The evolution of the orbit distance in the double averaged restricted 3-body problem with crossing singularities

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New perspectives on the N-body problem

Banff centre, Canada

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joining works in collaboration with:

A. Milani, C. Tardioli, G. Tommei

[1] G. and Milani, 1998: *'Averaging on Earth-crossing orbits'*, Cel. Mech. Dyn. Ast., **71/2**, 109–136

[2] G., 2002: *'On the stationary points of the squared distance between two ellipses with a common focus'*, SIAM Journ. Sci. Comp., **24/1**, 61–80

[3] G. and Tommei, 2007: *'On the uncertainty of the minimal distance between two confocal Keplerian orbits'*, DCDS-B, **7/4**, 755–778

[4] G. and Tardioli, 2012: *'The evolution of the orbit distance in the double averaged restricted 3-body problem with crossing singularities'*, submitted

3–body problem: Sun, Earth, asteroid

restricted problem: the asteroid does not influence the motion of the two larger bodies.

equations of motion of the asteroid:

$$\ddot{y} = -G\left[m_\odot \frac{(y - y_\odot(t))}{|y - y_\odot(t)|^3} + m_\oplus \frac{(y - y_\oplus(t))}{|y - y_\oplus(t)|^3}\right]$$

- y is the unknown position of the asteroid;
- $y_\odot(t), y_\oplus(t)$ are known functions of time, solutions of the two-body problem Sun-Earth.

The restricted 3–body problem

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The restricted 3–body problem

In heliocentric coordinates

$$\ddot{x} = -k^2 \left[\frac{x}{|x|^3} + \mu \left(\frac{(x - x')}{|x - x'|^3} - \frac{x'}{|x'|^3} \right) \right]$$

- $x = y - y_\odot$, $x' = y_\oplus - y_\odot$;
- $k^2 = Gm_\odot$, $\mu = \frac{m_\oplus}{m_\odot}$ is a small parameter;
- $-k^2\mu\frac{(x-x')}{|x-x'|^3}$ is the direct perturbation of the planet on the asteroid;
- $k^2\mu\frac{x'}{|x'|^3}$ is the indirect perturbation, due to the interaction Sun-planet.

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Canonical formulation of the problem

Use Delaunay's variables $\mathcal{Y} = (L, G, Z, \ell, g, z)$ for the motion of the asteroid:

$$\begin{cases} L = k\sqrt{a} \\ G = L\sqrt{1-e^2} \\ Z = G\cos I \end{cases} \qquad \begin{cases} \ell = n(t-t_0) \\ g = \omega \\ z = \Omega \end{cases}$$

These are canonical variables, representing the osculating orbit, solution of the 2-body problem Sun-asteroid.

Denote by $\mathcal{Y}' = (L', G', Z', \ell', g', z')$ Delaunay's variables for the planet.

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Canonical formulation of the problem

Hamilton's equations are

$$\dot{\mathcal{Y}} = \mathbb{J}_3 \, \nabla_{\mathcal{Y}} H \,,$$

where

$$\mathbb{J}_3 = \begin{bmatrix} \mathcal{O}_3 & -\mathcal{I}_3 \\ \mathcal{I}_3 & \mathcal{O}_3 \end{bmatrix} .$$

$H = H_0 - R$ is the Hamiltonian, $H_0 = -\frac{k^2}{2L^2}$ (unperturbed part),

$$R = k^2\mu \left(\frac{1}{|\mathcal{X} - \mathcal{X}'|} - \frac{\mathcal{X} \cdot \mathcal{X}'}{|\mathcal{X}'|^3} \right)$$ (perturbing function).

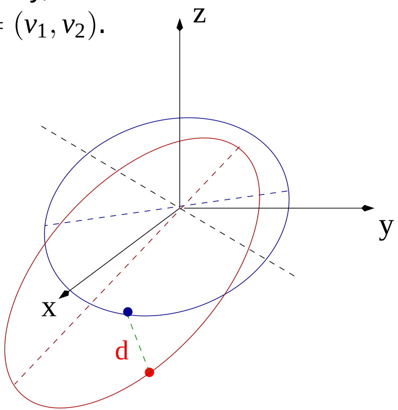
Here $\mathcal{X}, \mathcal{X}'$ denote x, x' as functions of $\mathcal{Y}$, $\mathcal{Y}'$.

Let (E_j, v_j), $j = 1, 2$ be the orbital elements of two celestial bodies on confocal Keplerian orbits:
E_j represents the trajectory of a body,
v_j is a parameter along it. Set $V = (v_1, v_2)$.

For a given two-orbit configuration $\mathcal{E} = (E_1, E_2)$, we introduce the Keplerian distance function

$$\mathbb{T}^2 \ni V \mapsto d(\mathcal{E}, V) = |\mathcal{X}_1 - \mathcal{X}_2|$$

We are interested in the local minimum points of d.



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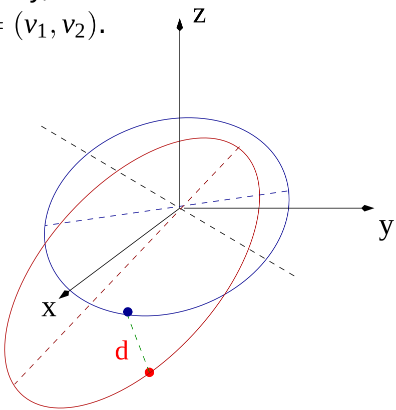

Is there still something that we do not know about distance of points on conic sections?

ἐθεώρουν σε σπεύδοντα μετασχεῖν
τῶν πεπραγμένων ἡμῖν κωνικῶν (1)
(Apollonius of Perga, *Conics*, Book I)

(1) I observed you were quite eager to be kept informed of the work I was doing in conics.

Some remarks on the critical points of d^2

- The local minimum points of d can be found by computing all the critical points of d^2.
- Apart from the case of two concentric coplanar circles, or two overlapping ellipses, d^2 has finitely many critical points.
- There exist configurations with 12 critical points, and 4 local minima of d^2.

  This is thought to be the maximum possible, but a proof is not known yet, see also Albouy, Cabral, Santos (2012).

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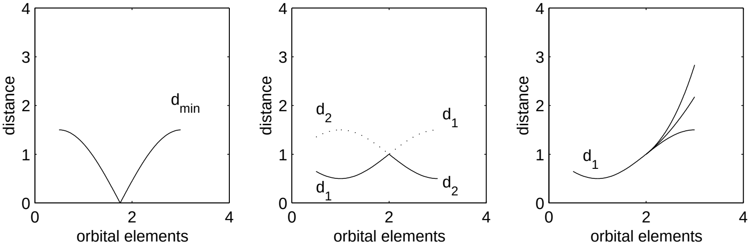
The orbit distance

Let $V_h = V_h(\mathcal{E})$ be a local minimum point of $V \mapsto d^2(\mathcal{E}, V)$. Consider the maps

$$\mathcal{E} \mapsto d_h(\mathcal{E}) = d(\mathcal{E}, V_h)\ ,$$
$$\mathcal{E} \mapsto d_{min}(\mathcal{E}) = \min_h d_h(\mathcal{E})\ .$$

The map $\mathcal{E} \mapsto d_{min}(\mathcal{E})$ gives the orbit distance.

Singularities of d_h and d_{min}



(i) d_h and d_{min} are not differentiable where they vanish;

(ii) two local minima can exchange their role as absolute minimum thus d_{min} loses its regularity without vanishing;

(iii) when a bifurcation occurs the definition of the maps d_h may become ambiguous after the bifurcation point.

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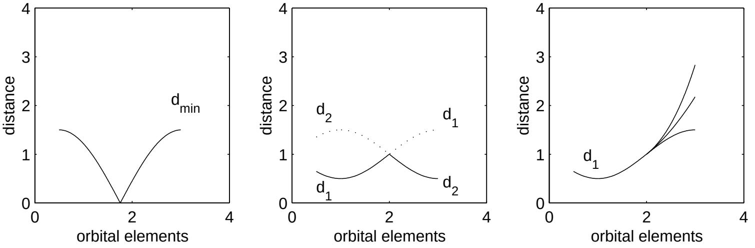

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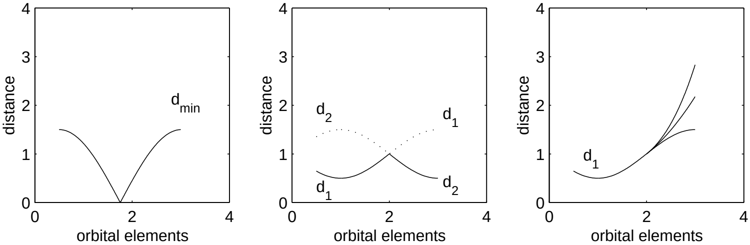

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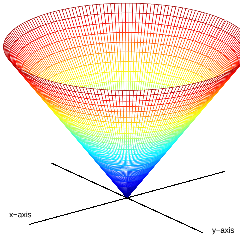
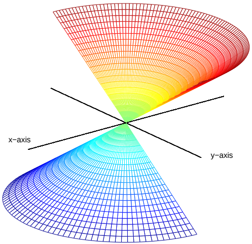
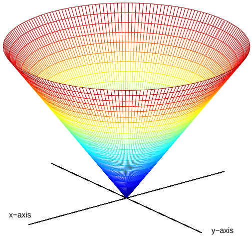


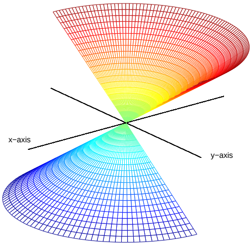


Model problem:

$$f(x,y) = \sqrt{x^2+y^2} \qquad \tilde{f}(x,y) = \begin{cases} -f(x,y) & \text{for } x>0 \\ f(x,y) & \text{for } x<0 \end{cases}$$

Can we smooth the maps $d_h(\mathcal{E})$, $d_{min}(\mathcal{E})$
through a change of sign?





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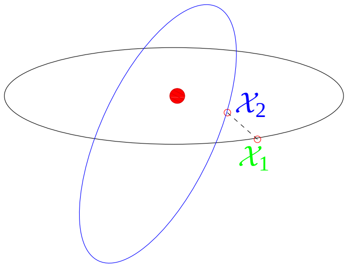
Local smoothing of d_h at a crossing singularity



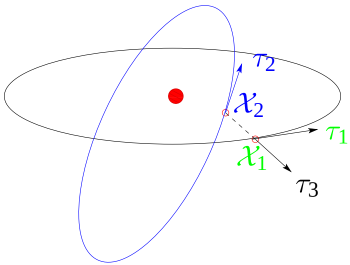
Smoothing d_h, the procedure for d_{min} is the same.

- Consider the points on the two orbits

$$\mathcal{X}_1^{(h)} = \mathcal{X}_1(E_1, v_1^{(h)})\,; \qquad \mathcal{X}_2^{(h)} = \mathcal{X}_2(E_2, v_2^{(h)})\,.$$

corresponding to the local minimum point
$V_h = (v_1^{(h)}, v_2^{(h)})$ of d^2;

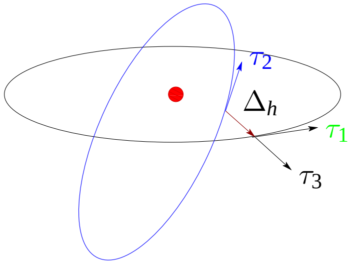
Local smoothing of d_h at a crossing singularity



- introduce the tangent vectors to the trajectories E_1, E_2 at these points:

$$\tau_1 = \frac{\partial \mathcal{X}_1}{\partial v_1}(E_1, v_1^{(h)})\,, \qquad \tau_2 = \frac{\partial \mathcal{X}_2}{\partial v_2}(E_2, v_2^{(h)})\,,$$

and their cross product $\tau_3 = \tau_1 \times \tau_2$;



- define also

$$\Delta = \mathcal{X}_1 - \mathcal{X}_2\,, \qquad \Delta_h = \mathcal{X}_1^{(h)} - \mathcal{X}_2^{(h)}\ .$$

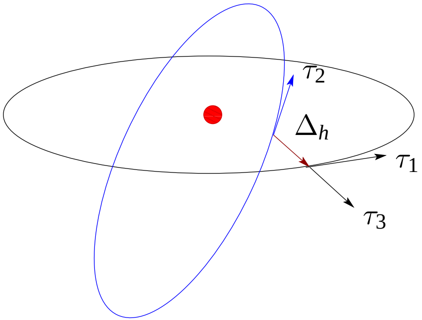
The vector Δ_h joins the points attaining a local minimum of d^2 and $|\Delta_h| = d_h$.

Note that $\Delta_h \times \tau_3 = 0$

Smoothing the crossing singularity

smoothing rule:

$$\tilde{d}_h = \text{sign}(\tau_3 \cdot \Delta_h) d_h$$



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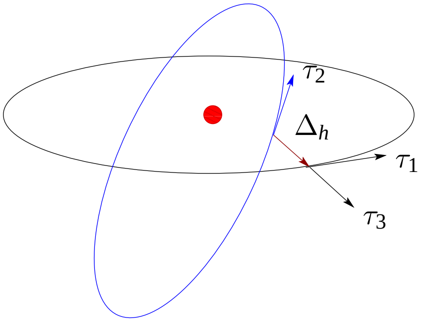

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The averaging method

The averaging principle is used to study the qualitative behavior of solutions of ODEs in perturbation theory, see Arnold, Kozlov, Neishtadt (1997).

unperturbed $\left\{ \begin{array}{l} \dot{\phi} = \omega(I) \\ \dot{I} = 0 \end{array} \right.$ $\qquad \phi \in \mathbb{T}^n, I \in \mathbb{R}^m$

perturbed $\left\{ \begin{array}{l} \dot{\phi} = \omega(I) + \epsilon f(\phi, I, \epsilon) \\ \dot{I} = \epsilon g(\phi, I, \epsilon) \end{array} \right.$

averaged $\dot{J} = \epsilon G(J)\,, \quad G(J) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} g(\phi, J, 0)\, d\phi_1 \ldots d\phi_n$

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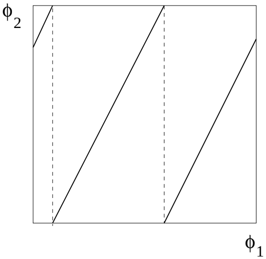
Averaging over 2 angular variables

Using the averaged equations corresponds to substituting the time average with the space average.

Case of 2 angles: a problem occurs if there are resonant relations of low order between the motions $\phi_1(t), \phi_2(t)$, i.e. if $k_1\dot{\phi}_1 + k_2\dot{\phi}_2 = 0$, with k_1, k_2 small integers.



Averaged Hamilton's equations:

$$\dot{Y} = -\mathbb{J}_2 \,\overline{\nabla_Y R}\,, \tag{1}$$

with $Y = (G, Z, g, z)$. We averaged over the fast angles ℓ, ℓ'.
If no orbit crossing occurs, (1) are equal to

$$\dot{Y} = -\mathbb{J}_2 \,\nabla_Y \overline{R} \tag{2}$$

with

$$\overline{R} = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} R\, d\ell\, d\ell' = \frac{\mu k^2}{(2\pi)^2}\int_{\mathbb{T}^2} \frac{1}{|\mathcal{X} - \mathcal{X}'|}\, d\ell\, d\ell'$$

The average of the indirect term of R is zero.

Crossing singularities

If there is an orbit crossing, then averaging on the fast angles ℓ, ℓ' produces a singularity in the averaged equations:

we take into account every possible position on the orbits, thus also the collision configurations.

$$\overline{R} = \frac{\mu k^2}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{1}{|\mathcal{X} - \mathcal{X}'|} \, d\ell \, d\ell'$$

and

$$\left| \mathcal{X}(E_1, v_1^{(h)}) - \mathcal{X}'(E_2, v_2^{(h)}) \right| = 0 \; .$$

Near-Earth asteroids and crossing orbits

(433) Eros: the first near-Earth asteroid (NEA, with $q = a(1-e) \leq 1.3$ AU), discovered in 1898; it crosses the trajectory of Mars.

from NEAR mission (NASA)

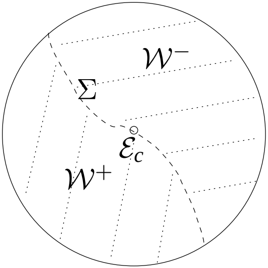
Today (January 15, 2013) we know about 9500 NEAs: several of them cross the orbit of the Earth during their evolution.

Main result

Let $\mathcal{E}_c$ be a non–degenerate crossing configuration for d_h, with only one crossing point.
Given a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$, we set

$$\mathcal{W}^+ = \mathcal{W} \cap \{\tilde{d}_h > 0\}\,,$$

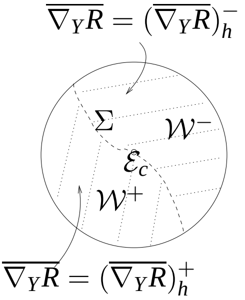
$$\mathcal{W}^- = \mathcal{W} \cap \{\tilde{d}_h < 0\}\,.$$



The averaged vector field $\overline{\nabla_Y R}$ is not defined on $\Sigma = \{d_H = 0\}$.

Main result

Theorem: The averaged vector field $\overline{\nabla_Y R}$ can be extended to two Lipschitz–continuous vector fields $(\overline{\nabla_Y R})_h^{\pm}$ on a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$. These extended vector fields, restricted to $\mathcal{W}^+$, $\mathcal{W}^-$ respectively, correspond to $\overline{\nabla_Y R}$.



Moreover the following relations hold:

$$\mathrm{Diff}_h\left(\frac{\overline{\partial R}}{\partial y_k}\right) \stackrel{def}{=} \left(\frac{\overline{\partial R}}{\partial y_k}\right)_h^- - \left(\frac{\overline{\partial R}}{\partial y_k}\right)_h^+ =$$

$$= \frac{\mu k^2}{\pi}\left[\frac{\partial}{\partial y_k}\left(\frac{1}{\sqrt{\det(\mathcal{A}_h)}}\right)\tilde{d}_h + \frac{1}{\sqrt{\det(\mathcal{A}_h)}}\frac{\partial \tilde{d}_h}{\partial y_k}\right],$$

where y_k is a component of Delaunay's elements Y, and

$$\mathcal{A}_h(\mathcal{E}) = \frac{1}{2}\frac{\partial^2 d^2}{\partial V^2}(\mathcal{E}, V_h(\mathcal{E}))\ .$$

Extraction of the singularity

We write

$$d^2(\mathcal{E},V) = d_h^2(\mathcal{E}) + (V - V_h)\cdot\mathcal{A}_h(\mathcal{E})(V - V_h) + \mathcal{R}_3^{(h)}(\mathcal{E},V)\,,$$

where

i) $2\mathcal{A}_h(\mathcal{E})$ is the Hessian matrix of $V \mapsto d^2(\mathcal{E},V)$ in V_h;

ii) $\mathcal{R}_3^{(h)}$ is Taylor's remainder in the integral form.

Introduce the approximated distance

$$\delta_h = \sqrt{d_h^2 + (V - V_h)\cdot\mathcal{A}_h(V - V_h)}\ .$$

Consider the following decomposition:

$$\mathcal{W}\setminus\Sigma \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k}\frac{1}{d}\,d\ell d\ell' = \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k}\Big(\frac{1}{d}-\frac{1}{\delta_h}\Big)\,d\ell d\ell' + \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k}\frac{1}{\delta_h}\,d\ell d\ell'$$

We prove that:

i) the two maps $\mathcal{W}^{\pm} \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k}\frac{1}{\delta_h}\,d\ell d\ell'$ admits two different analytic extensions to $\mathcal{W}$;

ii) the map $\mathcal{W}\setminus\Sigma \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k}\Big(\frac{1}{d}-\frac{1}{\delta_h}\Big)\,d\ell d\ell'$ admits a Lipschitz–continuous extension to $\mathcal{W}$.

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ii) the map $\mathcal{W}\setminus\Sigma \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k}\Big(\frac{1}{d}-\frac{1}{\delta_h}\Big)\,d\ell d\ell'$ admits a Lipschitz–continuous extension to $\mathcal{W}$.

Extraction of the singularity

Consider the following decomposition:

$$\mathcal{W}\setminus\Sigma \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k}\frac{1}{d}\, d\ell d\ell' = \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k}\Big(\frac{1}{d}-\frac{1}{\delta_h}\Big)\, d\ell d\ell' + \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k}\frac{1}{\delta_h}\, d\ell d\ell'$$

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idea of the proof of i)

$$\mathcal{W} \setminus \Sigma \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{\delta_h} \, d\ell \, d\ell' = \frac{\partial}{\partial y_k} \int_{\mathbb{T}^2} \frac{1}{\delta_h} \, d\ell \, d\ell'$$

Set

$$\mathcal{D} = \{V \in \mathbb{T}^2 : (V - V_h) \cdot \mathcal{A}_h (V - V_h) \leq r^2\}.$$

We have

$$\int_{\mathcal{D}} \frac{1}{\delta_h} \, d\ell \, d\ell' = \frac{2\pi}{\sqrt{\det \mathcal{A}_h}} \left(\sqrt{d_h^2 + r^2} - d_h \right) .$$

We obtain

$$\int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{\delta_h} \, d\ell \, d\ell' = \frac{\partial}{\partial y_k} \left(\frac{2\pi}{\sqrt{\det \mathcal{A}_h}} \right) \left(\sqrt{d_h^2 + r^2} - d_h \right) +$$
$$+ \frac{2\pi}{\sqrt{\det \mathcal{A}_h}} \frac{d_h}{\sqrt{d_h^2 + r^2}} \frac{\partial d_h}{\partial y_k} - \frac{2\pi}{\sqrt{\det \mathcal{A}_h}} \frac{\partial d_h}{\partial y_k} + \frac{\partial}{\partial y_k} \int_{\mathbb{T}^2 \setminus \mathcal{D}} \frac{1}{\delta_h} \, d\ell \, d\ell'$$

so that the formula

$$\left(\int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{\delta_h} \, d\ell \, d\ell' \right)_h^{\pm} = \frac{\partial}{\partial y_k} \left(\frac{2\pi}{\sqrt{\det \mathcal{A}_h}} \right) \left(\sqrt{d_h^2 + r^2} \mp \tilde{d}_h \right) +$$
$$+ \frac{2\pi}{\sqrt{\det \mathcal{A}_h}} \frac{\tilde{d}_h}{\sqrt{d_h^2 + r^2}} \frac{\partial \tilde{d}_h}{\partial y_k} \mp \frac{2\pi}{\sqrt{\det \mathcal{A}_h}} \frac{\partial \tilde{d}_h}{\partial y_k} + \frac{\partial}{\partial y_k} \int_{\mathbb{T}^2 \setminus \mathcal{D}} \frac{1}{\delta_h} \, d\ell \, d\ell'$$

defines analytic extensions of $\mathcal{W}^{\pm} \ni \mathcal{E} \mapsto \int_{\mathbb{T}^2} \frac{\partial}{\partial y_k} \frac{1}{\delta_h} \, d\ell \, d\ell'$ to $\mathcal{W}$.

Generalized solutions

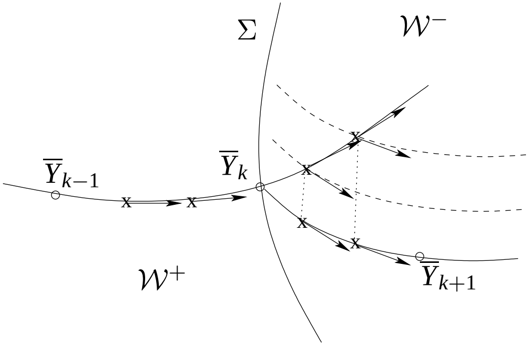
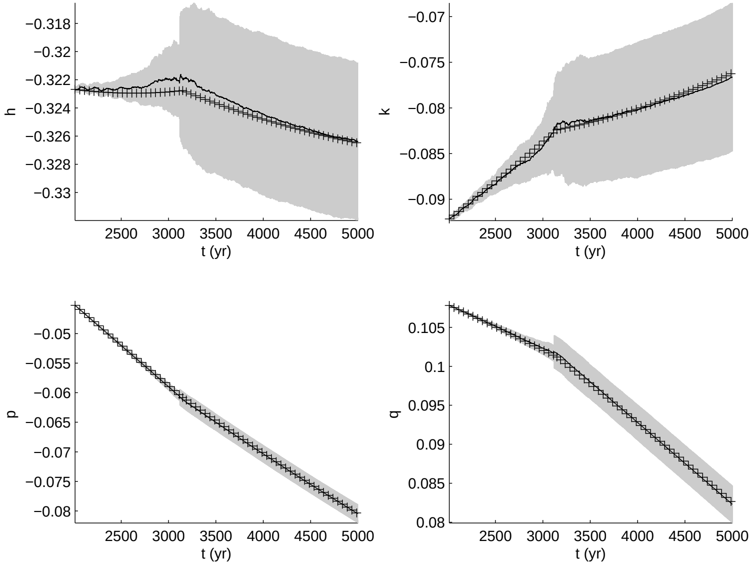


Figure: Runge-Kutta-Gauss method and continuation of the solutions of equations (1) beyond the singularity.

The averaged solutions are piecewise–smooth

Comparison of solutions for (1620) Geographos



Secular evolution of the orbit distance

Define the secular evolution of the minimal distances

$$\overline{d}_h(t) = \tilde{d}_h(\overline{\mathcal{E}}(t))\,, \qquad \overline{d}_{min}(t) = \tilde{d}_{min}(\overline{\mathcal{E}}(t))$$

in an open interval containing a crossing time t_c.

Proposition: Assume t_c is a crossing time and $\mathcal{E}_c = \overline{\mathcal{E}}(t_c)$ is a non-degenerate crossing configuration with only one crossing point, i.e. $d_h(\mathcal{E}_c) = 0$. Then there exists an interval (t_a, t_b), $t_a < t_c < t_b$ such that $\overline{d}_h \in C^1((t_a, t_b); \mathbb{R})$.

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Secular evolution of the orbit distance

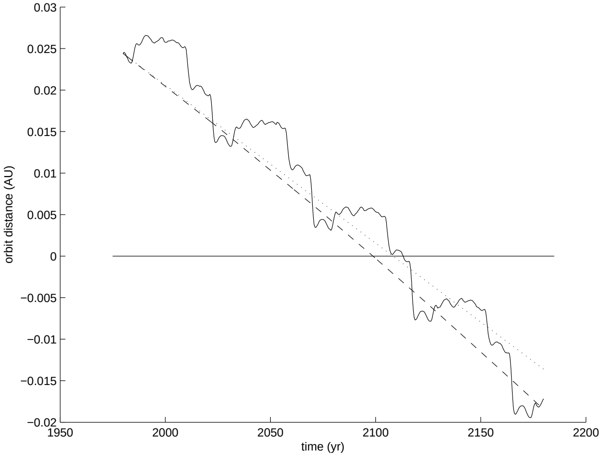
Proof:

$$\lim_{t\to t_c^+} \dot{\overline{d}}_h(t) - \lim_{t\to t_c^-} \dot{\overline{d}}_h(t) = \mathrm{Diff}_h\left(\overline{\nabla_Y R}\right) \cdot \mathbb{J}_2 \nabla_Y \tilde{d}_h\Big|_{\mathcal{E}=\mathcal{E}_c}$$

$$= \frac{\mu k^2}{\pi\sqrt{\det \mathcal{A}_h}} \{\tilde{d}_h, \tilde{d}_h\}_Y\Big|_{\mathcal{E}=\mathcal{E}_c} = 0\,,$$

The secular evolution of $\tilde{d}_{min}$ is more regular than that of the orbital elements in a neighborhood of a planet crossing time.

Evolution of the orbit distance for 1979 XB



Conclusions and future work

- We can compute the secular evolution of planet crossing asteroids, by averaging over the fast angles: the solutions are piecewise–smooth;
- the orbit distance along the averaged evolution is more regular than the orbital elements.

Open questions

- Can we prove that the averaged solutions are good approximation of the solutions of the full equations?
- What can we do in case of mean motion resonances?

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