- The UCPSP is a crucial ingredient for proving Wegner estimates for one and multi-particle Anderson Hamiltonians. The UCPSP replaces the covering condition.

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$$\left(\frac{\delta}{Q}\right)^{m_d\left(1+K^{\frac{2}{3}}\right)\left(Q^{\frac{4}{3}}+\log\frac{\|\psi\chi_\Omega\|_2}{\|\psi\chi_\Theta\|_2}\right)} \|\psi\chi_\Theta\|_2^2 \leq \left\|\psi\chi_{B(x_0,\delta)}\right\|_2^2 + \|\zeta\chi_\Omega\|_2^2,$$

where $m_d > 0$ is a constant depending only on d.

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$$\delta^{M_d\left(1+K^{\frac{2}{3}}\right)} \|\psi\|_2^2 \le \sum_{k \in \mathbb{Z}^d, \Lambda_1(k) \subset \Lambda} \left\|\psi \chi_{B(y_k,\delta)}\right\|_2^2 + \delta^2 \|H_\Lambda \psi\|_2^2$$

$$= \left\|W^{(\Lambda)} \psi\right\|_2^2 + \delta^2 \|H_\Lambda \psi\|_2^2.$$

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Let $E_0 > 0$ and $I \subset]-\infty, E_0]$ a closed interval; set $\beta = \frac{1}{2}|I|$. Since $H_\Lambda \geq -\|V\|_\infty$ for any box Λ, without loss of generality we assume $I = [E-\beta, E+\beta]$ with $E \in [-\|V\|_\infty, E_0]$, so

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Moreover, for any box Λ we have

$$\|(H_\Lambda - E)\psi\|_2 \leq \beta \|\psi\|_2 \quad \text{for} \quad \psi = \chi_I(H_\Lambda)\psi.$$

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$$\|V - E\|_\infty \leq \|V\|_\infty + \max\{E_0, \|V\|_\infty\} \leq K = 2\|V\|_\infty + E_0.$$

Moreover, for any box Λ we have

$$\|(H_\Lambda - E)\psi\|_2 \leq \beta \|\psi\|_2 \quad \text{for} \quad \psi = \chi_I(H_\Lambda)\psi.$$

Let Λ be a box as in the Corollary and $\psi = \chi_I(H_\Lambda)\psi$ real-valued. It follows from the Corollary applied to $H - E$ that

$$\delta^{M_d\left(1+K^{\frac{2}{3}}\right)} \|\psi\|_2^2 \leq \left\|W^{(\Lambda)}\psi\right\|_2^2 + \delta^2 \|(H_\Lambda - E)\psi\|_2^2 \leq \left\|W^{(\Lambda)}\psi\right\|_2^2 + \beta^2 \|\psi\|_2^2.$$

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If $\beta^2 \leq \gamma^2 := \frac{1}{2}\delta^{M_d\left(1+K^{\frac{2}{3}}\right)}$, i.e., $|I| \leq 2\gamma$, we get

$$\gamma^2 \|\psi\|_2^2 \leq \left\|W^{(\Lambda)}\psi\right\|_2^2, \quad \text{i.e.,} \quad \gamma^2\chi_I(H_\Lambda) \leq \chi_I(H_\Lambda)W^{(\Lambda)}\chi_I(H_\Lambda).$$

Proof of the Corollary

For simplicity we take a box $\Lambda = \Lambda_L(0)$ with $L \in \mathbb{N}_{\mathrm{odd}}$. We extend functions φ on Λ to functions $\widehat{V}$ and $\widetilde{\varphi}$ on $\mathbb{R}^d$ and V to a potential $\widehat{V}$ on $\mathbb{R}^d$ so

$$(-\widetilde{\Delta + V})\psi = (-\Delta + \widehat{V})\widetilde{\psi}.$$

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Take $Y \in \mathbb{N}_{\text{odd}}$, $9 \leq Y < \frac{L}{2}$. Since L is odd, we have $\overline{\Lambda} = \bigcup_{k \in \Lambda \cap \mathbb{Z}^d} \overline{\Lambda_1(k)}$. It follows that for all $\varphi \in \mathrm{L}^2(\Lambda)$ we have

$$\sum_{k \in \Lambda \cap \mathbb{Z}^d} \left\| \widetilde{\varphi}_{\Lambda_Y(k)} \right\|_2^2 \leq (2Y)^d \left\| \varphi_\Lambda \right\|_2^2 .$$

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We now fix $\psi \in \mathscr{D}(\Delta_\Lambda)$. Following Rojas-Molina and Veselić, we call a site $k \in \widehat{\Lambda} = \Lambda \cap \mathbb{Z}^d$ *dominating* (for ψ) if

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and note that, letting $\widehat{D} \subset \Lambda \cap \mathbb{Z}^d$ denote the collection of dominating sites,

$$\sum_{k \in \widehat{D}} \left\| \psi_{\Lambda_1(k)} \right\|_2^2 \geq \frac{1}{2} \left\| \psi_\Lambda \right\|_2^2 .$$

Proof of the Corollary-continued

If $k \in \widehat{D}$ we apply the QUCP with $\Omega = \Lambda_Y(k)$ and $\Theta = \Lambda_1(k)$, obtaining

$$\delta^{m'_d\left(1+K^{\frac{2}{3}}\right)} \left\|\psi_{\Lambda_1(k)}\right\|_2^2 \leq \left\|\psi_{B(y_{J(k)},\delta)}\right\|_2^2 + \delta^2 \left\|\widetilde{\zeta}_{\Lambda_Y(k)}\right\|_2^2,$$

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where $\zeta = (-\Delta + V)\psi$, Y is appropriately chosen, $Y \leq 40\sqrt{d} < \frac{L}{2}$, and the map $J\colon \widehat{D} \to \Lambda \cap \mathbb{Z}^d$ is defined appropriately so

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Summing over $k \in \widehat{D}$ and using $\sum_{k \in \widehat{D}} \left\| \psi_{\Lambda_1(k)} \right\|_2^2 \geq \frac{1}{2} \left\| \psi_\Lambda \right\|_2^2$, we get

$$\begin{aligned} \tfrac{1}{2}\delta^{m'_d\left(1+K^{\frac{2}{3}}\right)} \left\| \psi_\Lambda \right\|_2^2 &\leq 2 \sum_{k \in \Lambda \cap \mathbb{Z}^d} \left\| \psi_{B(y_k,\delta)} \right\|_2^2 + (2Y)^d \delta^2 \left\| \zeta_\Lambda \right\|_2^2 \\ &\leq 2 \sum_{k \in \Lambda \cap \mathbb{Z}^d} \left\| \psi_{B(y_k,\delta)} \right\|_2^2 + (80\sqrt{d})^d \delta^2 \left\| \zeta_\Lambda \right\|_2^2, \end{aligned}$$

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which implies (with a different constant $M_d > 0$)

$$\delta^{M_d\left(1+K^{\frac{2}{3}}\right)} \left\| \psi_\Lambda \right\|_2^2 \leq \sum_{k\in\Lambda\cap\mathbb{Z}^d} \left\| \psi \chi_{B(y_k,\delta)} \right\|_2^2 + \delta^2 \left\| \zeta_\Lambda \right\|_2^2.$$