Spectral Transition for Random Quantum Walks on Trees*

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Joint work with Eman HAMZA, Cairo University

CMP, to appear

Warm up : Quantum Walk on $\mathbb{Z} = \mathcal{T}_2$

Quantum particle with spin 1/2 on 1-dim lattice, i.e. $\mathcal{K}_2 = l^2(\mathbb{Z}) \otimes \mathbb{C}^2$

Spin evol.: C unitary op. on $\mathbb{C}^2$, "coin" space

Spin dependent shift:

Let $S_\pm$ shift to the right/left on $l^2(\mathbb{Z})$, $|\pm\rangle\langle\pm|$ proj. on $|\pm\rangle \in \mathbb{C}^2$

$S := S_+ \otimes |+\rangle\langle+| + S_- \otimes |-\rangle\langle-|$ on $l^2(\mathbb{Z}) \otimes \mathbb{C}^2$

One step dynamics: $\boxed{U := S(\mathbb{I} \otimes C)}$ s.t.

$$U|x \otimes \pm\rangle = \langle-|C \pm\rangle|(x-1) \otimes -\rangle + \langle+|C \pm\rangle|(x+1) \otimes +\rangle$$

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If $H = -\Delta + V$ then $U = e^{-iH}$ suggests $S \simeq e^{-i\Delta}$ and $C \simeq e^{iV}$

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Random Quantum Walk: '10 J. - Merkli

$C \rightsquigarrow \{C_\omega(x)\}_{x \in \mathbb{Z}}$, i.i.d. set of $C_\omega(x) \in U(2)$

$$U \rightsquigarrow U_\omega |x \otimes \pm\rangle = \langle -|C_\omega(x)) \pm\rangle |(x-1) \otimes -\rangle + \langle +|C_\omega(x)) \pm\rangle |(x+1) \otimes +\rangle$$

Homogeneous tree $\mathcal{T}_3$ with coord. number $q = 3$

$A_3 = \{a, b, c\}$ generators of a group with unit e s.t. $a^2 = b^2 = c^2 = e$.

Root: Origin e

Edges: Labels $\{a, b, c\}$ in trig. order

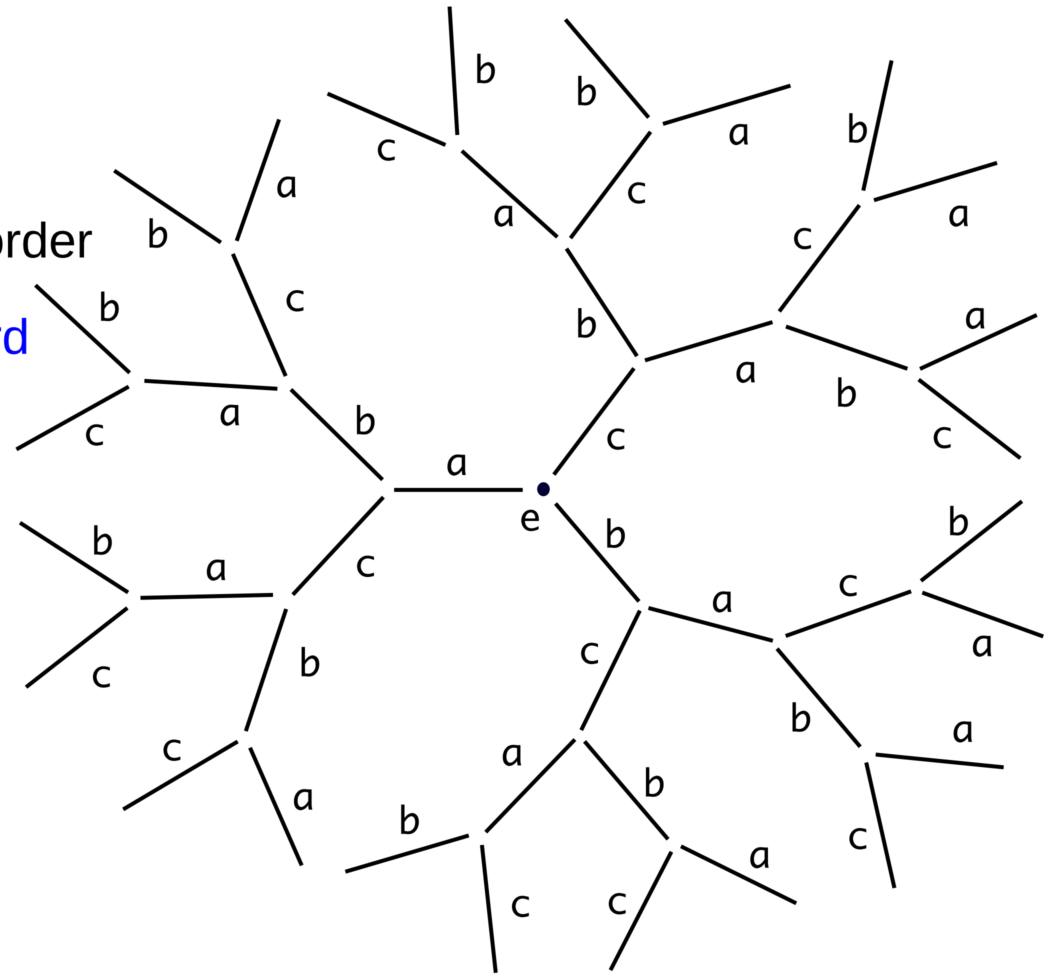
Any $x \in \mathcal{T}_3$: Finite reduced word

$$x = x_1 x_2 \cdots x_n, \; x_j \in A_3$$

Length: $|x| = n$

Distance: For any $x, y \in \mathcal{T}_3$,

$$d(x, y) = |x^{-1} y|$$



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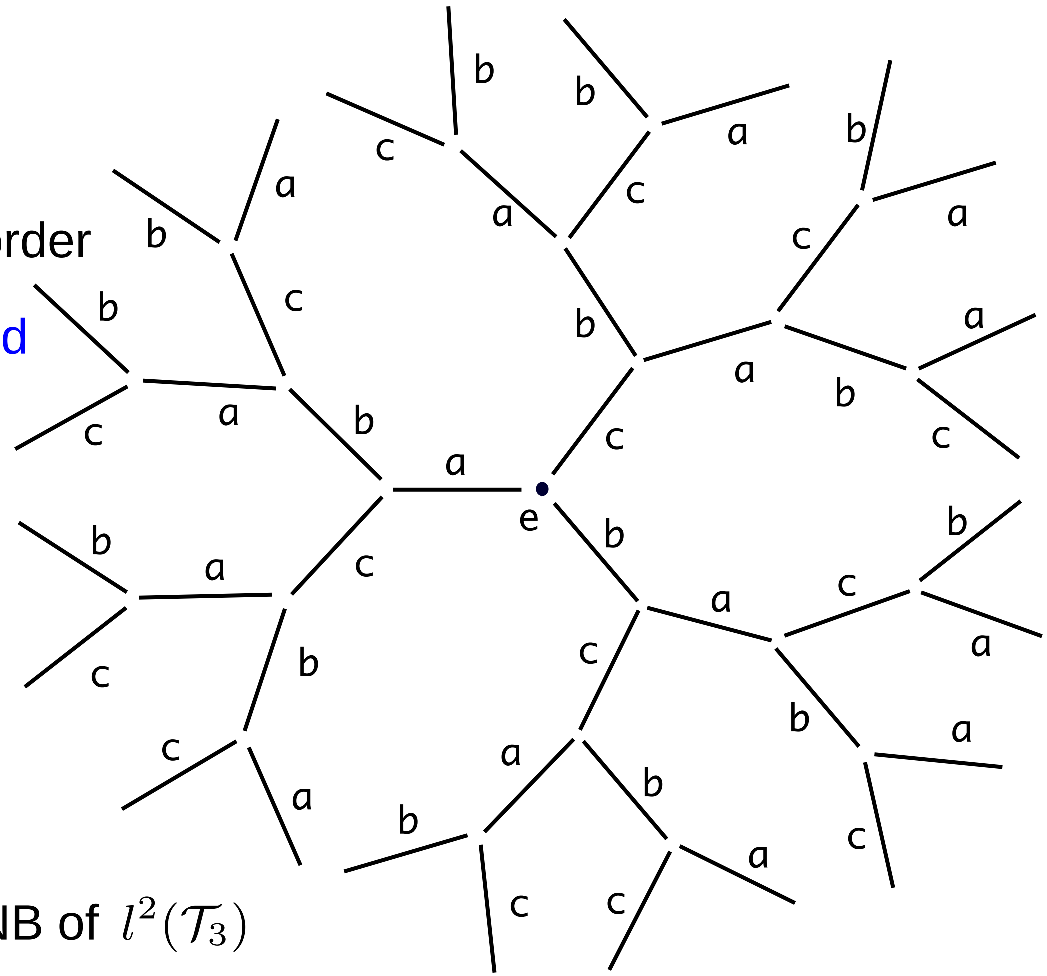

$l^2(\mathcal{T}_3)$:

$x \mapsto |x\rangle$, s.t. $\{|x\rangle\}_{x \in \mathcal{T}_3}$ ONB of $l^2(\mathcal{T}_3)$

$$\psi = \sum_{x \in \mathcal{T}_3} \psi(x)|x\rangle, \;\; \sum_{x \in \mathcal{T}_3} |\psi(x)|^2 < \infty .$$

Shifts on $\mathcal{T}_3$

Even / Odd: Sites x_e / x_o s.t. $|x_e|$ even / $|x_o|$ odd

Let $S_{ab} : l^2(\mathcal{T}_3) \to l^2(\mathcal{T}_3)$

$$S_{ab}|x_e\rangle = |x_e a\rangle$$
$$S_{ab}|x_o\rangle = |x_o b\rangle$$

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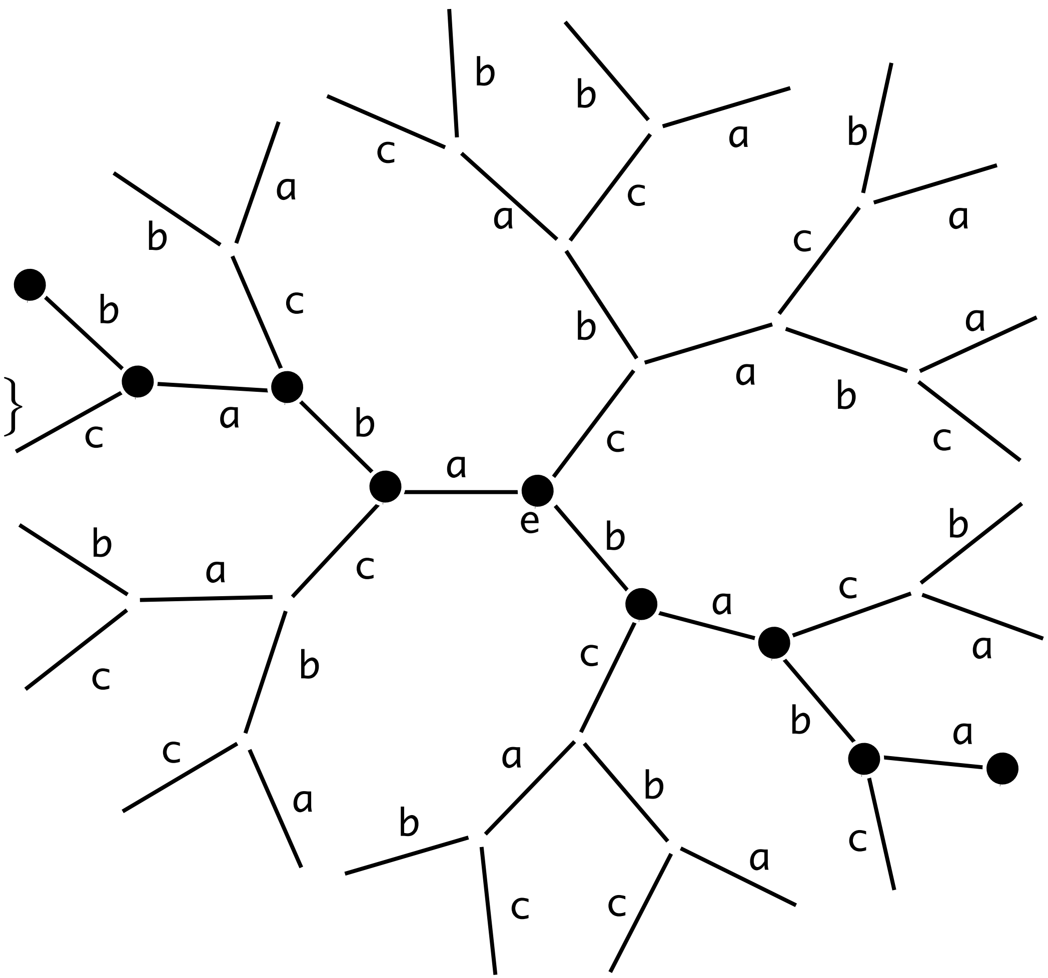
Cyclic subspaces:

$\mathcal{H}_x^{ab} =$ span $\{S_{ab}^n|x\rangle,\ n \in \mathbb{Z}\}$

$S_{ab}|_{\mathcal{H}_x^{ab}} \simeq$ shift on $l^2(\mathbb{Z})$.



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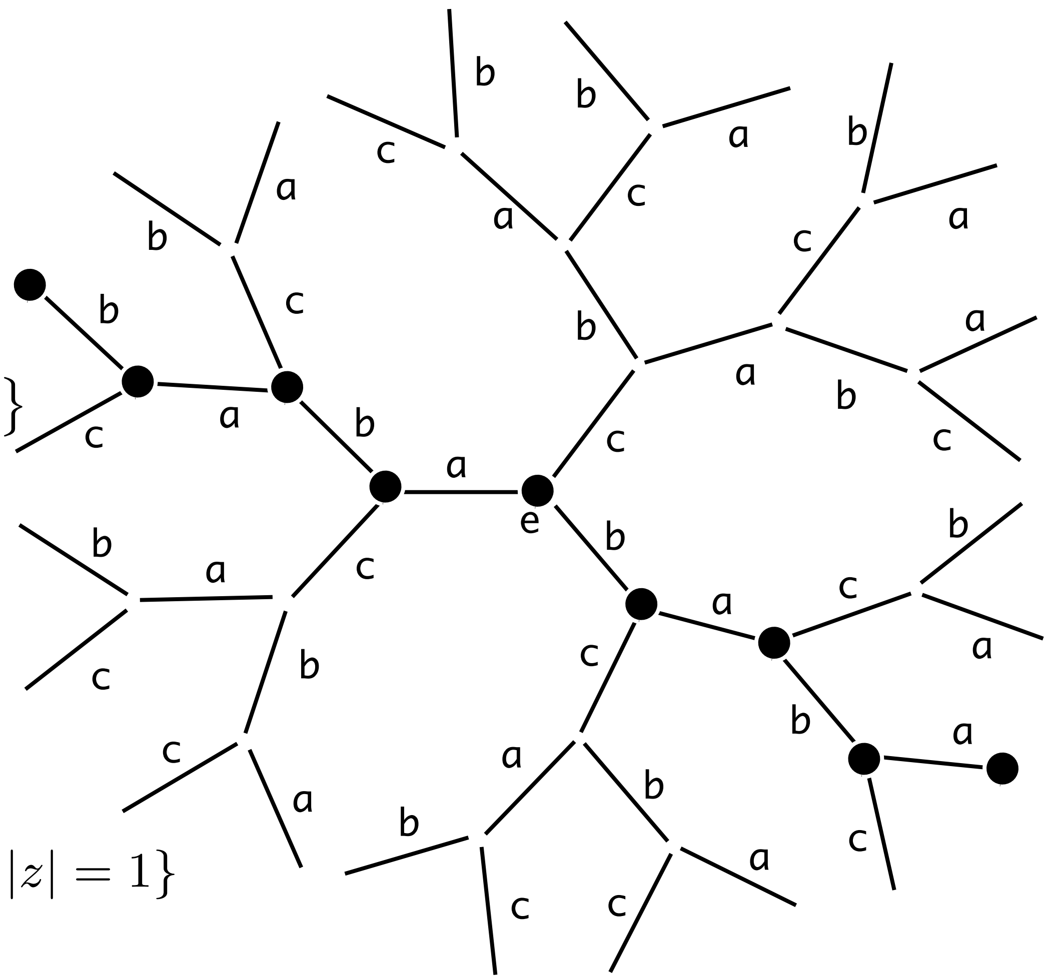

Spectrum:

$\sigma(S_{ab}) = \sigma_{ac}(S_{ab}) = \{z \text{ s.t. } |z| = 1\}$

Similarly for S_{bc} , S_{ca}

Quantum Walk on $\mathcal{T}_3$

- Unitary evolution:

  Particle with spin 1 on $\mathcal{T}_3$ jumping on nearest neighbors

  Hilbert space: $\mathcal{K}_3 = l^2(\mathcal{T}_3) \otimes \mathbb{C}^3$

  ONB of $\mathbb{C}^3$: $\{|a\rangle, |b\rangle, |c\rangle\}$,

  ONB of $\mathcal{K}_3$: $\{x \otimes \tau = |x\rangle \otimes |\tau\rangle\}_{x \in \mathcal{T}_3, \tau \in A_3}$

- Spin dep. shift on $\mathcal{K}_3$:

$$S = S_{bc} \otimes |a\rangle\langle a| + S_{ca} \otimes |b\rangle\langle b| + S_{ab} \otimes |c\rangle\langle c|$$

- Spin update: For $C \in U(3)$ a unitary op. on $\mathbb{C}^3$

  $\mathbb{I} \otimes C : \mathcal{K}_3 \to \mathcal{K}_3$

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- Time one dynamics of the QW:

$$\boxed{U(C) := S(\mathbb{I} \otimes C)}$$

s.t.

$$U(C)x_e \otimes a = C_{aa} x_e b \otimes a + C_{ba} x_e c \otimes b + C_{ca} x_e a \otimes c$$

$$U(C)x_o \otimes a = C_{aa} x_o c \otimes a + C_{ba} x_o a \otimes b + C_{ca} x_o b \otimes c \quad \text{etc...}$$

Random Environment $\Rightarrow$ RQW

Spatial disorder   Different coin op. C_ω at each site:

$$C \mapsto \{C_\omega(x)\}_{x\in\mathcal{T}_3} \quad \text{with} \quad \begin{array}{l} C_\omega(x_e)_{\tau,\sigma} = \exp(i\omega^\tau_{x_e\tau})C_{\tau,\sigma}, \\ C_\omega(x_o)_{\tau,\sigma} = \exp(i\omega^\tau_{x_o\sigma})C_{\tau,\sigma}\,,\ \tau,\sigma \in \{a,b,c\}. \end{array}$$

- Random Quantum Walk:

$$U(C) \mapsto U_\omega(C)$$

Property: Set $\mathbb{D}(\omega) = \text{diag}(\exp(i\omega^\tau_x))$, then random time one dynamics

$$\boxed{U_\omega(C) = \mathbb{D}(\omega)U(C)}$$

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Assumptions:

- $\{\omega^\tau_x\}^{\tau\in A_3}_{x\in\mathcal{T}_3}$ are i.i.d. $\mathbb{T}$–valued random variables
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Remarks:

- The transition prob. $|\langle x\otimes\tau|U_\omega(C)\ y\otimes\sigma\rangle|^2$ are deterministic.
- Spectral transition expected between "large" and "small" disorder regimes,
  Abou-Chacra, Anderson, Thouless ’73, Kunz Souillard ’83, Klein ’94

Landmarks in $U(3)$

Permutation mat. For $\pi \in \mathfrak{S}_3$ on $A_3 = \{a, b, c\}$ set $C_\pi = \sum_{\tau \in A_3} |\pi(\tau)\rangle\langle\tau|$

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- $C_{(abc)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad \Rightarrow U_\omega(C_{(abc)})$ p.p. $\leftrightarrow$ loc. (as is $U_\omega(C_{(acb)})$)

$\mathcal{H}_{x_0} =$ span$\{x_0 \otimes a,\ x_0 a \otimes b,\ x_0 \otimes c,\ x_0 c \otimes a,\ x_0 \otimes b,\ x_0 b \otimes c\}$
invariant under $U_\omega(C_{(abc)})$, $\forall x_o \in \mathcal{T}_3$

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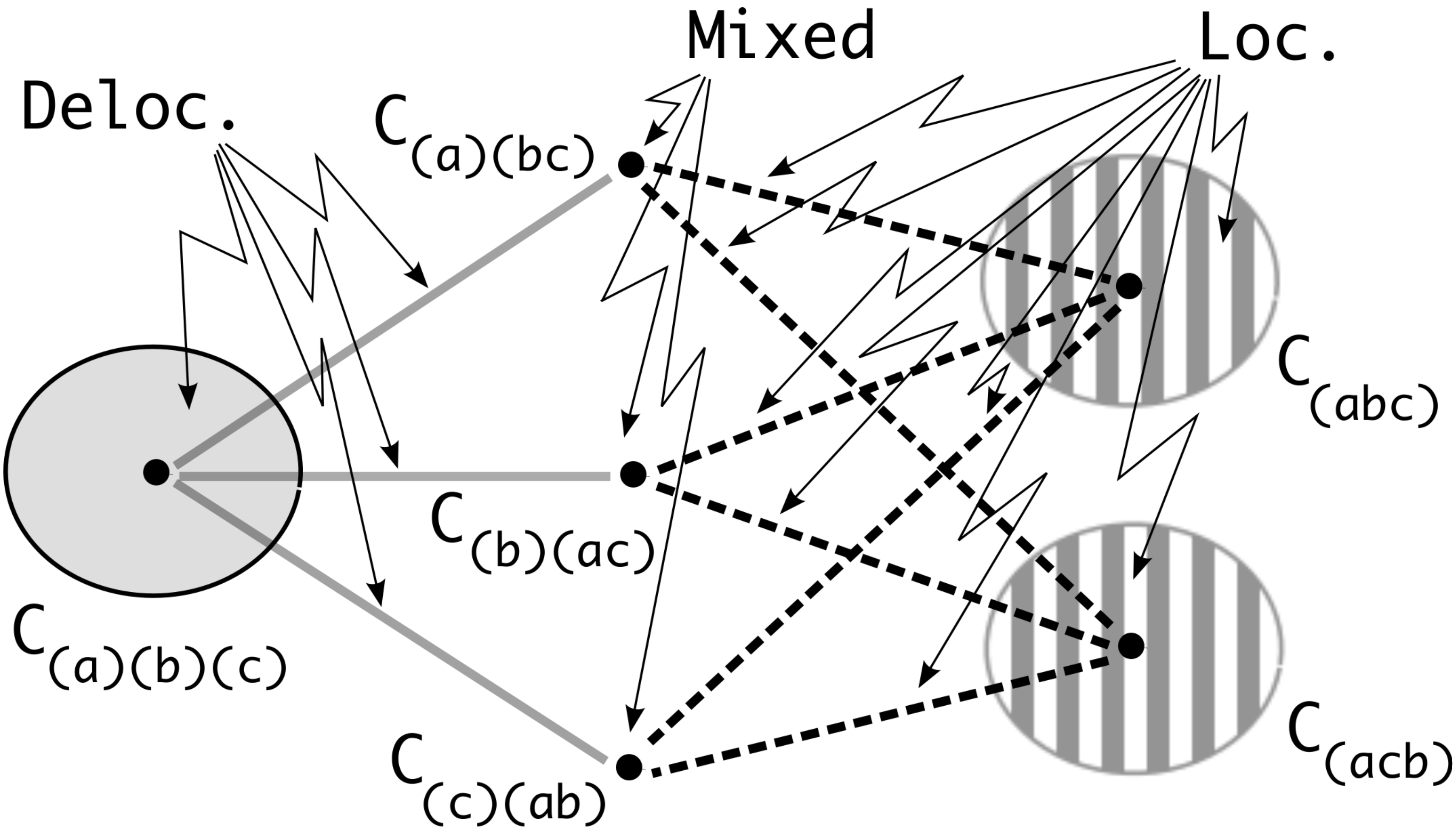
- $C_{(c)(ab)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\Rightarrow U_\omega(C_{(c)(ab)}) \simeq$ p.p. $\oplus$ a.c. $\leftrightarrow$ mixed
  (as are $U_\omega(C_{(a)(bc)})$, $U_\omega(C_{(b)(ac)})$)

  $\mathcal{H}_{x_0} =$ span$\{\cdots, x_o ba \otimes a, x_o b \otimes b, x_o \otimes a,\ x_o a \otimes b,\ x_0 ab \otimes a, \cdots\}$
  $\mathcal{H}_{x_e} =$ span$\{x_e \otimes a,\ x_e c \otimes b\}$ invar. under $U_\omega(C_{(ab)(c)})$, $\forall x_e, x_o \in \mathcal{T}_3$

Spectral Phase Diagram

Theorem:



Interpolating Matrices I

Localizing Matrices: For $0 \leq r \leq 1$ and $t = \sqrt{1 - r^2}$,

$$C_1^l(r) = \begin{pmatrix} 0 & r & t \\ 1 & 0 & 0 \\ 0 & t & -r \end{pmatrix}$$ s.t. $C_1^l(1) \simeq C_{(c)(ab)}$ and $C_1^l(0) = C_{(abc)}$

and similar $C_2^l(r), C_3^l(r), \ldots, C_6^l(r)$.

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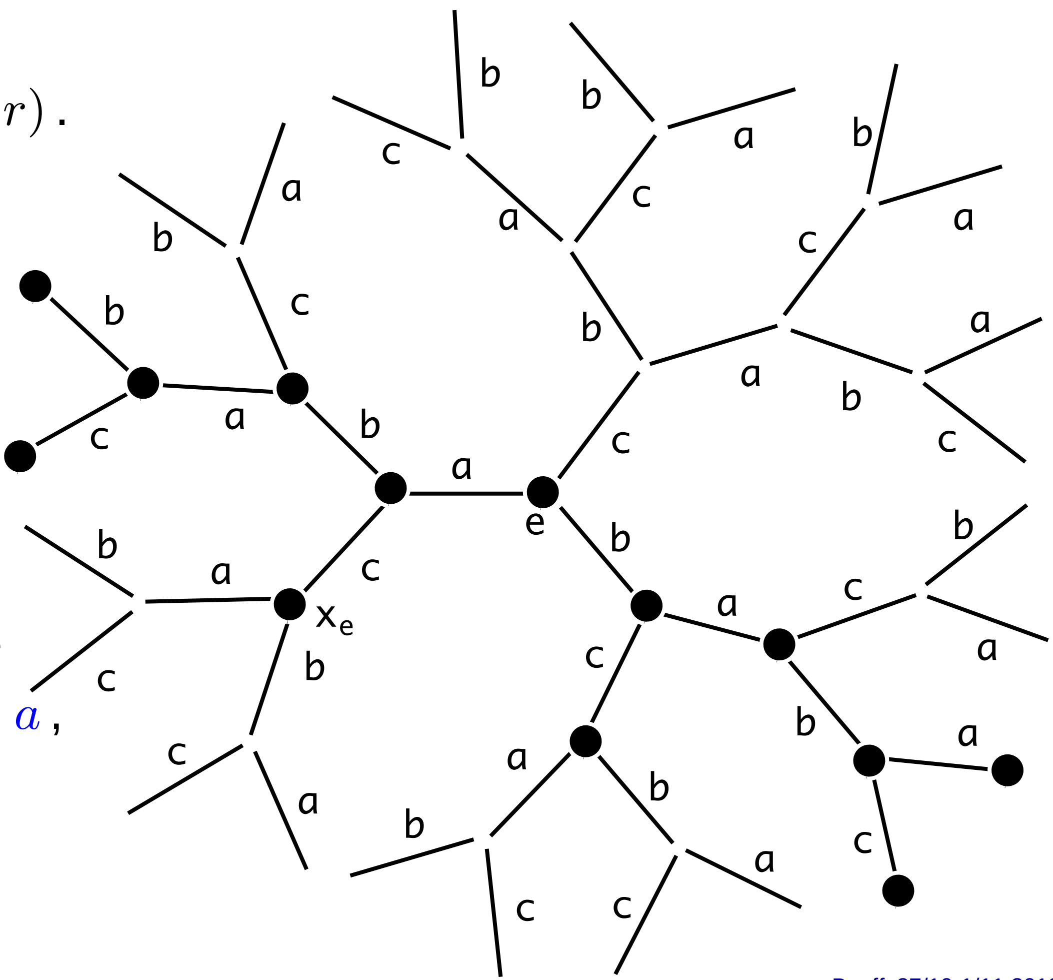

For all $x_e \in \mathcal{T}_3$, let $\mathcal{H}_{x_e \otimes a}$ be the $U_\omega(C_1^l(r))$-cyclic subsp. for $x_e \otimes a$,

Reduction to 1D model

Property For each $x_e \in \mathcal{T}_3$, $U_\omega(C_1^l(r))|_{\mathcal{H}_{x_e \otimes a}} \simeq V_\omega(r)$

where

$$V_\omega(r) = \mathbb{D}(\omega) \begin{pmatrix} \ddots & 0 & & & & & & & & \\ 0 & 0 & & & 1 & & & & & \\ 0 & 0 & 0 & t & 0 & r & & & & \\ r & -t & 0 & 0 & 0 & 0 & & & & \\ & & 0 & 0 & 0 & 0 & t & r & & \\ & & 1 & 0 & 0 & 0 & 0 & 0 & & \\ & & & r & & -t & 0 & 0 & & \\ & & & & & & 0 & 0 & & \\ & & & & & & 0 & 0 & 0 & t \\ & & & & & & r & -t & 0 & \ddots \end{pmatrix}$$

Allows for Transfer Matrix based analysis that yields localization $\forall r \in [0,1)$.

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Prop: $U_\omega(C_1^d(r))$ is purely ac. $\forall\ r \in (0, 1]$, and $\forall\ \omega$

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Path counting argument:

$\sum_{n\in\mathbb{Z}} |\langle x\otimes\tau|U^n x\otimes\tau\rangle|^2 < \infty \Rightarrow x\otimes\tau \in$ a.c. spectral subspace of U.

$$\langle x\otimes\tau|U^n x\otimes\tau\rangle = \sum_{\substack{y_j\in\mathcal{T}_3 \\ \sigma_j\in A_3}} \langle x\otimes\tau|Uy_1\otimes\sigma_1\rangle\langle y_1\otimes\sigma_1|Uy_2\otimes\sigma_2\rangle\cdots\langle y_{n-1}\otimes\sigma_{n-1}|Ux\otimes\tau\rangle$$

Spin $|b\rangle$ or $|c\rangle$:

s.t. $x_e \mapsto x_e c$ or $x_e a$ whereas $x_o \mapsto x_o a$ or $x_o b$.

Only possibility for return in $2n$ steps $x \mapsto xaaa\dots a \rightsquigarrow \sum_{n\in\mathbb{N}} t^{2n} < \infty$.

Neighborhood of $C_{(a)(b)(c)} = \mathbb{I}$

Perturbation: $C = \mathbb{I} + E$, with $\|E\| \leq \epsilon$

$$C_{\tau\tau} = O(1), \;\; C_{\tau\sigma} = O(\epsilon), \; \tau \neq \sigma$$

Path counting argument to show $\langle x \otimes \tau | U_\omega^n(C) \, x \otimes \tau \rangle \in l^2(\mathbb{Z})$ if ϵ small

Expansion $\rightsquigarrow$

Path: $x = x \; y_1 \; y_2 \; y_3 \; \cdots \; y_{2n}$ $\leftrightarrow$ Weight: $|C_{\tau\sigma_1} C_{\sigma_1\sigma_2} \cdots C_{\sigma_{2n}\tau}|$

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Lemma:

Weight of length $2n$ path from x to x contains at least n off diag. terms

Argument:

Strings with of m consecutive diag. elemts cannot reduce one another

$\Rightarrow$ the need for enough off-diag. elemts to ensure reduction.

Then: Weight $\leq \epsilon^n$, $\#\{\text{contrib. paths}\} \leq k^n$, $k \simeq 72 \;\Rightarrow$

$\sum_{n\in\mathbb{N}} |\langle x \otimes \tau | U_\omega^n(C)\, x \otimes \tau \rangle|^2 \leq \sum_{n\in\mathbb{N}} (\epsilon k)^n < \infty$, if $\epsilon > 0$ small enough.

Neighborhoods of $C_{(abc)}$ and $C_{(acb)}$

Theorem Let $\pi \in \{(abc),(acb)\}$, C_π and $U_\omega(C)$ be as above and $\mathbb{U} := \{|z|=1\}$.

For all $\gamma > 0$, there exists $\delta > 0,\ K < \infty$, s.t. $\forall C \in U(3)$, $\|C - C_\pi\| < \delta \implies \forall\, x, y \in \mathcal{T}_3$ and $\forall \sigma, \tau \in A_3$

$$\mathbb{E}\left[\sup_{f \in C(\mathbb{U}),\, \|f\|_\infty \le 1} |\langle x \otimes \tau | f(U_\omega(C))\, y \otimes \sigma\rangle|\right] \le K e^{-\gamma d(x,y)}$$

à la "Aizenman-Molchanov" '09 Hamza-J.-Stolz
Similar approach Asch, Bourget, J. '11, J. '12

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$\Rightarrow$ Spectral localization:

$\sigma(U_\omega(C))$ is pure point, a.s. (Enss-Veselic ’83)

Remarks

- Generalizations to homog. trees of coord. number $q \geq 3$:

  $\exists$ open sets $\mathcal{L} \subset \mathbb{C}^q$, resp. $\mathcal{D} \subset \mathbb{C}^q$ s.t.
  $C \in \mathcal{L} \Rightarrow \sigma(U_\omega(C))$ is p.p., resp. $C \in \mathcal{D} \Rightarrow \sigma(U_\omega(C))$ is a.c.

- RQW "analog" of weak disorder Anderson deloc. on trees '94 Klein

- $q = 1$: $\Rightarrow$ dyn. localization $\forall$ non-diag. C. '10 J.-Merkli,

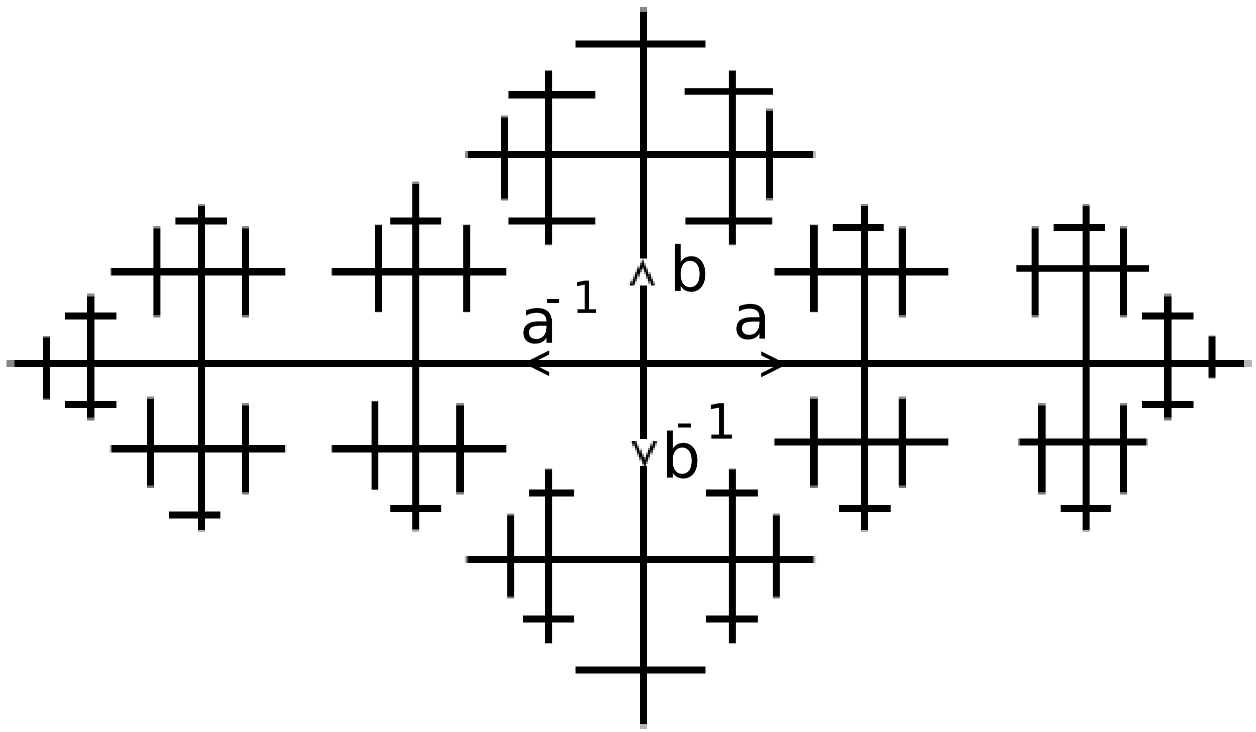
- Large disorder localization: RQW on $\mathbb{Z}^d$: '12 J.

- Spectral analysis for the deterministic case $(q = 3)$ '13 J.-Marin

Highlights of RQW in $\mathcal{T}_4$

The tree $\mathcal{T}_4$:

Hilbert space: $\mathcal{K}_4 = l^2(\mathcal{T}_4) \otimes \mathbb{C}^4$



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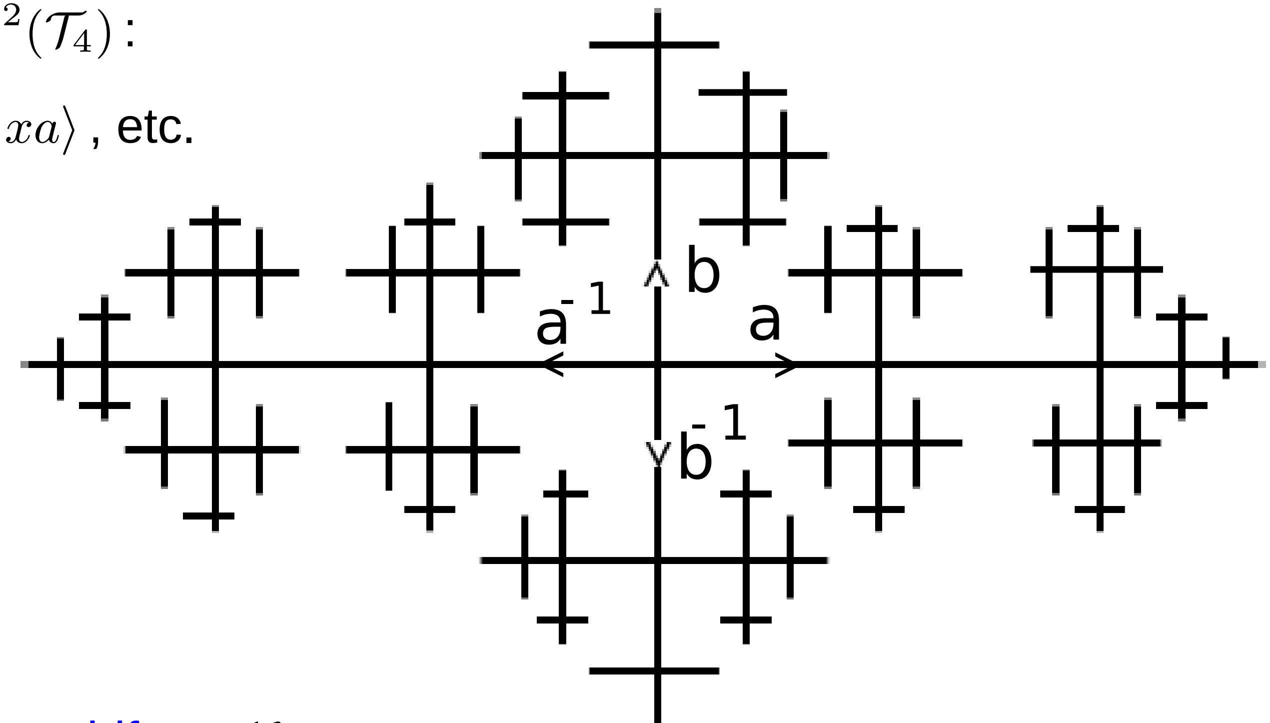

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$$S = S_a \otimes |a\rangle\langle a| + S_b \otimes |b\rangle\langle b| + S_{a^{-1}} \otimes |a^{-1}\rangle\langle a^{-1}| + S_{b^{-1}} \otimes |b^{-1}\rangle\langle b^{-1}|$$

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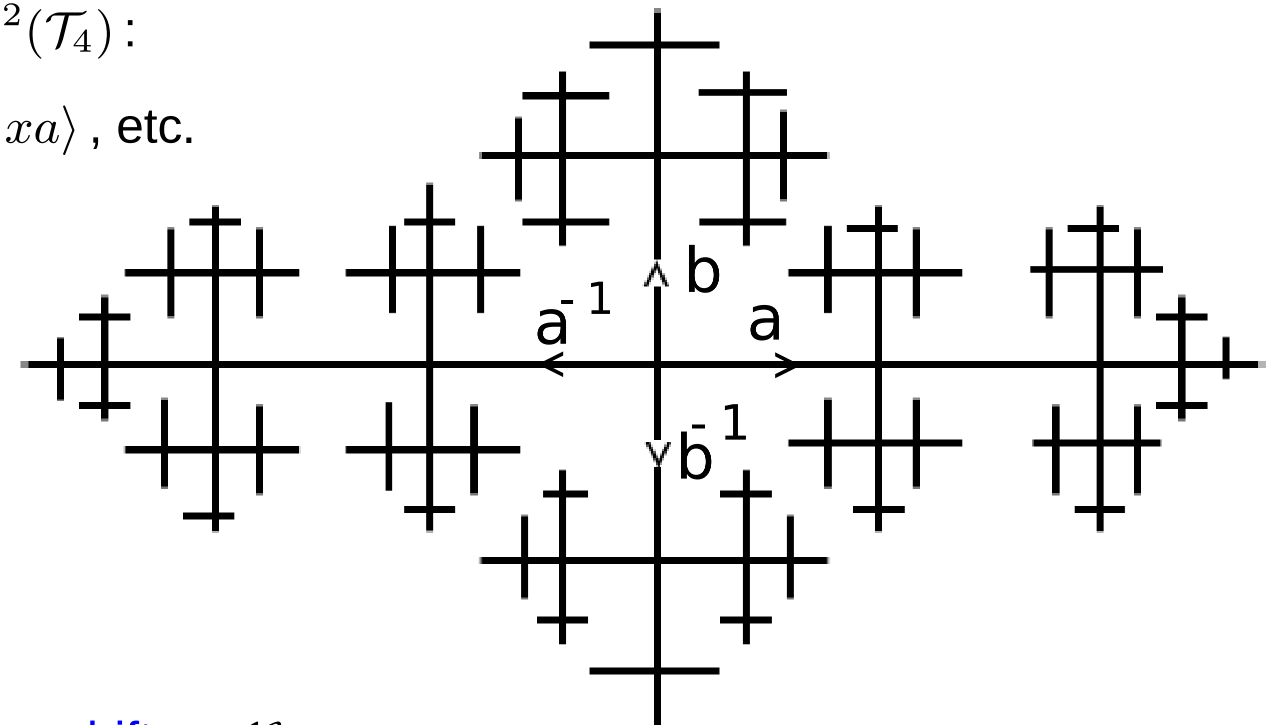

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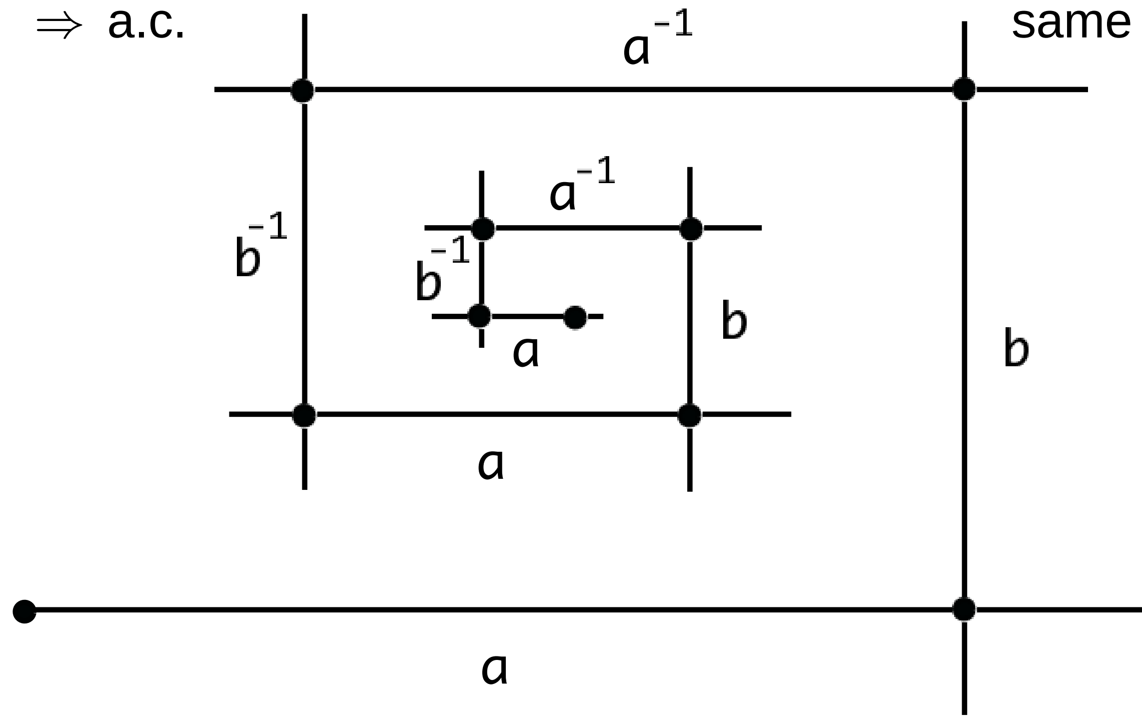
- Spin update: Via $C \in U(4)$

- RQW: $\boxed{U_\omega(C) := \mathbb{D}(\omega)S(\mathbb{I} \otimes C)}$

Highlights of RQW in $\mathcal{T}_4$

- $C_{(a)(b)(a^{-1})(b^{-1})} = \mathbb{I} \;\Rightarrow$ a.c. $\quad U_\omega(\mathbb{I}) \simeq S$.

- $C_{(aba^{-1}b^{-1})} \;\Rightarrow$ a.c. $\quad$ same for $C_{(ab^{-1}a^{-1}b)}$



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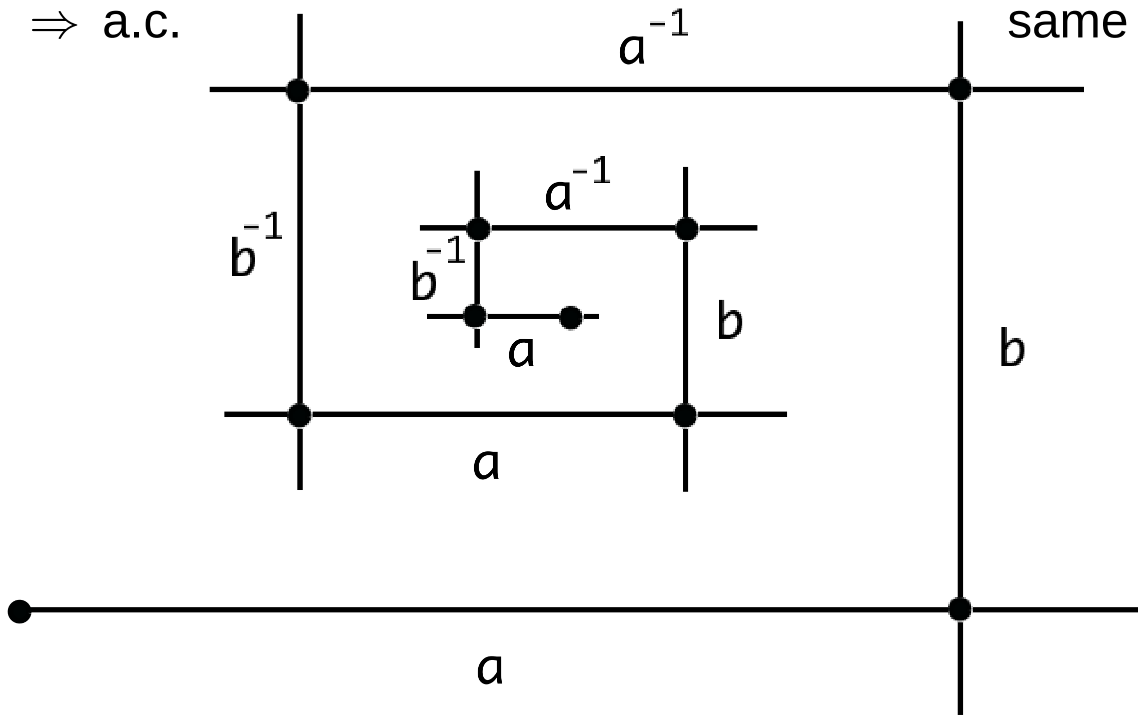

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QRW in $\mathcal{T}_4$: Highlights

Theorem: