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Symbolic dynamics: from the N-centre to the $(N+1)$-body problem, a preliminary study

New perspectives on the N-body problem

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Joint work with S. Terracini.

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The planar N-centre problem

We consider N heavy bodies (the centres) with masses $m_1, \ldots, m_N > 0$ and positions $c_1, \ldots, c_N \in \mathbb{R}^2$. We study the motion of a moving test particle which interacts with the centres trough a Newtonian-like potential

$$V(x) = \sum_{j=1}^{N} \frac{m_j}{\alpha |x - c_j|^\alpha}, \qquad x \in \mathbb{R}^2 \setminus \{c_1, \ldots, c_N\},$$

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homogeneous of degree $-\alpha < 0$, $\alpha \in [1, 2)$. Motion equation:

$$\ddot{x}(t) = -\sum_{j=1}^{N} \frac{m_j}{\left|x(t) - c_j\right|^{\alpha+2}} (x(t) - c_j) = \nabla V(x(t)), \tag{1}$$

$\Rightarrow$ Energy first integral: if $x : I \subset \mathbb{R} \to \mathbb{R}^2$ is a solution then

$$\frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = h \qquad \forall t \in I.$$

Some known results ($\alpha = 1$, planar problem)

- $N = 1, 2$: the system is integrable.
- [S. V. Bolotin, 1984] Non existence of analytic first integrals independent by the energy in case $N \geq 3$ and $h > 0$.
- [M. Klein, A. Knauf, 1992] The dynamical system associated to the N-centre problem on the non negative energy level has symbolic dynamics and positive entropy. Description of scattering phenomena.
- [A. Knauf, I. A. Taimanov, 2003] Spatial case: non existence of analytic first integrals independent by the energy in case $N \geq 3$ and h greater then a certain threshold.

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- [S. V. Bolotin, Negrini, 2003] Chaotic motion for the 3-centre problem with $h < 0$ and h small, with a centre far away from the others .
- [L. Dimare, 2010] Chaotic motion for the 3-centre problem with $h < 0$ and h small, with a centre having small mass with respect to the others.

Positive energy vs. negative energy

Periodic solutions of the N-centre problem with a fixed energy h are closed geodesics in the Riemannian manifold

$$\{V(x) > -h\} \quad \text{Hill's region}$$

endowed with metric

$$g_{ij}(x) := (V(x) + h)\delta_{ij} \qquad \text{Jacobi metric.}$$

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Remark

If $h \geq 0$ the set $\{V(x) > -h\}$ is the punctured plane. To find closed geodesics, one has to "regularize the manifold" and then to use global geometric techniques based on the fundamental group of the regularized manifold [M. Klein, A. Knauf, 1992].
If $h < 0$ the set $\{V(x) > -h\}$ is bounded and the metric vanishes on the boundary. In particular, any arc of the boundary has null length so that it is a minimal geodesic for the related fixed ends problem.
The use of global geometric techniques is impossible.

Assumptions

Energy $h < 0$, $|h|$ sufficiently small, $N \geq 3$, $c_1, \ldots, c_N \in \mathbb{R}^2$, $m_1, \ldots, m_N > 0$, $\alpha \in [1, 2)$.

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Possible partitions of the centres in two non-empty sets: there are $2^{N-1} - 1$ partitions.

$$\mathcal{P} := \{P_j : j = 1, \ldots, 2^{N-1} - 1\} \qquad \text{Labels.}$$

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Possible partitions of the centres in two non-empty sets: there are $2^{N-1} - 1$ partitions.

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Partitions which isolate one centre:

$$P_j := \{\{c_j\}, \{c_1, \ldots, c_N\} \setminus \{c_j\}\} \qquad j = 1, \ldots, N.$$
$$\mathcal{P}_1 := \{P_j \in \mathcal{P} : j = 1, \ldots, N\} \subset \mathcal{P}.$$

Existence of periodic solutions

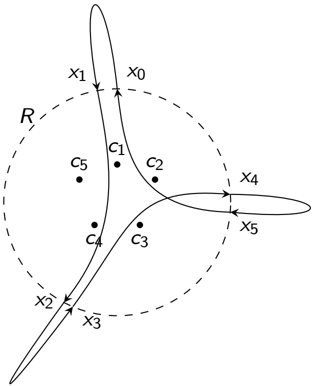
Theorem [N.S. and S.Terracini,DCDS-A 2012]

Let $\alpha \in [1,2)$, $N \geq 3$, $c_1, \dots, c_N \in \mathbb{R}^2$, $m_1, \dots, m_N \in \mathbb{R}^+$.
There exists $\bar{h} > 0$ such that for every $h \in (-\bar{h}, 0)$, $n \in \mathbb{N}$ and $(P_{j_1}, \dots, P_{j_n}) \in \mathcal{P}^n$ there exists a periodic solution $x_{((P_{j_1},\dots,P_{j_n}),h)}$ of the N-centre problem (1) with energy h, which depends on $(P_{j_1}, \dots, P_{j_n})$ in the following way.
There are $R, \delta > 0$ (independent on $(P_{j_1}, \dots, P_{j_n})$) such that $x_{((P_{j_1},\dots,P_{j_n}),h)}$ crosses $2n$ times within one period the circle $\partial B_R(0)$, at times $(t_k)_{k=0,\dots,2n-1}$ and

- in (t_{2k}, t_{2k+1}) the solution stays outside $B_R(0)$ and

$$|x_{((P_{j_1},\dots,P_{j_n}),h)}(t_{2k}) - x_{((P_{j_1},\dots,P_{j_n}),h)}(t_{2k+1})| < \delta.$$

- in (t_{2k+1}, t_{2k+2}) the solution lies inside $B_R(0)$, and, if it does not collide against any centre, then it separates them according to the partition P_{j_k}.


x_1
x_0
R
c_1
c_5
c_2
x_4
x_5
c_4
c_3
x_2
x_3

For every $\alpha \in [1,2)$ and $N \geq 3$ it is possible to give sufficient conditions on the sequence $(P_{j_1}, \ldots, P_{j_n}) \in \mathcal{P}^n$ in order to get collision-free solutions:

For every $\alpha \in [1,2)$ and $N \geq 3$ it is possible to give sufficient conditions on the sequence $(P_{j_1}, \ldots, P_{j_n}) \in \mathcal{P}^n$ in order to get collision-free solutions:

- $\alpha = 1$ and $N \geq 4$: $P_{j_k} \in \mathcal{P} \setminus \mathcal{P}_1$ for every k.
- $\alpha = 1$ and $N = 3$: we require $n = 4m$ and

$$(P_{j_{k+1}}, \ldots, P_{j_{k+4}}) \in \{(P_1 P_1 P_2 P_3), (P_2 P_2 P_3 P_1)\}.$$

- $\alpha \in (1,2)$: no restrictions.

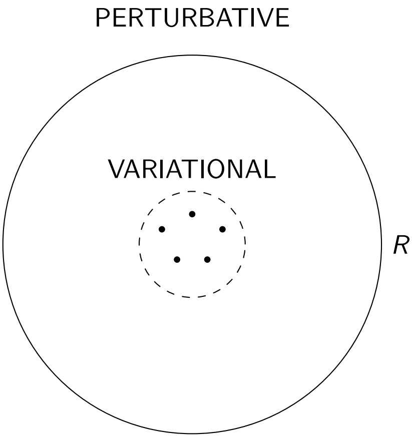
Sketch of the proof: step 1) Preliminaries

If $|h|$ is not too large, outside a ball of radius $R > \max\{|c_k| : k = 1, \dots, N\} > 0$ we can consider the problem as a small perturbation of the α-Kepler's problem:

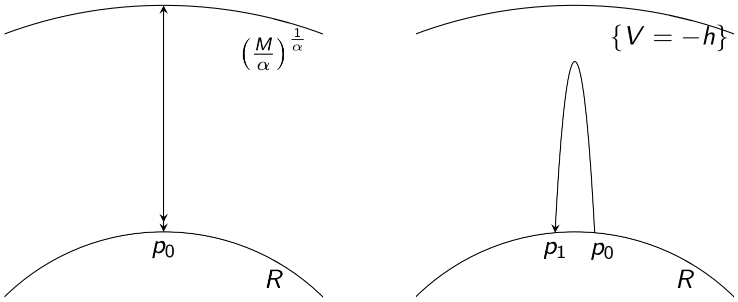
$$V(x) = \frac{M}{\alpha |x|^\alpha} + W_h(x) = V_0(x) + W_h(x)$$

with $M = \sum_{j=1}^N m_j$ and $W_h(x) = o(1)$ as $h \to 0$ in the $\mathcal{C}^1$ topology for $|x| \geq R$.

Plan



Step 2) Outer dynamics



Proposition (outer solutions)

There exist $\delta > 0$ and $h_1 > 0$ such that for every $h \in (-h_1, 0)$, for every $p_0, p_1 \in \partial B_R(0) : |p_1 - p_0| < 2\delta$ we find $T > 0$ and a unique solution $x_{\text{ext}}(\cdot\,; p_0, p_1; h)$ of

$$\begin{aligned} \ddot{x}(t) = \nabla V(x(t)) \qquad & t \in [0, T], \\ \frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = h \qquad & t \in [0, T], \\ |x(t)| > R \qquad & t \in (0, T), \\ x(0) = p_0, \quad & x(T) = p_1. \end{aligned}$$

Step 3) Inner dynamics

We are going to search solutions of

$$\begin{aligned} \ddot{x}(t) = \nabla V(x(t)) \qquad & t \in [0, T], \\ \frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = h \qquad & t \in [0, T], \\ |x(t)| < R \qquad & t \in (0, T), \\ x(0) = p_1, \quad & x(T) = p_2, \end{aligned} \tag{2}$$

with $p_1, p_2 \in \partial B_R(0)$, and $T > 0$ to be determined. We wish the solution to be collision-free and such that it divides the centres according with a prescribed partition.

We fix $p_1, p_2 \in \partial B_R(0)$, $[a,b] = [0,1]$.
For every $P_j \in \mathcal{P}$ we set

$$\begin{aligned} \widehat{K}_{P_j} = \widehat{K}_{P_j}^{p_1 p_2}([0,1]) := \{ & u \in H^1([0,1]) : u(0) = p_1, u(1) = p_2, \\ & |u(t)| \le R \ \forall t \in (0,1), \ u \text{ is collision-free}, \\ & u \text{ separates the centres according to the partition } P_j \} \end{aligned}$$

and

$$K_{P_j} = K_{P_j}^{p_1 p_2}([0,1]) := \overline{\widehat{K}_{P_j}}^{\sigma(H^1,(H^1)^*)}.$$

Theorem (inner solutions)

There exists $h_2 > 0$ such that for every $h \in (-h_2, 0)$, $p_1, p_2 \in \partial B_R(0)$ and $P_j \in \mathcal{P}$, there exist $T > 0$ and a solution $x_{P_j}(\cdot\,; p_1, p_2; h) \in K_{P_j}^{p_1 p_2}([0, T])$ of problem (2) which is a reparametrization of a local minimizer of the length functional

$$L_h(u) = \int_0^1 \sqrt{(V(x)+h)}|\dot{u}|$$

in $K_{P_j}^{p_1 p_2}([0,1])$.

Theorem (inner solutions)

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$$L_h(u) = \int_0^1 \sqrt{(V(x) + h)}|\dot{u}|$$

in $K_{P_j}^{p_1 p_2}([0, 1])$. Moreover, a minimizer is collision-free except in case $p_1 = p_2$ and $P_j \in \mathcal{P}_1$; in such a situation y_{P_j} can be a collision-free, or can be an ejection-collision solution with a unique collision in c_j.

Step 4) A finite dimensional reduction

We fix $h \in (-\min\{h_1, h_2\}, 0)$, $n \in \mathbb{N}$, and $(P_{k_1}, P_{k_2}, \ldots, P_{k_n}) \in \mathcal{P}^n$.
We define

$$D = \Big\{ (p_0, \ldots, p_{2n}) \in (\partial B_R(0))^{2n+1} : |p_{2j+1} - p_{2j}| \leq \delta \\ \text{for } j = 0, \ldots, n-1, \; p_{2n} = p_0 \Big\}.$$

We alternate outer and inner trajectories, according to the given sequence of partitions. Let $(p_0, \dots, p_{2n}) \in D$.

- For every $j \in \{0, \dots, n-1\}$, we take an outer arc and we pose:

$$x_{2j}(t) := x_{\text{ext}}(t; p_{2j}, p_{2j+1}; h)$$

  for $t \in [0, T_{2j}]$.

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$$x_{2j}(t) := x_{\text{ext}}(t; p_{2j}, p_{2j+1}; h)$$

  for $t \in [0, T_{2j}]$.

- On the other hand, for every $j = 0, \ldots, n-1$, we can find an inner arc associated with the partition $P_{k_{j+1}}$. We set

$$x_{2j+1}(t) := y_{P_{k_{j+1}}}(t; p_{2j+1}, p_{2j+2}; h),$$

  where $T_{2j+1} := T_{P_{k_{j+1}}}(p_{2j+1}, p_{2j+2}; h)$.

Let us set $\mathfrak{T}_k := \sum_{j=0}^{k} T_j$, $k = 0, \dots, 2n-1$. We define

$$\gamma^{((P_{j_1},\dots,P_{j_n}),h)}_{(p_0,\dots,p_{2n})}(s) := \begin{cases} x_0(s) & s \in [0, \mathfrak{T}_0] \\ x_1(s - \mathfrak{T}_0) & s \in [\mathfrak{T}_0, \mathfrak{T}_1] \\ \vdots & \\ x_{2n-2}\left(s - \mathfrak{T}_{2n-3}\right) & s \in \left[\mathfrak{T}_{2n-3}, \mathfrak{T}_{2n-2}\right] \\ x_{2n-1}\left(s - \mathfrak{T}_{2n-2}\right) & s \in \left[\mathfrak{T}_{2n-2}, \mathfrak{T}_{2n-1}\right]. \end{cases}$$

It is piecewise differentiable and $\mathfrak{T}_{2n-1}$-periodic; in general, it is not $\mathcal{C}^1$ in $\{0, \mathfrak{T}_0, \dots, \mathfrak{T}_{2n-1}\}$. If $\alpha = 1$ it is possible also that $\gamma^{((P_{j_1},\dots,P_{j_n}),h)}_{(p_0,\dots,p_{2n})}$ has a finite number of collisions.

We introduce $F = F_{((P_{k_1},\ldots,P_{k_n});h)} : D \to \mathbb{R}$

$$F(p_0,\ldots,p_{2n}) := \sum_{j=0}^{2n-1} \int_0^{T_j} \sqrt{(V(x_j)+h)}|\dot{x}_j|$$

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Proposition

There exists $(p_0,\ldots,p_{2n}) \in D$ which minimizes F.
There exists $\bar{h} > 0$ such that, if $h \in (-\bar{h}, 0)$, then the associated function $\gamma_{(p_0,\ldots,p_{2n})}^{((P_{j_1},\ldots,P_{j_n}),h)}$ is a periodic solution of the N-centre problem (1) with energy h. The value $\bar{h}$ depends neither on n, nor on $(P_{k_1},\ldots,P_{k_n}) \in \mathcal{P}^n$.

Remark

We found non-minimal closed geodesics with Morse index which can be arbitrarily large.
No hyperbolicity.

A perturbed problem

A perturbed problem

We assume that the centres are not fixed, but rotate according to the law $c_k(t) = \exp\{i\nu t\} c_k$

$$\Rightarrow \ddot{x}(t) = -\sum_{k=1}^{N} \frac{m_k}{|x(t) - e^{i\nu t} c_k|^{\alpha+2}} (x(t) - e^{i\nu t} c_k).$$

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Letting $x(t) = \exp\{i\nu t\} z(t)$ we have

$$\ddot{z}(t) + 2\nu i \dot{z}(t) = \nu^2 z(t) - \sum_{k=1}^{N} \frac{m_k}{|z(t) - c_k|^{\alpha+2}} (z(t) - c_k).$$

If $|\nu|$ is small, this problem can be seen as a perturbation of the N-centre problem.

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If $|\nu|$ is small, this problem can be seen as a perturbation of the N-centre problem.

Jacobi constant: if $z : I \subset \mathbb{R} \to \mathbb{R}^2$ is a solution then

$$\frac{1}{2}|\dot{z}(t)|^2 - \frac{\nu^2}{2}|z(t)|^2 - V(z(t)) = h \qquad \forall t \in I.$$

We term $\Phi_\nu(z) = \frac{\nu^2}{2}|z|^2 + V(z)$.

Motivations

- toy model towards an extension to the restricted $(N+1)$-body problem.
- the trajectories that we constructed are not stable under perturbation a priori.

Existence of periodic solutions

We are interested in collision-free solutions.
Thus (recall the distinction made at the beginning) we fix our minds on the case $\alpha = 1$ and $N \geq 4$.

For every $\alpha \in [1,2)$ and $N \geq 3$ it is possible to give sufficient conditions on the sequence $(P_{j_1}, \ldots, P_{j_n}) \in \mathcal{P}^n$ in order to get collision-free solutions:

- $\alpha = 1$ and $N \geq 4$: $P_{j_k} \in \mathcal{P} \setminus \mathcal{P}_1$ for every k.
- $\alpha = 1$ and $N = 3$: we require $n = 4m$ and

$$(P_{j_{k+1}}, \ldots, P_{j_{k+4}}) \in \{(P_1 P_1 P_2 P_3), (P_2 P_2 P_3 P_1)\}.$$

- $\alpha \in (1,2)$: no restrictions.

Existence of periodic solutions

Let $\alpha = 1$, $N \geq 4$, $c_1, \ldots, c_N \in \mathbb{R}^2$, $m_1, \ldots, m_N \in \mathbb{R}^+$. There exists $\tilde{h} > 0$ such that, given $h \in (-\tilde{h}, 0)$, there is $\tilde{\nu} = \tilde{\nu}(h) > 0$ such that to each $\nu \in (-\tilde{\nu}, \tilde{\nu})$, $n \in \mathbb{N}$ and $(P_{j_1}, \ldots, P_{j_n}) \in (\mathcal{P} \setminus \mathcal{P}_1)^n$ we can associate a collision-free periodic solution $z_{((P_{j_1},\ldots,P_{j_n}),h,\nu)}$ of

$$\begin{cases} \ddot{z}(t) + 2\nu i \dot{z}(t) = \nabla \Phi_\nu(z(t)) \\ \frac{1}{2}|\dot{z}(t)|^2 - \Phi_\nu(z(t)) = h, \end{cases}$$

which depends on $(P_{j_1}, \ldots, P_{j_n})$ in the following way. There exist $R, \delta > 0$ (depending on h only) such that $z_{((P_{j_1},\ldots,P_{j_n}),h,\nu)}$ crosses $2n$ times within one period the circle $\partial B_R(0)$, at times $(t_k)_{k=0,\ldots,2n-1}$, and

- in (t_{2k}, t_{2k+1}) the solution stays outside $B_R(0)$ and

$$|z_{((P_{j_1},\ldots,P_{j_n}),h,\nu)}(t_{2k}) - z_{((P_{j_1},\ldots,P_{j_n}),h,\nu)}(t_{2k+1})| < \delta.$$

- in (t_{2k+1}, t_{2k+2}) the solution lies inside $B_R(0)$ and separates the centres according to the partition P_{j_k}.

Sketch of the proof: step 1) Preliminaries

Outside a ball of radius $R > \max\{|c_k| : k = 1, \dots, N\} > 0$ we can consider the new problem as a small perturbation of the α-Kepler's problem:

$$\Phi_\nu(x) = \frac{M}{\alpha |x|^\alpha} + W_{h,\nu}(x) = V_0(x) + W_{h,\nu}(x)$$

with $M = \sum_{j=1}^N m_j$ and $W_{h,\nu}(x) = o(1)$ as $h, \nu \to 0$ in the $\mathcal{C}^1$ topology for $|x| \geq R$.

step 2) Outer dynamics

The perturbative approach works for the outer dynamics.

Proposition (outer solutions)

There exist $\delta > 0$, $h_1 > 0$ and $\nu_1 > 0$ such that for every $(h, \nu_1) \in (-h_1, 0) \times (-\nu_1, \nu_1)$, for every $p_0, p_1 \in \partial B_R(0) : |p_1 - p_0| < 2\delta$, there exist a unique solution $z_{\text{ext}}(\cdot\,; p_0, p_1; h, \nu)$ of

$$\begin{aligned} \ddot{z}(t) + 2i\nu\dot{z}(t) = \nabla\Phi_\nu(z(t)) \qquad & t \in [0, T], \\ \frac{1}{2}|\dot{z}(t)|^2 - \Phi_\nu(z(t)) = h \qquad & t \in [0, T], \\ |z(t)| > R \qquad & t \in (0, T), \\ z(0) = p_0, \quad & z(T) = p_1. \end{aligned}$$

step 3) Inner dynamics

It is not possible to adapt the approach used for the N-centre problem: **the alternative "collision-free" or "ejection-collision" does not hold anymore**.

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It is not possible to adapt the approach used for the N-centre problem: **the alternative "collision-free" or "ejection-collision" does not hold anymore**.
However, a variational formulation can be recovered.
We fix $p_1, p_2 \in \partial B_R(0)$, $[a, b] = [0, 1]$.
We recall that for every $P_j \in \mathcal{P}$ we termed

$$\widehat{K}_{P_j} = \widehat{K}_{P_j}^{p_1 p_2}([0,1]) := \{u \in H^1([0,1]) : u(0) = p_1, u(1) = p_2, \\ |u(t)| \leq R \ \forall t \in (0,1), \ u \text{ is collision-free}, \\ u \text{ separates the centres according to the partition } P_j\}$$

and

$$K_{P_j} = K_{P_j}^{p_1 p_2}([0,1]) := \overline{\widehat{K}_{P_j}}^{\sigma(H^1,(H^1)^*)}.$$

Variational formulation

Theorem

Let $u \in \widehat{K}_{P_j}$ be a non-constant critical point of

$$L_{h,\nu}(u) = \int_0^1 \sqrt{(\Phi_\nu(u) + h)}|\dot{u}| + \frac{1}{\sqrt{2}}\nu \int_0^1 \langle iu, \dot{u} \rangle,$$

such that $|u(t)| < R$ for every $t \in (0,1)$. Then there exists a reparameterization z of u which is a classical solution of

$$\begin{cases} \ddot{z}(t) + 2\nu i\dot{z}(t) = \nabla\Phi_\nu(z(t)) & t \in [0,T] \\ \frac{1}{2}|\dot{z}(t)|^2 - \Phi_\nu(z(t)) = h & t \in [0,T] \\ z(0) = p_1 \qquad z(T) = p_2. \end{cases} \tag{3}$$

Weak solutions

Let $u \in K_{P_j}$ be a local minimizer of $L_{h,\nu}$. If $|u(t)| < R$ for every $t \in (0,1)$, there exists a reparameterization z of u which is a solution almost everywhere of (3). We say that z is a weak solution of (3).

Weak solutions

Let $u \in K_{P_j}$ be a local minimizer of $L_{h,\nu}$. If $|u(t)| < R$ for every $t \in (0,1)$, there exists a reparameterization z of u which is a solution almost everywhere of (3). We say that z is a weak solution of (3).

The existence of weak solutions can be proved by means of the direct methods of the calculus of variations.

Proposition (inner solutions)

There exist $h_2, \nu_2 > 0$ such that, for every $(p_1, p_2, h, \nu, P_j) \in (\partial B_R(0))^2 \times (-h_2, 0) \times (-\nu_2, \nu_2) \times \mathcal{P}$, problem (3) has a weak solution $z_{P_j}(\cdot\,; p_1, p_2; h, \nu) \in K_{P_j}^{p_1 p_2}([0, T])$.

step 4) A finite dimensional reduction

Let $\widehat{h} = \min\{h_1, h_2\}$, $\widehat{\nu} = \min\{\nu_1, \nu_2\}$.
We fix $h \in (-\widehat{h}, 0)$, $\nu \in (-\widehat{\nu}, \widehat{\nu})$, $n \in \mathbb{N}$ and $(P_{k_1}, P_{k_2}, \ldots, P_{k_n}) \in \mathcal{P}^n$.
We define

$$D = \Big\{ (p_0, \ldots, p_{2n}) \in (\partial B_R(0))^{2n+1} : |p_{2j+1} - p_{2j}| \leq \delta \\ \text{for } j = 0, \ldots, n-1, \ p_{2n} = p_0 \Big\}.$$

We alternate outer and inner trajectories, according to the given sequence of partitions. Let $(p_0, \ldots, p_{2n}) \in D$.

- For every $j \in \{0, \ldots, n-1\}$, we take an outer arc and we pose:

$$z_{2j}(t) := z_{\text{ext}}(t; p_{2j}, p_{2j+1}; h, \nu)$$

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  for $t \in [0, T_{2j}]$.

- On the other hand, for every $j = 0, \ldots, n-1$, we can find an inner arc associated with the partition $P_{k_{j+1}}$. We set

$$z_{2j+1}(t) := z_{P_{k_{j+1}}}(t; p_{2j+1}, p_{2j+2}; h, \nu),$$

  for $t \in [0, T_{2j+1}]$.

Let us set $\mathfrak{T}_k := \sum_{j=0}^{k} T_j,\ k = 0, \ldots, 2n-1$. We define

$$\gamma_{((p_0,\ldots,p_{2n})}^{((P_{j_1},\ldots,P_{j_n}),h,\nu)}(s) := \begin{cases} z_0(s) & s \in [0, \mathfrak{T}_0] \\ z_1(s - \mathfrak{T}_0) & s \in [\mathfrak{T}_0, \mathfrak{T}_1] \\ \vdots & \\ z_{2n-2}\left(s - \mathfrak{T}_{2n-3}\right) & s \in [\mathfrak{T}_{2n-3}, \mathfrak{T}_{2n-2}] \\ z_{2n-1}\left(s - \mathfrak{T}_{2n-2}\right) & s \in [\mathfrak{T}_{2n-2}, \mathfrak{T}_{2n-1}]. \end{cases}$$

It is piecewise differentiable and $\mathfrak{T}_{2n-1}$-periodic; in general, it is not $\mathcal{C}^1$ in $\{0, \mathfrak{T}_0, \ldots, \mathfrak{T}_{2n-1}\}$. If $\alpha = 1$ it is possible also that $\gamma_{(p_0,\ldots,p_{2n})}^{((P_{j_1},\ldots,P_{j_n}),h,\nu)}$ has some collisions.

We introduce $F = F_{((P_{k_1},\ldots,P_{k_n});h,\nu)} : D \to \mathbb{R}$

$$F(p_0,\ldots,p_{2n}) := \sum_{j=0}^{2n-1} \int_0^{T_j} \sqrt{(\Phi_\nu(z_j)+h)}|\dot{z}_j| + \frac{1}{\sqrt{2}}\nu \int_0^1 \langle i z_j, \dot{z}_j \rangle.$$

We will simply write F.

Proposition

There exists $(p_0, \ldots, p_{2n}) \in D$ which minimizes F.
There exist $\bar{h}, \bar{\nu} > 0$ such that, if $(h, \nu) \in (-\bar{h}, 0) \times (-\bar{\nu}, \bar{\nu})$, then the associated function $\gamma_{(p_0,\ldots,p_{2n})}^{((P_{j_1},\ldots,P_{j_n}),h,\nu)}$ is a periodic weak solution of

$$\begin{cases} \ddot{z}(t) + 2\nu i \dot{z}(t) = \nabla \Phi_\nu(z(t)) \\ \frac{1}{2}|\dot{z}(t)|^2 - \Phi_\nu(z(t)) = h. \end{cases}$$

The values $\bar{h}$ and $\bar{\nu}$ depend neither on n, nor on $(P_{k_1}, \ldots, P_{k_n}) \in \mathcal{P}^n$.

Proposition

There exist $\bar{h}, \bar{\nu} > 0$ such that, if $(h, \nu) \in (-\bar{h}, 0) \times (-\bar{\nu}, \bar{\nu})$ and $(P_{j_1}, \ldots, P_{j_n}) \in \mathcal{P}^n$ then there exists a periodic weak solution with Jacobi constant h of

$$\ddot{z}(t) + 2\nu i \dot{z}(t) = \nabla \Phi_\nu(z(t))$$

denoted $\gamma^{((P_{j_1},\ldots,P_{j_n}),h,\nu)}$.

step 5) From weak to classical solutions

Here we fix $h \in (-\bar{h}, 0)$ but not ν.
Goal: to find $\nu_{th}(h)$ such that if $|\nu| < \nu_{th}(h)$ then $\gamma^{((P_{j_1},\dots,P_{j_n}),h,\nu)}$ is collision-free.

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Idea

If $\nu \to 0$ then the "minimizer" $\gamma^{((P_{j_1},\dots,P_{j_n}),h,\nu)}$ tends to $\gamma^{((P_{j_1},\dots,P_{j_n}),h,0)}$ in the weak topology of H^1.

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In this way, the threshold $\nu_{th}(h)$ depends on n.
The truth is that it is sufficient to prove a continuity for the inner arcs of solutions.

The set of the inner minimizers

For every $h \in (-\bar{h}, 0)$ we term

$$\mathcal{IM}_h := \{u_{P_j}(\cdot; p_1, p_2; h, \nu) : \ p_1, p_2 \in \partial B_R(0), \ P_j \in \mathcal{P}, \ |\nu| < \bar{\nu}\},$$

the set of the *inner minimizers* of $\{L_{h,\nu}\}_{|\nu|<\bar{\nu}}$ for a fixed value of h, and

$$\mathcal{IS}_h := \{y_{P_j}(\cdot; p_1, p_2; h, \nu) : \ p_1, p_2 \in \partial B_R(0), \ P_j \in \mathcal{P}, \ |\nu| < \bar{\nu}\},$$

the set of the corresponding *inner solutions* for a fixed value of h.

One can prove that $\mathcal{IM}_h$ **is relatively compact**: let $\nu_m \to 0$ and let us consider

$$\{u_{P_j}(\cdot\,; p_1^m, p_2^m; h, \nu_m) : \; p_1^m, p_2^m \in \partial B_R(0), \; P_j \in \mathcal{P}\} \subset \mathcal{IM}_h.$$

There exist subsequences $p_1^m \to \widetilde{p}_1 \in \partial B_R(0)$, $p_2^m \to \widetilde{p}_2 \in \partial B_R(0)$, and

$$u_{P_j}(\cdot\,; p_1^m, p_2^m; h, \nu_m) \rightharpoonup \widetilde{u} \in K_{P_j}^{\widetilde{p}_1 \widetilde{p}_2}([0,1]).$$

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Question

Is it true that $\widetilde{u} = u_{P_j}(\cdot\,;\widetilde{p}_1,\widetilde{p}_2;h,0) \in \mathcal{IM}_h$?

Continuity Property for the minimizers

Lemma

Let $h \in (-\bar{h}, 0)$, $P_j \in \mathcal{P}$, $((p_1^m, p_2^m)) \subset (\partial B_R(0))^2$ and $(\nu_m) \subset (-\bar{\nu}, \bar{\nu})$. Let $u_m = u_{P_j}(\cdot\,; p_1^m, p_2^m; h, \nu_m)$ be a minimizer for the following variational problem:

$$\min\left\{ L_{h,\nu_m}(u) : u \in K_{P_j}^{p_1^m p_2^m}([0,1]) \right\}.$$

Assume $(p_1^m, p_2^m) \to (\widetilde{p}_1, \widetilde{p}_2)$, $\nu_m \to 0$, and $u_m \rightharpoonup \widetilde{u}$ weakly in H^1. Then $\widetilde{u}$ is a minimizer for

$$\min\left\{ L_{h,0}(u) : u \in K_{P_j}^{\widetilde{p}_1 \widetilde{p}_2}([0,1]) \right\}.$$

It is not difficult to conclude from here, recalling what we know about the inner minimizers of the N-centre problem:

- for an inner minimizer $u_{P_j}(\cdot\,; p_1, p_2; h, 0) = u_{P_j}(\cdot\,; p_1, p_2, h)$ a collision can occur only if $p_1 = p_2$ and $P_j \in \mathcal{P}_1$.

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- for an inner minimizer $u_{P_j}(\cdot\,; p_1, p_2; h, 0) = u_{P_j}(\cdot\,; p_1, p_2, h)$ a collision can occur only if $p_1 = p_2$ and $P_j \in \mathcal{P}_1$.

Consequently, if $\alpha = 1$, $N \geq 4$ and $h \in (-\bar{h}, 0)$, there exists $\widetilde{\nu}(h)$ such that each minimizer $u_{P_j}(\cdot\,; p_1, p_2; h, \nu) \in \mathcal{IM}_h$ is collision-free provided $P_j \in \mathcal{P} \setminus \mathcal{P}_1$ and $|\nu| < \widetilde{\nu}$.

Proposition

Let $\alpha = 1$ and $N \geq 4$. Let $h \in (-\bar{h}, 0)$. There exists $\tilde{\nu}(h)$ such that for every $\nu \in (-\tilde{\nu}(h), \tilde{\nu}(h))$, $n \in \mathbb{N}$ and $(P_{j_1}, \dots, P_{j_n}) \in (\mathcal{P} \setminus \mathcal{P}_1)^n$, the function $\gamma^{((P_{j_1},\dots,P_{j_n}),h,\nu)}$ is collision-free.

Remark

The Continuity Lemma permits to restrict the attention on a unique passage inside $B_R(0)$; in particular the argument is independent on n, which can be arbitrarily large.

The circular restricted $(N+1)$-body problem $(\alpha = 1)$

We assign the masses $m_1, \ldots, m_N > 0$. Let $c_1, \ldots, c_N$ be a central configuration for the planar N-body problem. We study the motion of a particle of null mass under the gravitational attraction of the primaries:

$$\ddot{x}(t) = -\sum_{k=1}^{N} \frac{m_k}{|x(t) - e^{it} c_k|^3} (x(t) - e^{it} c_k).$$

If $x(t) = \exp\{it\} z(t)$ then

$$\ddot{z}(t) + 2i\dot{z}(t) = z(t) - \sum_{k=1}^{N} \frac{m_k}{|z(t) - c_k|^3} (z(t) - c_k). \tag{4}$$

The Jacobi constant is a first integral of the system.

Problem

Apparently, the main problem is the lack of a perturbative structure.

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Proposition

$z(t)$ is a solution of (4) with Jacobi constant $h < 0$ if and only if $y(t) = (-h)z((-h)^{-3/2}t)$ is a solution of

$$\ddot{y}(t)+2i(-h)^{-3/2}\dot{y}(t) = (-h)^{-3}y(t)-\sum_{k=1}^{N}\frac{m_k}{|y(t)-(-h)c_k|^3}(y(t)-(-h)c_k)$$

with Jacobi constant -1.

Setting $w(t) = \exp\{i(-h)^{-3/2}t\}y(t)$, we see that

$$\ddot{w}(t) = -\sum_{k=1}^{N}\frac{m_k}{|w(t)-(-h)e^{i(-h)^{-3/2}t}c_k|^3}(w(t)-(-h)e^{i(-h)^{-3/2}t}c_k). \tag{5}$$

Through $y \leftrightarrow w$ it makes sense to say that w "has Jacobi constant -1".

step 1) Preliminaries

Outside a ball of radius $R > \max_k\{|(-h)c_k|\}$ the problem can be seen as a small perturbation of the Kepler's problem:

$$-\sum_{k=1}^{N} \frac{m_k}{|w(t)-(-h)e^{i(-h)^{-3/2}t}c_k|^3}(w(t)-(-h)e^{i(-h)^{-3/2}t}c_k) = -\frac{M}{|w|^3}w + \frac{h^2}{2}O(1).$$

As a consequence,

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- The inner dynamics works **in the rotating frame of reference** (for y).
- The finite dimensional reduction works **in the fixed frame of reference** (for w).

This permits to find weak solutions to (5) "with Jacobi constant -1".

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- The outer dynamics works **in the fixed frame of reference** (for w).
- The inner dynamics works **in the rotating frame of reference** (for y).
- The finite dimensional reduction works **in the fixed frame of reference** (for w).

This permits to find weak solutions to (5) "with Jacobi constant -1".

Open problem

We cannot rule out the occurrence of collisions.

Thank you for the attention!