New Perspectives on the N-body Problem

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January 13, 2013–January 18, 2013

1 Overview of the Field

The problem of the motion of $N \geq 2$ point–masses (i.e., ideal bodies with no physical dimension identified with points in the Euclidean three–dimensional space) interacting only through Newton's law of mutual gravitational attraction, has been a central issue in astronomy, physics and mathematics since the early developments of modern calculus. When $N = 2$ the problem has been completely solved ("integrated") by Newton: the motion takes place on conics, whose focus is occupied by the center of mass of the two bodies; but for $N \geq 3$ a complete understanding of the problem is still far away, notwithstanding the efforts of many excellent mathematicians over the last centuries (Newton, Euler, d'Alembert, Lagrange, Laplace followed by H. Poincaré, G.D. Birkhoff, C.L. Siegel, J.K. Moser, V.I. Arnold, M. Herman,...)

The dynamics f the N–body problem has the fascinating feature of hosting regular and chaotic regimes and has often inspired the elaboration of new mathematical theories of local, perturbative and global nature. Indeed, to be attacked efficiently, it requires a sophisticated mixture of algebraic, analytical, geometrical, topological and probabilistic techniques.

While the original impulse, coming from astronomy, has been somehow shaded by the massive use of machines for computing orbits of celestial bodies or satellites, the mathematical richness and beauty of the N–body problem has retained most of its original attraction, and still may be considered as one of the most fascinating and difficult branch of mathematics (see, below, for a list of recent excellent contributions, most of the authors being present at this conference).

2 Recent Developments and Open Problems

Some of the most important recent developments in the theory of the N–body problem have been given by participants to this workshop. In particular, we select the following subjects:

1. *Existence and classification of stable orbits in the planetary N-body problem.* In [10] Arnold's theorem on the existence of quasi-periodic motions for the planetary N + 1-body problem (1 big and N small masses) has been, for the first time, completely proved. Later, [7], a new description of the phase space, allowed to check certain non-degeneracies, to complete Arnold's project ([1]) and to obtain new results concerning stable periodic orbits and lower dimensional elliptic invariant tori.

2. *The variational approach to the periodic problem.* The problem of the existence and the qualitative properties of periodic orbits for the N-body problem (from the classical celestial mechanics point of view to more recent advances in molecular and quantum models) has been extensively studied over the decades, and, more recently, new tools and approaches have given a significant boost to the field. In particular, nontrivial action minimizing periodic trajectories have been recently found by imposing suitable symmetry and/or topological constraints ([6], [12], [4], [13]).

3. *The number of central configurations.* The finiteness of the number of central configurations is a long standing conjecture in Celestial Mechanics and has been recently confirmed as Smale's 6th question for the 21st century list of problem. As known, the structure of the set of central configurations plays a central role in the analysis of collisions and the final evolution of parabolic trajectories. Recent breakthrough are the papers [14] and [2] where it is proved that the number of central configurations for the 4 and 5 body problems is finite, up to a codimension 2 set of masses.

Still many interesting and very difficult problems have to been pursued. A selection of these include:
– coexistence of stable and unstable regions;
– analysis of mean-orbital resonances;
– Nekhoroshev analysis of the secular approximate invariants (mutual inclinations and eccentricities);
– long-period periodic orbits near invariant tori; intermediate stable/unstable invariant tori; resonant tori;
– instabilities (including special instances of Arnold diffusion), possible applications of Aubry–Mather theory
– symbolic dynamics;
– study of selected trajectories such as periodic, bounded and parabolic orbits;
– applications of weak KAM theory to the N-body problem;
– stability issues for periodic trajectories found by variational methods;
– regularization of collisions.

3 Presentation Highlights

In this section we report a selection of extended abstracts among the 25 seminars that took place in the conference.

Vivina Barutello (Università di Torino, vivina.barutello@unito.it)
Parabolic trajectories as minimal phase transitions
We consider the conservative dynamical system

$$\ddot{x}(t) = \nabla V(x(t)), \qquad x \in \mathbb{R}^d \setminus \mathcal{X}, \tag{1}$$

where $d \geq 2$, the potential V is smooth outside –and goes to infinity near– the collision set $\mathcal{X}$. A (global) *parabolic trajectory* for (1) is a collisionless solution which has null energy. In the Kepler problem ($V(x) = 1/|x|$) all global zero-energy trajectories are indeed parabola; for the class of anisotropic Kepler problems in $\mathbb{R}^d \setminus \{0\}$ with homogeneous potentials, we seek parabolic trajectories having prescribed asymptotic directions at infinity and which, in addition, are Morse minimizing geodesics for the Jacobi metric. Such trajectories correspond to saddle heteroclinics on the collision manifold, are structurally unstable and appear only for a codimension-one submanifold of such potentials. We give them a variational characterization in terms of the behavior of the parameter-free minimizers of an associated obstacle problem. We then give a full characterization of such a codimension-one manifold of potentials and we show how to parameterize it with respect to the degree of homogeneity.

We then focus on the planar case: we deepen and complete the analysis for homogeneous singular potentials characterizing all parabolic orbits connecting two minimal central configurations as free-time Morse minimizers (in a given homotopy class of paths). These may occur for at most one value of the homogeneity exponent. In addition, we link this threshold of existence of parabolic trajectories with the absence of collisions for all the minimizers of fixed-ends problems. Also the existence of action minimizing periodic trajectories with nontrivial homotopy type can be related with the same threshold.

Kuo-Chang Chen (National Tsing Hua University, kchen@math.nthu.edu.tw)
Keplerian action functional, convex optimization, and n-body problems with only boundary and topological

constraints
Variational methods have been applied to construct various types of solutions for the n-body problem, under various types of symmetry constraints. However, there were not much success with similar approaches for n-body problems without symmetry and equal-mass constraints, especially when $n > 3$. In this talk we will introduce an apparatus to construct periodic solutions for the n-body problem with only boundary and topological constraints. This approach is a combination of variational arguments in our works on retrograde orbits for the three-body problem and certain constraint convex optimization problems. Our method has no restriction on equal masses. We will illustrate this approach by constructing relative periodic solutions for the planar four-body problems within several topological classes.
The first key step of our construction is to consider the minimization problem for the Keplerian action functional confined to the space

$$\Gamma_T(\Omega_0, \Omega_1) = \{\mathbf{r} \in H^1([0,T], \mathbb{C}) : \mathbf{r}(0) \in \Omega_0,\ \mathbf{r}(T) \in \Omega_1\},$$

where boundary constraints Ω_0, Ω_1 are singleton or rays from the origin. When both Ω_0 is the positive real line, $\Omega_1 = \rho e^{i\phi}$, we show that the infimun action value is convex in ρ and increasing in $\phi \in (0, \pi/2]$.
The next key step of our construction is to consider minimization problem with n bodies which move from one collinear configuration to another, and we show that the action functional is bounded from below by

$$\frac{3}{2}\left[\sum_{(i,j)\in\blacktriangle_0'} m_i m_j \pi^{\frac{2}{3}} + \sum_{(i,j)\in\blacktriangle_1'} m_i m_j \phi^{\frac{2}{3}} + \sum_{(i,j)\in\blacktriangle_2'} m_i m_j (\pi - \phi)^{\frac{2}{3}}\right]$$

plus another summation which is the positive combination of the above mentioned infimum Keplerian action value. Here $\blacktriangle_0'$, $\blacktriangle_1'$, $\blacktriangle_2'$ is a suitable partition for subscript pairs. We provide lower bound estimates for the action functional by convex optimization methods. This allows us to construct many solutions for the n-body problem that are not covered by previous methods.

Jacques Féjoz (Université Paris-Dauphine, fejoz@ceremade.dauphine.fr)
Punctured invariant tori in the spatial three-body problem (after Zhao Lei)
Consider the spatial three-body problem in the regime where the outer body revolves far away around the other two, after reduction by translations and rotations. The phase space has dimension 8.

Theorem [Zhao L.] *There exists a subset of initial conditions of positive Lebesgue measure leading to inner almost collisions i.e., motions along which the two inner bodies pass infinitely many times arbitrarily close to each other. Up to time reparametrization, these solutions are quasiperiodic with 4 frequencies.*
Such solutions are oscillatory in the velocities, in the sense that the limsup of the velocities is infinite, while the liminf is finite and non-zero. On the other hand, positions remain bounded.

This theorem generalizes former results of Chenciner-Llibre [5] in the planar restricted problem and of [9] in the planar full problem, where almost collisions produce syzygy stutterings (for any possible value of the angular momentum) in the sense of Kaplan–Levi–Montgomery [15].
Zhao's proof consists in:
- analyzing the surprising commutative interplay between the averaging and regularizing (for the inner binary collisions) procedures
- understanding the local geometry of collisions
- making sense of the study of Lidov-Ziglin in the neighborhood of degenerate Keplerian ellipses
- applying some kind of equivariant KAM theorem, showing the existence of invariant Lagrangian tori in the regularized problem
- showing some transversality property, which ensures that most solutions lying on these tori actually do not ever lead to collisions.

This work is part of the PhD thesis of Zhao Lei.

Davide L. Ferrario (Università di Milano Bicocca, davide.ferrario@unimib.it)
Dynamics of some symmetric n-body problems
We study n-body problems which are symmetric with respect to the action of suitable extensions of finite rotation groups. Such problems occur either as showcases of typical dynamical problems and features (such as

Devaney isosceles problem, Sitnikov problem, ...), or as remarkable and interesting families of periodic orbits (Chenciner-Montgomery figure-eight, Ferrario-Terracini symmetric orbits, Fusco-Gronchi-Negrini platonic orbits, ...) The space of symmetric configurations is the complement of an arrangement of linear subspaces in a Euclidean space, and blow-up, McGehee coordinates and variational methods can be in some cases used to understand local dynamics (around the space of collisions) and some properties of periodic orbits.

Giovanni Gronchi (Università di Pisa, gronchi@dm.unipi.it)
The evolution of the orbit distance in the double averaged restricted 3-body problem
We study the long term evolution of the distance between two Keplerian confocal trajectories in the framework of the averaged restricted 3-body problem. The bodies may represent the Sun, a solar system planet and an asteroid. The secular evolution of the orbital elements of the asteroid is computed by averaging the equations of motion over the mean anomalies of the asteroid and the planet. When an orbit crossing with the planet occurs the averaged equations become singular. However, it is possible to define piecewise differentiable solutions by extending the averaged vector field beyond the singularity from both sides of the orbit crossing set and to compute an explicit formula for the difference of the extended vector fields at planet crossings. We generalize the previous results by improving the singularity extraction technique and show that the extended vector fields are Lipschitz-continuous. Moreover, we consider the distance between the Keplerian trajectories of the small body and the planet. Apart from exceptional cases, we can select a sign for this distance so that it becomes an analytic map of the orbital elements near to crossing configurations. We prove that the evolution of the 'signed' distance along the averaged vector field is more regular than that of the elements in a neighborhood of the crossing times. A comparison between averaged and non-averaged evolutions and applications of these results are shown using orbits of near-Earth asteroids.

Marcel Guardia (University of Maryland, marcel.guardia@upc.edu)
Oscillatory motions for the restricted planar circular three body problem
In 1980 J. Llibre and C. Simó [16] proved the existence of oscillatory motions for the restricted planar circular three body problem, that is, of orbits which leave every bounded region but which return infinitely often to some fixed bounded region. To prove their existence they had to assume that the ratio between the masses of the two primaries was exponentially small with respect to the Jacobi constant. In the present work, we generalize their work proving the existence of oscillatory motions for any value of the mass ratio.

We show that, for any mass ratio and large enough Jacobi constant, there exist transversal intersections between the stable and unstable manifolds of infinity which guarantee the existence of a symbolic dynamics that creates the so called oscillatory orbits. The main achievement is to rigorously prove the transversality of the invariant manifolds without assuming the mass ratio small, since then this transversality can not be checked by using classical perturbation theory respect to the mass ratio. Finally, we show that in a curve in the two dimensional parameter space formed by the mass ratio and the Jacobi constant, the invariant manifolds of infinity undergo a cubic tangency. This is a joint work with P. Martin and T. M. Seara.

Yiming Long (Nankai University, longym@nankai.edu.cn)
Stability of elliptic Lagrangian solutions of the classical three body problem via index theory
Lagrange found his famous elliptic equilateral triangle solutions of the classical planar three body problem in 1772 which depend on the mass parameter and eccentricity of the ellipse. Linear stability of such solutions has been investigated by perturbation methods or numerical methods. But we are not aware of any rigorous analytical method which relate this stability to the parameters in their full range. In this lecture, I shall give a brief introduction on the new rigorous analytical method and recent results jointly obtained by Xijun Hu, Shanzhong Sun and myself on this linear stability problem for the full range of the masses and eccentricity via index theories for symplectic matrix paths. We proved the existence of three curves in the parameter rectangle, which separate the full parameter domain precisely according to the linear stability of elliptic Lagrangian solutions.

Ezequiel Maderna (Universidad de la República, emaderna@cmat.edu.uy)
Abundance of complete parabolic motions via weak KAM theory
In this talk I will consider the general N-body problem. I will show a Hölder estimate for the minimal action

which enable us to prove the existence of weak solutions for the Hamilton-Jacobi equation. The calibrating curves of a given such solution produces a lamination of the configuration space composed by rays of free time minimizers which, as we will see, they are complete parabolic motions.

Richard Montgomery (University of California at Santa Cruz, rmont@ucsc.edu)
Paradoxes and Opportunities from Global Regularization.
Specific beautiful features of the N-body problem exclude most results from the modern theory of dynamical systems and of symplectic topology from being applicable. These features are collisions and symmetries. They imply that the N-body flow violates the most basic assumptions of modern theory, namely that the flow is complete and that all its periodic orbits are non-degenerate in the sense that the linearized Poincare return map about such orbits has no 1's in its spectrum, and hence is isolated. Collisions between bodies in the N-body problem renders the flow incomplete. The N-body problem has a large symmetry group, the group of Euclidean motions, and these symmetries imply that periodic orbits are never isolated. In order to apply modern theory, and to obtain a global qualitative picture of the problem, it will be useful, and probably necessary, to (A) regularize the flow so as to make it complete, and (B) eliminate its symmetries.

There are well-known methods in place for achieving goals (A) and (B). Levi-Civita showed us how to regularize binary collisions in the 1920s, thereby eliminating those simplest collisions as a source of non-completeness. McGehee showed us in 1970 how to regularize triple collisions, more properly how to blow-up these collisions, replacing them with a sphere. Meyer (a conference participant), Marsden and Weinstein set up the formal mechanism of symplectic reduction for eliminating symmetries by a quotient procedure in 1974. In this talk we describe how Rick Moeckel and I systematically and democratically apply the methods of regularization, blow-up, and symplectic reduction to the N-body problem. Democratically means without singling out any one body as being special. We call the simultaneous democratic application of all three methods 'global regularization'. We explicitly describe the global regularization of the planar 3-body problem, showing that the result is a complete flow on a 6-dimensional phase space having only discrete symmetries remaining. We hint at the structure of the globally regularized phase space and its flow for the planar 4-body problem and the spatial 3-body problem.

At the heart of the Levi-Civita's regularization is a complex squaring map $z \mapsto z^2$. At the heart of reduction in the planar 3-body problem is the shape sphere S^2 whose points represent oriented similarity classes of triangles and which comes with three marked points: the three types of binary collisions. Combining reduction and regularization leads to a regularized shape sphere S^2_{reg} together with a regularization map $S^2_{reg} \to S^2$ which is a $4:1$ branched cover, branched over the 3 binary collision points. The associated group of deck transformations of this covering is a 4-element group whose appearance in the 3-body problem is new. We use this group to pose new variational problems of potential use for finding new solutions in the three body problem.

The new group, and the regularization map also leads to apparent paradoxes between known results. One such paradox is between Marchal's theorem and a basic theorem of differential geometry asserting the local minimality of geodesics. Marchal's theorem is the most powerful recent theorem in the application of the direct method of the calculus of variations to the Newtonian analysis and asserts that action minimizers cannot have interior collision points. But regularized solutions do pass through collision and there is a regularized geometric Jacobi-Maupertuis variational principle for which these solutions are minimizers with interior collision points. So on the one hand minimizers can have no interior collisions, while on the other hand, they can have interior collisions. In conversations with Andrea Venturelli during the conference this apparent paradox was resolved. The resolution resulted in new topological insights regarding the nature of near-collision orbits.

The methods of global regularization for the planar N-body problem lead inexorably to algebraic geometry. Maps are rational. Varieties encountered are algebraic. Branched covers appear frequently. McGehee blow-up is replaced by the blow-up of algebraic geometry. In the case of the planar 4-body problem the final globally regularized phase space appears to be the cotangent bundle of the famous K3 surface from the theory algebraic surfaces.

We believe global regularization may lead to a new renaissance of interactions between algebraic geometry and celestial mechanics. For general N the new symmetries arising out of Levi-Civita's local 2:1 branched cover are products of a large number $d(N)$ copies of two element sign change groups, and may lead to many new collision and near-collision orbits in the planar N-body problem.

The talk was largely based on paper (arXiv: 1202.0972) , the planned 1st part of a 3 part series.

Dan Offin (Queen's University, offind@mast.queensu.ca)
Dynamics, symmetry and hyperbolicity.
We begin by discussing the connection between absolute minimization of the action and hyperbolic structures on an invariant set for a convex Lagrangian system on a compact complete manifold. Limiting cases of absolutely minimizing periodic orbits and the resulting hyperbolic structure are considered. Next we consider conditionally minimizing periodic orbits in the N-body problem with symmetry constraints. Although this context does not the earlier case of absolutely minimizing trajectories on compact manifolds, we show how the argument may be modifed to apply in certain cases of the N-body problem. We give a sufficient condition using the spatio-temporal structure of symmetric minimizing curves for the N-body problem to guarantee hyperbolicity of the orbits.
This is a joint work with Gonzalo Contreras.

Rafael Ortega (Universidad de Granada, rortega@ugr.es)
Stable periodic solutions in the forced pendulum equation
Consider the equation

$$x'' + \beta \sin x = f(t)$$

where $\beta > 0$ is a parameter and the forcing $f : \mathbb{R} \to \mathbb{R}$ is a continuous and 2π-periodic function with zero average, that is

$$f(t + 2\pi) = f(t), \quad \int_0^{2\pi} f(t)dt = 0.$$

It is well known that this equation has at least two different 2π-periodic solutions $\varphi(t)$ and $\psi(t)$. Due to the periodicity of the sine function we are assuming that $\varphi \neq \psi + 2\pi N$. The original proof of this result was of variational nature. Hamel proved the existence of φ by minimization ($\sim$ 1920) and Mawhin and Willem obtained ψ via mountain pass ($\sim$ 1980). Later Franks presented a symplectic proof ($\sim$ 1980) based on Poincaré-Birkhoff theorem. The goal of this talk is to discuss the properties of stability of these solutions (without imposing conditions of the type "small/large parameters").

The solution φ is a minimizer of the action functional and so it is always unstable. The main result of this talk says that if $\beta \leq \frac{1}{4}$ then there exists a stable 2π-periodic solution for almost all forcings f. The stability is understood in the sense of Lyapunov. The sentence "for almost all forcings" is understood in the sense of prevalence; that is, for all functions f in a set P that is prevalent in the Banach space

$$X = \{f \in C(\mathbb{R}/2\pi\mathbb{Z}) : \int_0^{2\pi} f(t)dt = 0\}$$

endowed with the uniform norm. The number $\frac{1}{4}$ is optimal. For any $\beta > \frac{1}{4}$ it is possible to construct an open set G in X such that if $f \in G$ then all 2π-periodic solutions are unstable. The result is not valid in this case because any prevalent set intersects G. Even if $\beta \leq \frac{1}{4}$ but $\beta > \frac{1}{9}$ it is possible to find some $f \in X$ producing only unstable 2π-periodic solutions. This shows the need of the sentence "for almost all forcings". The proof is structured in four steps: 0) non-degeneracy of periodic solutions (tool: parametric transversality in prevalent version) 1) existence of an elliptic solution (Sturm comparison, minimization, topological degree) 2) stability and arithmetic conditions (diophantine numbers, Russmann's theorem) 3) conclusion (regular values of the discriminant, transversality conditions, Haar null sets under diffeomorphisms).

Gabriella Pinzari (Università Roma Tre, pinzari@mat.uniroma3.it)
On certain instabilities occurring in the planetary three–body problem, with closely spaced planets
The planetary N–body problem has a positive measure invariant set (*Kolmogorov set*) for the motion, consisting into the union of Lagrangian, KAM tori, with maximal number of frequencies. This statement goes back to Arnold [1] but its proof, to be completed, took about 50 years and strong efforts [10], [7], due to the degeneracies of the problem. Outside this set, the question of stability of motions is essentially open. The stability of semi–major axes up to exponentially long times has been proved by Nekhoroshev in 1977. The stability of the whole system, including eccentricities and mutual inclinations is a more subtle question, deeply related to mean–motion resonances. In a region of phase space where these are absent, stability con

be proven at least for polynomially long times [8].
The presence of mean–motion resonances in general changes drastically the aspect of the effective Hamiltonian and instability may occur: in [11] a displacement of the eccentricity in the planar, restricted, elliptic three–body problem along a suitable resonance has been proved.
I shall talk about a case of instability occurring in the planetary three–body problem, where the planets are very close one to the other and revolve in opposite verses. This corresponds to the very special resonance 1:1, with the mean motions $n_1 \sim n_2$. Along this resonance the equilibrium point of the effective perturbing function bifurcates from elliptic to partially hyperbolic and hence calls for lower dimensional hyperbolic KAM tori. Such tori should have co–dimension 1 in the planar problem, 2 in the spatial problem, with hyperbolic directions corresponding to one of the eccentricities and the mutual inclination.
I shall discuss a Graff–type normal form for this system. The proof uses a new reduction by the SO(3)–symmetry and extreme averaging techniques; the major difficulty being to avoid collision singularity. As future direction of research, it would be desirable to check whether this normal form is sufficient to prove existence of unstable tori and (eventually) Arnold instability for semi–axes. Remarkably, in a similar setting, some kind of instability for semi–axes seems to be numerically detected: [19] and references therein.

Pablo Roldan (Universitat Politècnica de Catalunya, pablo.roldan@upc.edu)
Numerical study of a normally hyperbolic cylinder in the RTBP
We consider the circular planar Restricted Three-Body Problem modeling a Sun-Jupiter-Asteroid system, and we show the existence of a normally hyperbolic cylinder composed of resonant periodic orbits using numerical methods. Then we study the splitting of the associated invariant manifolds.
This is a key step in our proof of diffusion along mean-motion resonance in the elliptic planar RTBP.

Tere M. Seara (Universitat Politècnica de Catalunya, tere.m-seara@upc.edu)
Orbits with increasing angular momentum in the elliptic restricted three body problem: combining two scattering maps
The goal of the talk is to show the existence of global instability in the elliptic restricted three body problem. In this model, we find orbits whose angular momentum changes between two a priori stablished values. The main tool is to combine two different scattering maps associated to the normally parabolic manifold of infinity to build trajectories whose angular momentum increases. This is a joint work with A. Delshams, V. Kaloshin and A. de la Rosa.

Nicola Soave (Università di Milano Bicocca)
Symbolic dynamics: from the N-centre to the $(N+1)$-body problem
In a previous paper with S. Terracini [20], we proved the existence of periodic solutions with negative energies for the planar N-centre problem. That result is based upon a broken geodesics method, which, roughly speaking, consists in gluing together different arcs of solutions of some fixed ends problems. In this talk we discuss the generalization of the method on an ideal (i.e. different from the physical one) restricted $(N+1)$-body problem, with $N \geq 3$. We prove the existence of infinitely many collision-free periodic solutions with negative and small *Jacobi constant* and small values of the angular velocity, for any initial configuration of the centres. We will introduce a Maupertuis' type variational principle; major difficulties arise from the fact that, contrary to the classical Jacobi length, the related functional does not come from a Riemaniann structure but from a Finslerian one. Our existence result allows us to characterize the associated dynamical system with a symbolic dynamics, where the symbols are given partitions of the centres in two non-empty sets. Further developments towards an application to the real restricted $(N+1)$-body problem are discussed.

Alfonso Sorrentino (Università di Roma Tre, Sorrentino@mat.uniroma3.it)
Symplectic and variational methods for the study of invariant Lagrangian graphs.
In this talk I would like to describe some properties of Hamiltonian and Lagrangian systems, with particular attention to the relation between their *action-minimizing* properties, their *symplectic* nature and their *dynamics*. More specifically, I shall illustrate what kind of information the *principle of least Lagrangian action* conveys into the study of the integrability of these systems, and, more generally, how these information relate to the existence or to the non–existence of invariant Lagrangian graphs. These very interesting (and difficult) questions can be tackled from different perspectives.

I - REGULARITY OF THE MINIMAL AVERAGE LAGRANGIAN ACTION.
In the study of Tonelli Lagrangian and Hamiltonian systems, a central role in understanding the dynamical and topological properties of the action-minimizing sets is played by the so-called *Mather's average action* (sometimes referred to as *β-function* or *effective Lagrangian*), with particular attention to its differentiability and non-differentiability properties. Roughly speaking, this is a convex superlinear function on the first homology group of the base manifold, which represents the minimal action of invariant probability measures within a prescribed *homology class*, or *rotation vector*. Except for the trivial cases $\dim H_1(M;\mathbb{R}) = 0$ or 1, understanding whether or not this function is differentiable, or even smoother, and what are the implications of its regularity to the dynamics of the system is a formidable problem, which is still far from being understood. In particular: if $\dim H_1(M;\mathbb{R}) \geq 2$, does the regularity of β imply the integrability of the system? Or more generally, is the existence of an invariant Lagrangian graph detected by some regularity property of this function?
In a joint work with Daniel Massart [17], we address the above problem and provide positive answers to both questions in the case of Tonelli Lagrangians on closed surfaces, not necessarily orientable (in this latter case, one considers the lifted Lagrangian to the orientable double cover).

II - WEAK LIOUVILLE–ARNOL′D THEOREM.

It is natural to expect that "sufficiently" symmetric systems ought to possess an abundance of invariant Lagrangian graphs. This is indeed the content of a very classical result in the study of Hamiltonian systems, *Liouville–Arnol′d Theorem*, which is is concerned with the integrability of a Hamiltonian system with as many *independent* "symmetries" as its degrees of freedom, with the extra assumption that they are in *involution*.

A natural question is the following: is it possible to weaken the assumptions in Liouville–Arnol′d theorem? In particular: what happens when the involution hypothesis on the integrals of motion is dropped? Clearly, the involution hypothesis is fundamental in deducing the non-trivial fact that these invariant sets are Lagrangian and that the motion on them is conjugate to a rigid rotation. This is not just a sufficient condition, it is also somehow necessary: the Lagrangianeity of these submanifolds, in fact, is essentially equivalent to the involution hypothesis. Hence, it seems almost hopeless to deduce interesting results without assuming it. However, in the case of Tonelli Hamiltonians it turns out to be possible.

In [21], I introduced the definition of *Weak Integrability*. A Hamiltonian $H \in C^2(T^*M)$ is said to be weakly integrable if there exists a C^2 map $F : T^*M^n \longrightarrow \mathbb{R}^n$ whose singular set is nowhere dense, and such that F Poisson-commutes with H (i.e., each component is an integral of motion for H). Observe that there exist examples of Hamiltonian systems that are weakly integrable but not Liouville Integrable. For example, let G be a compact semi-simple Lie group of rank at least 2; in any neighbourhood of the bi-invariant metrics, there are left-invariant metrics with positive topological entropy that are not completely integrable. Nevertheless, these metrics are weakly integrable.

In [21] and in a subsequent joint work with Leo Butler [3], we proved a version of Liouville–Arnol′d Theorem for weakly integrable Tonelli Hamiltonians. This theorem shows the existence of a family of smooth invariant Lagrangian graphs $\{\Lambda_{c'}\}_{c'\in\mathcal{O}}$ – where $\mathcal{O} \subset H^1(M,\mathbb{R})$ and c' represents the cohomology class of $\Lambda_{c'}$ – which form a lamination of the space. These Lagrangian graphs admit the structure of a smooth $\mathbb{T}^d$-bundle over a base B^{n-d} that is parallelisable, for some $d > 0$, and the motion on them is recurrent; in particular, orbits on each of them are conjugate by a smooth diffeomorphism isotopic to the identity.
Moreover – amongst other topological properties of the underlying manifold – we prove that if $\dim H^1(M;\mathbb{R}) \geq \dim M$, then $\dim H^1(M;\mathbb{R}) = \dim M$ and M is diffeomorphic to $\mathbb{T}^n$; it follows that H is integrable in the sense of Liouville and therefore the integrals of motion are in involution. In other words, if $\dim H^1(M;\mathbb{R}) \geq \dim M$ then the notion of weak integrability is equivalent to the classical notion of Liouville integrability.

Jinxin Xue (University of Maryland, jinxinxue@gmail.com)
Noncollision singularities in a simplified four-body problem
In this work we study a model of simplified four-body problem called planar two-center-two-body problem. In the plane, we have two fixed centers $Q_1 = (-\chi, 0), Q_2 = (0, 0)$ of masses 1, and two moving bodies Q_3 and Q_4 of masses $\mu \ll 1$. They interact via Newtonian potential. Q_3 is captured by Q_2, and Q_4 travels back and forth between two centers. Based on a model of Gerver, we prove that there is a Cantor set of initial conditions which lead to solutions of the Hamiltonian system whose velocities are accelerated to infinity

within finite time avoiding all early collisions. We consider this model as a simplified model for the planar four-body problem case of the Painlevé conjecture. This is a joint work with Dmitry Dolgopyat.

4 Scientific Progress Made

The above selection of reports, together with other seminars which have been recorded and are available on the web site of the conference, testifies a series of impressive progresses in the subject of the N–body problem from many different points of view ranging from variational approaches to perturbative techniques, from weak KAM theory to geometric Arnold diffusion, from topological approaches to symbolic dynamical constructions. Several informal sessions, holding discussions at various locations, have contributed to significant progress in the sharing of ideas and perspectives among participants. Moeckel and Montgomery's approach to the global regularization of the 3-body problem, with the remarkable algebraic-geometric extension to the 4-body problem, has brought new insight to the variational approach; minimizing suitable action and geometric functionals on a global regularized branched cover of the configuration space (for parabolic, elliptic or hyperbolic energy levels) is a promising new approach. Another very interesting approach that has stimulated discussions and attempts to connect different branches in the N-body problem community has been Long's use of index theory for the stability of elliptic Lagrangian solutions. The idea of using symplectic matrix paths has been analyzed and discussed in several occasions during the workshop, and will be a definite ingredient in the progress in the field, for the N-center and the N-body problems. For example, among the progresses it was proved that for weak singular Lagrangian system, it is possible to obtain a (symplectic) decomposition of the phase space into invariant subspaces, in order to simplify the computation of the spectrum of the variational matrix. In particular an analogous of the Meyer-Schmidt decomposition holds. In the same line, a linear instability result was presented in terms of a suitable Brouwer degree associated to a central configuration, by using a trace-type formula for analytic paths of Hermitian matrices.

5 Outcome of the Meeting

Having brought together most of the main specialists on the N–body problem together with leaders of different areas in dynamical systems (Variational Methods, Mather Theory, Hamiltonian dynamics, Ergodic Theory) has produced a stimulating atmosphere where many new potential interactions and collaborations have been started (for example, Knauf–Montgomery, Guardia–Pinzari on Arnold diffusion for the planetary N–body problem, and many others). Also, various cross–invitations have been made (e.g., Shibayama–Chierchia) and several mathematical problems have been solved (compare, e.g., the extended abstract by Montgomery). The groups of people working on the linear stability for singular Lagrangian systems (e.g., Y. Long, S. Sun, G. Pinzari, D.L. Ferrario, S. Terracini, A. Portaluri, V. Barutello, and others), after the seminars and informal discussion sessions, which agreeably found new directions and collaborations, started planning to attack an extended array of cross-field problems, among which we quote the following conjectures/ideas: 1) that very probably it should holds that to each central configuration with odd Morse index or odd nullity it corresponds an unstable relative equilibrium both for the logarithmic potential as well as for the α-homogeneous potential, with $\alpha \in (-2, 0)$. 2) It should be possible to give some insight to the Moeckel conjectures about the linear stability of relative equilibria with a dominant mass by using some symplectic techniques. 3) Once linear stability is established, the KAM stability of relative equilibria should be investigated.

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