The Wegner estimate and the integrated density of states for N-particle random Schrödinger operators

BIRS: Disordered quantum many-body systems

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related work by A. Klein and S. Nguyen, UCI

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Outline of the talk

1. The model of N-particle random Schrödinger operators
2. Main results:
   - Wegner estimate
   - Integrated density of states
3. Related works
4. Sketch of the proof
   - Klein's scale-free unique continuation principle for spectral projectors
   - scUCPSP for N-particle random Schrödinger operators
   - spectral averaging and exponential decay
5. Two region Wegner estimates
6. Application to the Delone-Anderson model

N-body random Schrödinger operators-1

Model of N-quantum particles in $\mathbb{R}^d$ moving in a random potential.

Hilbert space: $L^2(\mathbb{R}^{Nd})$ or symmetric or antisymmetric subspaces.

- The *unperturbed* N-body interacting Hamiltonian $H_{0,N}$:

$$H_{0,N} = -\sum_{j=1}^{N} \Delta_j + U(x_1, \ldots, x_N),$$

  where U is a bounded and vanishes at infinity, for example, pair interactions:

$$\sum_{1 \leq j < k \leq N} U(x_j - x_k).$$

- The full N-body random Hamiltonian:

$$H_{\omega,N} = H_{0,N} + \sum_{i=1}^{N} V_\omega(x_i),$$

  where $V_\omega(x_i)$ is a one-body random Anderson-type potential.

N-body random Schrödinger operators-2

Anderson-type random potential:

$$V_\omega(x_i) = \sum_{j \in \mathbb{Z}^d} \omega_j u(x_i - y_j).$$

- Random variables: $\{\omega_j\}_{j \in \mathbb{Z}^d}$ independent random variables with probability measures having compact support $[0, M]$.
- Single-site potential: $u \in L_0^\infty(\mathbb{R}^d)$ and strictly positive on a ball: $u > \varepsilon \chi_{\Lambda_\ell(0)}$, some finite $\varepsilon > 0$ and $\ell > 0$.
- Points: $\{y_j\}$ located in $\Lambda_1(j)$, for $j \in \mathbb{Z}^d$, so that for some $0 < \delta < 1/2$, $B_\delta(y_j) \subset \Lambda_1(j)$.

Mostly interested in u with small support: $\mathrm{supp}\ u \subset \Lambda_1(0)$.

N-body random Schrödinger operators-3

Wegner estimate: Restrict $H_{\omega,N}$ to a region $\Lambda \subset \mathbb{R}^{Nd}$: $H^{\Lambda}_{\omega,N}$
Fix E_0, What is $\mathbb{P}\{\mathrm{dist}\,(\sigma(H^{\Lambda}_{\omega,N}), E_0) < \varepsilon\}$?
GOAL: Bound in terms of $|\Lambda|$ and ε

Local operators

Definition

An N-particle rectangle is a product

$$\Lambda_{\mathbf{L}}((j_1,\ldots,j_N)) = \Lambda_{L_1}(j_1) \times \cdots \times \Lambda_{L_N}(j_N)$$

where $j_i \in \mathbb{Z}^d$ and $L_j > 0$. For a box, write Λ_L if $L_1 = L_2 = \cdots = L_N$.

Finite volume operators $H^{\Lambda}_{\omega,N}$: Restrict $H_{0,N}$ to Λ and take

$$V^{(\Lambda)}_{\omega}(x_1,\ldots,x_N) = \sum_{i=1}^{N} \left(\sum_{j \in \Lambda_{L_i} \cap \mathbb{Z}^d} \omega_j v_j(x_i - y_j) \right)$$

Sum over random variables in each component box.

Wegner estimate for N-body random Schrödinger operators

Restriction of the Hamiltonian $H_{\omega,N}$ to N-particle rectangles in $\mathbb{R}^{Nd}$ with Dirichlet or periodic boundary conditions on $\partial\Lambda$.

- Levy concentration. Define conditional probability measures:

$$\mu_j(E, E+|I|) = \mathbb{P}\{\omega_j \in [E, E+|I|] \mid (\omega_k)_{k\neq j}\}.$$

  and set

$$s(|I|) = \sup_{j\in\mathbb{Z}^d} \mathbb{E}\{\sup_{E\in\mathbb{R}} \mu_j([E, E+|I|])\}$$

- Wegner estimate:

Theorem

Let $\Lambda \subset \mathbb{R}^{Nd}$ be an N-particle rectangle with $L = \max_{i=1,\ldots,N} L_i$ sufficiently large. For any $E_0 > 0$, let $I = [I_-, I_+] \subset (-\infty, E_0] \subset \mathbb{R}$ be an energy interval with $|I$ sufficiently small. There exists a constant $0 < C(E_0, d, N, u, U) < \infty$ so that

$$\mathbb{P}\{\sigma(H^{\Lambda}_{\omega,N}) \cap I \neq \emptyset\} \leq C(E_0, d, N, u, U) s(|I|)|\Lambda|.$$

Integrated density of states for N-body random Schrödinger operators-1

- Require: more regularity assumptions on probability distribution: $\{\omega_j\}$ are *iid* with $d\mu_0(\omega_0) = g(\omega_0)d\omega_0$, $g \in L_0^\infty(\mathbb{R})$, so that $s(|I|) \leq \|g_0\|_\infty |I|$.
- Require: the regularity assumption that $y_j = j \in \mathbb{Z}^d$
- Require: Interaction potential U is a pair-interactions potential

Let $N_\Lambda(E)$ be the number of eigenvalues of $H_{\omega,N}^\Lambda$ less than or equal to E.

Definition

The integrated density of states (IDS) $N(E)$ is defined by

$$N(E) = \lim_{|\Lambda|\to\infty} \frac{N_\Lambda(E)}{|\Lambda|},$$

where Λ is an N-particle cube.

Integrated density of states for N-body random Schrödinger operators-2

Theorem

Assume that the random variables are independent and identically distributed and that the probability distribution is absolutely continuous with a bounded density. The integrated density of states $N(E)$ for $H_{\omega,N}$ is locally uniformly Lipschitz continuous. The density of states dN/dE exists and is locally bounded.

Form of the IDS from Klopp and Zenk:
The IDS $N(E)$ is given by the free particle IDS
Let $N_1(E)$ be the IDS for $H_{\omega,1} = -\Delta + V_\omega$ and let ν_1 be the corresponding DOS measure.

$$N(E) = (N_1 * \nu_1 * \nu_1 \cdots * \nu_1)(E), \quad N-\text{times}. \tag{1}$$

Other recent results on N-body random Schrödinger operators

1. Wegner estimate:
   - Chulaevsky and Suhov on $\mathbb{Z}^d$
   - Kirsch on $\mathbb{Z}^d$
   - Boutet de Monvel, Chulaevsky, Stollmann, Suhov on $\mathbb{R}^d$
   - Klopp and Zenk on $\mathbb{Z}^d$ and $\mathbb{R}^d$
2. Properties of the IDS
   - Klopp and Zenk
3. Localization
   - Chulaevsky and Suhov on $\mathbb{Z}^d$
   - Boutet de Monvel, Chulaevsky, Stollmann, Suhov on $\mathbb{R}^d$
   - Aizenman and Warzel on $\mathbb{Z}^d$
   - Klein and S. T. Nguyen on $\mathbb{Z}^d$ and $\mathbb{R}^d$

Analog of Kirsch on $\mathbb{Z}^d$; extends Klopp and Zenk in removing the covering condition on $\mathbb{R}^d$, and the other works with the correct volume and Levy concentration dependance.

Scale-free unique continuation principle for spectral projectors-1

A result of Klein (**CMP**, 2013):

1. Hamiltonian: $H = -\Delta + V$ be a self-adjoint Schrödinger operator on $L^2(\mathbb{R}^D)$, where V is a bounded potential.
2. Let $\Lambda_{\mathbf{L}}(x_0) \subset \mathbb{R}^D$ be a rectangle so that L_i are large enough.
3. Potential: Fix $\delta \in]0, 1/2]$ and let $\{y_j\}$, for $j \in \mathbb{Z}^D$ be points so that $B_\delta(y_j) \subset \Lambda_1(j)$ for all $j \in \mathbb{Z}^D$.
4. Define a nonnegative potential $W_{\Lambda_{\mathbf{L}}(x_0)}(x)$ by

$$W_{\Lambda_{\mathbf{L}}(x_0)}(x) = \sum_{j : \Lambda_1(j) \subset \Lambda_{\mathbf{L}}(x_0)} \chi_{B_\delta(y_j)}(x) \geq 0. \tag{2}$$

5. For any $E_0 > 0$, define a constant $K = K(V, E_0) = 2\|V\|_\infty + E_0$.

Scale-free unique continuation principle for spectral projectors-2

Theorem (Klein)

There exists a finite positive constant $M_D > 0$, such that, if we define a constant $\gamma = \gamma(D, K, \delta) > 0$ by

$$\gamma^2 = (1/2)e^{M_D(1+K^{2/3})}, \tag{3}$$

Let $E_\Lambda(I)$ be the spectral projector for $H_{\Lambda_{\mathbf{L}}(x_0)}$ and interval I. Then for any closed interval $I \subset]-\infty, E_0]$, with $|I| < 2\gamma$, we have

$$E_\Lambda(I) W_{\Lambda_L(x_0)} E_\Lambda(I) \geq \gamma^2 E_\Lambda(I).$$

- Improves a result of Rojas-Molina and Veselić who proved the result for eigenfunctions rather than spectral projectors
- Based on a quantitative UCP result of Bourgain and Kenig

sfUCPSP for N-body random Schrödinger operators-1

N-particle random potential $V_\omega^{(N)}$ on $L^2(\mathbb{R}^{Nd})$:

$$V_\omega^{(N)}(x_1,\ldots,x_N) = V_\omega(x_1)\times 1\times\cdots\times 1+\cdots+1\times 1\times\cdots\times V_\omega(x_N), \tag{4}$$

where

$$V_\omega(x_i) = \sum_{j\in\mathbb{Z}^d}\omega_j u(x_i-y_j), \quad x_i, y_j\in\mathbb{R}^d. \tag{5}$$

Denote by $\tilde{V}^{(N)}$, respectively, $\tilde{V}$, the Anderson potentials (4), respectively, (5), with all random variables set equal to one.
For example,

$$\tilde{V}(x_i) = \sum_{j\in\mathbb{Z}^d} u(x_i-y_j), \quad x_i, y_j\in\mathbb{R}^d.$$

sfUCPSP for N-body random Schrödinger operators-2

Theorem

- *Let $H^{\Lambda}_{\omega,N}$ be the restriction to Λ of the N-body Schrödinger operator with a bounded N-body potential U and random potential $V^{(N)}_{\omega}$.*
- *Let $\tilde{V}^{(N)}_{\Lambda}$ be the random N-body potential obtained from $V^{(N)}_{\omega}|\Lambda$ with all $\omega_j = 1$.*

For any $E_0 > 0$, let $\gamma \equiv \gamma(d, N, U, u, E_0) > 0$ be the constant in the sfUCPSP depending on E_0, the dimension d, the N-body potential U, particle number $N \geq 1$, and the single-site potential u, but independent of Λ.

Then, for any interval $I \subset (-\infty, E_0]$ with $|I| \leq 2\gamma$, we have

$$E^{(N)}_{\Lambda}(I)\tilde{V}^{(N)}_{\Lambda}E^{(N)}_{\Lambda}(I) \geq N\gamma^2 E^{(N)}_{\Lambda}(I).$$

sfUCPSP for N-body random Schrödinger operators-3

The single-site potential u is chosen so that $u(x) \geq \varepsilon \chi_{\Lambda_\ell(0)}(x)$, for $x \in \mathbb{R}^d$. Suppose $0 < \delta \leq \ell$ so $B_\delta(y_j) \subset \Lambda_\ell(j)$.

Basic idea:

$$
\begin{aligned}
\tilde{V}_\Lambda^{(N)}(x_1, \ldots, x_N) &= \sum_{i=1}^{N} \tilde{V}_{\Lambda_i}(x_i) \\
&= \sum_{i=1}^{N} \left(\sum_{y_{j_i} \in \tilde{\Lambda}_{L_i}} u(x_i - y_{j_i}) \right) \\
&\geq \sum_{i=1}^{N} \left(\sum_{j_i \in \tilde{\Lambda}_{L_i}} \chi_{B(y_{j_i}, \delta)}(x_i) \right) \\
&\geq N \sum_{\mathbf{j} \in \tilde{\Lambda}} \chi_{\Lambda_\delta(\mathbf{y_j})}(x_1, \ldots, x_N),
\end{aligned}
$$

where $\mathbf{y_j} = (y_{j_1}, \ldots, y_{j_N})$ and $\tilde{\Lambda} = \Lambda \cap \mathbb{Z}^{Nd}$.

Proof of the Wegner estimate for N-body random Schrödinger operators-1

For a small energy interval Δ we must estimate from above:

$$\mathbb{P}\{\sigma(H^{\Lambda}_{\omega,N}) \cap \Delta \neq \emptyset\} \leq \mathbb{E}\{TrE^{(N)}_{\Lambda}(\Delta)\}.$$

- sfUCPSP involving the spectral projectors for $H^{\Lambda}_{N,\omega}$
- Allows improvement of Combes, Hislop, Klopp

Use sfUCPSP twice:

$$\begin{aligned}
\mathbb{E}\{TrE^{(N)}_{\Lambda}(\Delta)\} &\leq (N\gamma^2)^{-1}\mathbb{E}\{TrE^{(N)}_{\Lambda}(\Delta)\tilde{V}^{(N)}_{\Lambda}\} \\
&\leq (N\gamma^2)^{-2}\mathbb{E}\{TrE^{(N)}_{\Lambda}(\Delta)\tilde{V}^{(N)}_{\Lambda}E^{(N)}_{\Lambda}(\Delta)\tilde{V}^{(N)}_{\Lambda}\}
\end{aligned}$$

Proof of the Wegner estimate for N-body random Schrödinger operators-2

Use positivity:

$$\begin{aligned}
&\mathbb{E}\{ Tr E_{\Lambda}^{(N)}(\Delta)\} \\
&\quad\leq \quad (E_0 + M)(N\gamma^2)^{-2}\mathbb{E}\{ Tr E_{\Lambda}^{(N)}(\Delta)\tilde{V}_{\Lambda}^{(N)}(H_{0,N}^{\Lambda} + M)^{-1}\tilde{V}_{\Lambda}^{(N)}\}.
\end{aligned}$$

We must estimate:

$$\mathbb{E}\{ Tr E_{\Lambda}^{(N)}(\Delta)\tilde{V}_{\Lambda}^{(N)}(H_{0,N}^{\Lambda} + M)^{-1}\tilde{V}_{\Lambda}^{(N)}\}$$

Main tools

1. Spectral averaging result of Combes, Hislop, Klopp (2007):
2. Exponential decay of the resolvent localized between disjoint regions

Proof of the Wegner estimate for N-body random Schrödinger operators-3

Spectral Averaging:

Lemma

Let $\mathbf{y_j} \equiv (y_{j_i}, \ldots, y_{j_N}) \in \mathbb{R}^{Nd}$ and let $\Phi_{\mathbf{y_j}}$ be a nonnegative compactly supported function with support in a ball around $\mathbf{y_j}$. Let K be a non-random bounded operator so that $\Phi_{\mathbf{y_j}} K \Phi_{\mathbf{y_k}}$ is trace class. Let $E_\Lambda(\Delta)$ be the random spectral projector for a random Schrödinger operator restricted to Λ. We then have

$$\mathbb{E}\{\mathrm{Tr} E_\Lambda(\Delta) \Phi_{\mathbf{y_j}} K \Phi_{\mathbf{y_k}}\} \leq 8 s(|\Delta|) \|\Phi_{\mathbf{y_j}} K \Phi_{\mathbf{y_k}}\|_1, \tag{6}$$

where $s(\cdot)$ is the Levy concentration.

Reduces the problem to finding suitable K and computing $\|\Phi_{\mathbf{y_j}} K \Phi_{\mathbf{y_k}}\|_1$. This is obtained by expanding the potentials $\tilde{V}_\Lambda^{(N)}$.

Proof of the Wegner estimate for N-body random Schrödinger operators-4

Exponential decay of the localized resolvent:

Resolvents localized between functions with disjoint supports exhibit exponential decay:

Lemma

Suppose that χ_1 and χ_2 are two functions with disjoint and compact supports in $\Lambda \subset \mathbb{R}^{Nd}$ and d_{12} is the distance between their supports. There exist finite, positive constants $A_{12}, \alpha > 0$, depending on $H_{0,N}$ and M, so that

$$\|\chi_1(H_{0,N}^{\Lambda} + M)^{-1}\chi_2\|_1 \leq A_{12} e^{-\alpha d_{12}}. \tag{7}$$

Proof of these known results follows from the Combes-Thomas method.

Two region Wegner estimates-1

These were introduced by Chulaevsky and Suhov. These estimates relate the eigenvalues of the local Hamiltonians associated with two regions that are sufficiently separated. The potentials associated with such regions need not be independent.

Projections:

- Let $\Pi_j : \mathbb{R}^{Nd} \to \mathbb{R}^d$ be the projection onto the j^{th}-coordinate.
- For an N-particle rectangle $\Lambda \subset \mathbb{R}^{Nd}$, the set $\Pi_j \Lambda$ is the j^{th}-component rectangle of Λ in $\mathbb{R}^d$.
- $\Pi\Lambda$ is the subset of $\mathbb{R}^d$ defined by

$$\Pi\Lambda \equiv \bigcup_{j=1}^{N} \Pi_j \Lambda \subset \mathbb{R}^d. \qquad (8)$$

- For a subset $\mathscr{J} \subset \{1, \ldots, N\}$, we define

$$\Pi_{\mathscr{J}} \Lambda = \bigcup_{j \in \mathscr{J}} \Pi_j \Lambda \subset \mathbb{R}^d. \qquad (9)$$

Two region Wegner estimates-2

Definition

Two rectangles $\Lambda, \Lambda' \subset \mathbb{R}^{Nd}$ are **partially separated** if there exists a nonempty subset $\mathscr{J} \subset \{1, \ldots, N\}$ of indices so that either

$$\Pi_{\mathscr{J}} \Lambda \cap \left[\Pi_{\mathscr{J}^c} \Lambda \cup \Pi \Lambda' \right] = \emptyset,$$

or

$$\Pi_{\mathscr{J}} \Lambda' \cap \left[\Pi_{\mathscr{J}^c} \Lambda' \cup \Pi \Lambda \right] = \emptyset.$$

Two region Wegner estimates-3

The notion of partial separation guarantees that there are random variables in one rectangle that are independent of the random variables in the second rectangle.

Theorem

Let $H^{\Lambda}_{\omega,N}$ be an N-particle Hamiltonian restricted to an N-particle rectangle Λ. Let $I_0 \subset (-\infty, E_0]$ be an energy interval for any $E_0 > 0$ fixed. If Λ and Λ' are two partially separated N-particle rectangles, then for any $\varepsilon > 0$,

$$\mathbb{P}\{\mathrm{dist}\,(\sigma(H^{\Lambda}_{\omega,N}) \cap I_0, \sigma(H^{\Lambda'}_{\omega,N}) \cap I_0) < \varepsilon\}$$

$$\leq C_W C(E_0, d, N, u, U) E_0^d |\Lambda'||\Lambda| s(2\varepsilon),$$

where the constant $C(E_0, d, N, u, U)$ is as in the Wegner estimate and C_W is the constant from Weyl's law.

Application to Delone-Anderson Model

Definition

For positive constants $0 < m < M < \infty$, an (m, M) **Delone set** $\Gamma_{m,M}$ is a discrete subset $\{z_j\} \subset \mathbb{R}^d$ so that

1. Any cube of side-length m contains no more that one point,
2. Any cube of side length M contains at least one point.

Lattice $M\mathbb{Z}^d$ contains at least one point in each basic cube of side length M.

Decompose $\Gamma_{m,M} = \Gamma_1 \cup \Gamma_2$, with $\{y_j\} \subset \Gamma_1$ so $y_j \in \Lambda_M(j)$.

Decompose V_ω into two pieces according to Γ_1 and Γ_2.

Treat the piece obtained by summing over $z_j \in \Gamma_1$ according to the sfUCPSP.

Conclusions for N-body random Schrödinger operators

- IDS is as well-behaved as the Levy concentration is well-behaved.
- Systems have pair-interactions that are bounded and decaying at infinity.
- One-particle random potentials may be Anderson-Delone type as in Rojas-Molina and Veselić.
- Method yields two-region Wegner estimates used in multi-scale analysis