Entire Parabolic Trajectories as Minimal Phase Transitions

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Zero-energy trajectories

Let us consider the conservative dynamical system

$$(DS) \qquad \ddot{x}(t) = \nabla V(x(t)), \qquad x \in \mathbb{R}^d \setminus \mathcal{X}, \quad d \geq 2,$$

where $\mathcal{X}$ is a singular (or collision) set and the potential V satisfies

- $V \in \mathcal{C}^2(\mathbb{R}^d \setminus \mathcal{X})$, $V(x) \to \infty$ as $\mathrm{dist}(x, \mathcal{X}) \to 0$;
- the normalized condition $0 = \liminf_{|x| \to \infty} V(x) < V(x)$ for every x.

Definition

A (global) parabolic trajectory for (DS) is a solution which:

- is defined on $\mathbb{R}$;
- is collisionless, i.e. $x(t) \notin \mathcal{X}\ \forall t \in \mathbb{R}$;
- has null energy

$$\frac{1}{2}|\dot{x}(t)|^2 = V(x(t)), \qquad \text{for every } t \in \mathbb{R}.$$

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Asympthotic Behaviour of Parabolic Trajectories with Homogeneous Potentials

We assume that V is homogeneous of degree $-\alpha$, $\alpha \in (0,2)$.

In this setting parabolic trajectories can be equivalently defined as solutions satisfying

$$|\dot{x}(t)| \to 0 \text{ as } t \to \pm\infty.$$

Furthermore, they enjoy some asymptotic properties, regarding both their "radial" and "angular" part:

$$r(t) := |x(t)| > 0 \text{ and } s(t) := x(t)/|x(t)| \in \mathbb{S}^{d-1}.$$

(A1) There exists $\gamma > 0$ such that

$$\lim_{t\to+\infty} V(s(t)) = \gamma.$$

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DEFINITION

Recall that a central configuration (c.c.) for V is a unitary vector which is a critical point of the restriction of V to the sphere $\mathbb{S}^{d-1}$.

(A3) $s(t)$ has infinitesimal distance from the set of c.c. of V as $t \to \pm\infty$.

Starting from Chazy in 1920, many authors worked on this kind of motions: Devaney, Pollard, Saari, Marchall, Hulkhover, Knauf, Chenciner, Maderna, Venturelli, Da Luz]).

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Parabolic Trajectories as Transition Orbits

In particular, whenever the set of c.c. of V is discrete and $\xi^{\pm}$ are c.c., a parabolic trajectory x which satisfies

$$s(t) \to \xi^{\pm}, \qquad \text{as } t \to \pm\infty,$$

can be read as a transition orbit between ξ^{-} and ξ^{+}. Since to every central configuration ξ is associated the zero-energy homothetic motion

$$x(t) = t^{\frac{2}{2+\alpha}}\xi,$$

when the set of c.c. of V is discrete parabolic trajectories are indeed asympthotic to the justapposition of two homothetic motions.

We address to the following problem:

Under which conditions on V does a global parabolic solution connecting two different (minimizing) c.c. exist?

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Under which conditions on V does a global parabolic solution connecting two different (minimizing) c.c. exist?

- Such estimates are in some sense dual to the ones that describe the behaviour of trajectories approaching a collision.

If $x = rs$ solves (DS) on $(a, t^*) \subset \mathbb{R}$ and $r(t^*) \to 0$ as $t \to t^*$, then
(A1) $\lim_{t\to+\infty} V(s(t)) = \gamma$, for some $\gamma > 0$.
(A2)' As $t \to t^*$

$$r(t) \sim (K(\alpha)(t^* - t))^{\frac{2}{2+\alpha}}, \text{ and } \dot{r}(t) \sim -\sqrt{2\gamma}(K(\alpha)(t^* - t))^{-\frac{\alpha}{2+\alpha}}.$$

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- Estimates (A1),(A2),(A2)' and (A3) (or quite similar ones) actually hold for huger classes of potentials including quasi-homogeneous and logarithmic ones ([B.-Ferrario-Terracini (2008)]).

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Action Functional

Given any V as above (homogeneous, smooth outside $\mathcal{X}$, positive....), $a < b$, and $x \in H^1\left((a,b);\mathbb{R}^d\right)$, let us consider the (possibly infinite) lagrangian action functional with lagrangian $\mathcal{L}$:

$$\mathcal{A}(x) = \mathcal{A}([a,b];x) := \int_a^b \mathcal{L}(\dot{x}(t), x(t))\,\mathrm{d}t,$$

$$\mathcal{L}(\dot{x}, x) := \frac{1}{2}|\dot{x}|^2 + V(x).$$

- The action may be finite on solutions interacting with the singular set.
- Admitting their existence, entire parabolic trajectories would have infinite action.

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Given two minimal c.c. for V, ξ^- and ξ^+, ingoing and outgoing asymphotic directions, we consider the following class of minimizers for $\mathcal{A}$

We say that $x \in H^1_{\mathrm{loc}}(\mathbb{R})$ is a free-time Morse minimizer of $\mathcal{A}$, if

- $x(t) \notin \mathcal{X}$, $\forall t \in \mathbb{R}$;
- $r(t) \to +\infty$, $s(t) \to \xi^\pm$ as $t \to \pm\infty$;
- for every $a < b$, $a' < b'$, and $z \in H^1(a', b')$, there holds

$$z(a') = x(a),\ z(b') = x(b) \qquad \Longrightarrow \qquad \mathcal{A}([a,b];x) \le \mathcal{A}([a',b'];z).$$

A free-time Morse minimizer minimizes all fixed-endpoints problems both in space and time, hence, by virtue of *Maupertuis' principle*, it satisfies the Euler-Lagrange equation (DS) and it has null energy.

Free-time Morse minimizers are indeed global parabolic trajectories.

Variational Solutions: Free Morse Minimizers

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1. **Existence of free-time Morse minimizers.**

- In general a potential V does not need to admit global parabolic trajectories.
- Dealing with the N-body potential, in 2008, `E. Maderna` and `A. Venturelli` proved that parabolic arcs -i.e. defined only on the half line- asymphthotic to a minimal c.c. exist for every starting configuration. On the other hand, in 2011, `A. da Luz` and `E. Maderna` showed that if no topological constraints are imposed, there are no global parabolic trajectories.
- Parabolic motions are indeed very unstable objects. Their existence will be related to a specific threshold for a suitable parameter.

We provide a necessary and sufficient condition for the existence of global parabolic trajectories restricting our analysis to the class of anisotropic keplerian potentials in any dimension ($\mathcal{X} = \{0\}$).

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2. Connection with colliding solutions.

- In celestial mechanics, and more in general in the theory of singular hamiltonian systems, parabolic trajectories play a central role and they are known to carry precious information on the behavior of solutions near collisions.
- In 2011, `A. da Luz` and `E. Maderna` proved the absence of entire parabolic trajectories which are Morse minimal follows from the absence of collisions in Bolza minimizers (Marchal's Principle).

For planar anisotropic keplerian potentials we connect the presence of free Morse minimizers to the presence/absence of collisions to other classes of problems (fixed-time Bolza problems, periodic trajectories...) obtained by minimizing the action under topological constraints.

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Devaney's Work: the Anisotropic Kepler Problem in $\mathbb{R}^2$

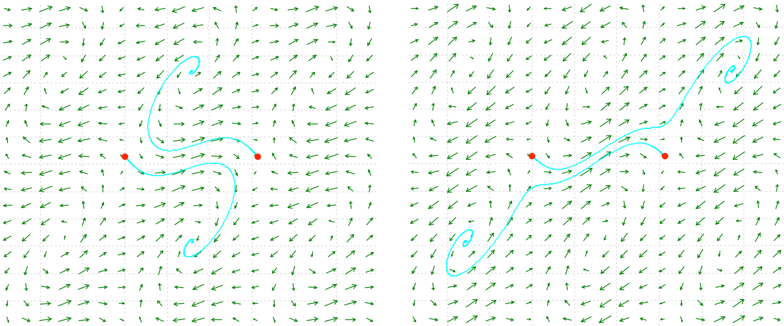
- In 1978 R.L. Devaney (*Invent. Math., 45*), considered the planar anisotropic Kepler problem

$$V(r\cos\vartheta, r\sin\vartheta) = \frac{U(\vartheta)}{r^\alpha}, \qquad \vartheta \in \mathbb{R},\ r > 0,$$

  where U is a 2π-periodic function such that $U(\vartheta) \geq U_{\min} > 0$, $\forall \vartheta \in \mathbb{R}$.

- In this setting parabolic trajectories connecting minima correspond to saddle-saddle heteroclinic connections for a planar dynamical system he obtained after a suitable variable change (a variant of McGehee coordinates).

- But generically the unstable manifold at a saddle falls into a sink, while the stable one emanates from a source, implying that parabolic motions do no exist.



The two pictures represent the phase portrait of the planar dynamical system studied by Devaney with $U(\vartheta) = 2 - \cos(2\vartheta)$*, when* $\alpha = 0.5$ *(at left) or* $\alpha = 1$ *(at right). We focus on the saddles* $(0, \pi)$ *and* (π, π)*: from the mutual positions of the heteroclinic departing from* $(0, \pi)$ *and the one ending in* (π, π) *we deduce that the two vector fields are not topologically equivalent. Using standard arguments in the theory of structural stability, we infer the existence, for some* $\bar{\alpha} \in (0.5, 1)$*, of a saddle connection between* $(0, \pi)$ *and* (π, π)*.*

Introducing a Bifurcation Parameter

- Working in the class of anisotropic Keplerian potentials (not necessarily planar), we choose as parameter the homogeneity exponent $-\alpha$.
- To clarify the role of such parameter, it may be helpful to let the potential vary in a class and look for parabolic orbits as pairs trajectory-parameter.
  More precisely, let us fix $\xi^+ \neq \xi^-$ in $\mathbb{S}^{d-1}$ and $V_{\min} > 0$, and let us define the metric spaces

$$\mathcal{U} = \left\{ V \in \mathcal{C}^2(\mathbb{S}^{d-1}) : \begin{array}{l} s \in \mathbb{S}^{d-1} \text{ implies } V(s) \geq V(\xi^\pm) = V_{\min}; \\ \exists \delta > 0, \mu > 0 \text{ such that } |s - \xi^\pm| < \delta \\ \text{implies } V(s) - V(\xi^\pm) \geq \mu |s - \xi^\pm|^2 \end{array} \right\},$$

$$\mathcal{V} = \left\{ (V, \alpha) \in \mathcal{C}^2(\mathbb{S}^{d-1}) \times (0,2) : V \in \mathcal{U} \right\},$$

the latter being equipped with the product distance.

An auxiliary Bolza problem

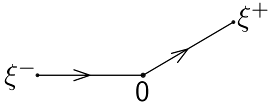
Fixed V in the described class, the existence of entire parabolic solutions is related to its behaviour with respect to the following fixed-endpoints problem:

$$c(V) := \inf \left\{ \mathcal{A}([a,b];x) : a < b,\ x \in H^1(a,b),\ x(a) = \xi^-,\ x(b) = \xi^+ \right\}.$$

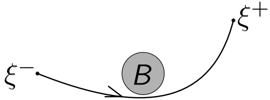
- We minimize among path (and their re-parametrization) connecting ξ^- to ξ^+.
- It turns out not only that $c(V)$ is achieved, but also that it can be achieved only by two different kind of paths.

Inner and Outer Potentials

$\mathrm{In} := \{V : c(V)$ is achieved by the juxtaposition of two homothetic motions, the first connecting ξ^- to the origin and the second the origin to $\xi^+\}$



$\mathrm{Out} := \{V : c(V)$ is achieved by motions which are uniformly bounded away from the origin $\}$



The sets In and Out enjoy the following properties:

- $\mathrm{In} \cap \mathrm{Out} = \emptyset$, $\mathrm{In} \cup \mathrm{Out} = \mathcal{V}$;
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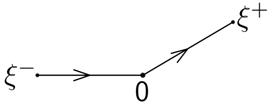

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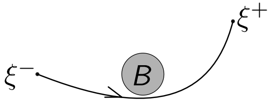

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Structure Theorem

The role of the homogeneity parameter can be now clarified by the following property. Let $\Pi := \partial\mathrm{In} \cap \partial\mathrm{Out}$.

Lemma (Separation Property)

There exists an open nonempty set $\Sigma \subset \mathcal{U}$, and a continuous function $\bar{\alpha} : \Sigma \to (0,2)$ such that

$$\Pi = \{(V, \bar{\alpha}(V)) : \; V \in \Sigma\}\,.$$

We can now characterize the set of potentials admitting parabolic Morse minimizers as the graph of the above function.

Main Theorem.

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Back to $\mathbb{R}^2$: Topological Constraints

- Let $d = 2$ and $U(\vartheta) := V(\cos\vartheta, \sin\vartheta)$.
- For the sake of simplicity, let U be a positive, $\mathcal{C}^2$ Morse function such that every local minimum is indeed a global one.
- Since $\mathbb{R}^2 \setminus \{0\}$ is not simply connected, we can search for minimizers with respect to a given homotopy class:

connecting ξ^- and ξ^+ with $h \in \mathbb{Z}$ rotations around 0 $\iff$ connecting $\vartheta^- := \arg \xi^-$, $\vartheta^+ := \arg \xi^+ + 2h\pi$ in the universal covering

Motivated by this, we introduce the set

$$\Theta := \{\vartheta \in \mathbb{R} : \vartheta \text{ is a (non-degenerate global) minimum for } U\}.$$

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The Parabolic Threshold

Theorem

Let $\vartheta^-, \vartheta^+ \in \Theta$, $\vartheta^- \neq \vartheta^+$; then

1. *there exists at most one $\bar{\alpha} \in (0,2)$ such that $V = (U, \alpha)$ admits a corresponding parabolic trajectory with asympthotic directions ϑ^- and ϑ^+ if and only if $\alpha = \bar{\alpha}$;*
2. *every parabolic trajectory is indeed a free-time Morse minimizer.*

Furthermore:

3. *if $|\vartheta^+ - \vartheta^-| > \pi$ then there exists exactly one $\bar{\alpha}$.*

- While the first two assertions follow from the Main Theorem, the third one is peculiar for the planar case.
- In order to guarantee the existence of $\bar{\alpha}$, we force the Bolza problem to have a collision (i.e. the potential to be In) when α is small. Increasing α necessarely the potential becomes Out, crossing the boundary Π.

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Absence of Collisions for Bolza Problems with Topological Constraint

- When a topological constraint is imposed, the Marchal's averaged variation argument does not work, and other devices has to be designed to avoid the occurrence of collisions.
- Gordon's Theorem is a first result in this direction: *The keplerian ellipses minimize the action among loops having nontrivial winding number about the origin.*

Theorem

Given any integer $k \neq 0$ and period $T > 0$, if

$$\alpha > \bar{\alpha}(U, \vartheta^*, \vartheta^* + 2k\pi)\,, \quad \textit{for every minimum } \vartheta^* \textit{ of } U,$$

then there exists an action minimizing collisionless T–periodic trajectory winding k times around zero.

- To construct Morse minimizers of parabolic type, we will first consider analogous problems on bounded intervals (Bolza problems), and then pass to the limit.
- This procedure may fail for two main reasons: sequences of approximating trajectories may either converge to the singularity, or escape to infinity.
- This naturally leads to introduce some constraints and to study the constrained minimization problem

$$m = m(\varepsilon, x_1, x_2) := \inf_{x\in\Gamma} \mathcal{A}(x) \quad \text{where } \Gamma := \bigcup_{T>0} \Gamma_T, \quad \text{and}$$

$$\Gamma_T := \left\{ x \in H^1(-T,T) : x(-T) = x_1, x(T) = x_2, \min_{t\in[-T,T]} |x(t)| = \varepsilon \right\}$$

where $\varepsilon > 0$ and $x_1, x_2 \in \mathbb{R}^d \setminus B_\varepsilon(0)$ are fixed.

Construction of Parabolic Trajectories

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$$\Gamma_T := \left\{ x \in H^1(-T,T) : x(-T) = x_1, x(T) = x_2, \min_{t \in [-T,T]} |x(t)| = \varepsilon \right\}$$

where $\varepsilon > 0$ and $x_1, x_2 \in \mathbb{R}^d \setminus B_\varepsilon(0)$ are fixed.

- To construct Morse minimizers of parabolic type, we will first consider analogous problems on bounded intervals (Bolza problems), and then pass to the limit.
- This procedure may fail for two main reasons: sequences of approximating trajectories may either converge to the singularity, or escape to infinity.
- This naturally leads to introduce some constraints and to study the constrained minimization problem

$$m = m(\varepsilon, x_1, x_2) := \inf_{x\in\Gamma} \mathcal{A}(x) \quad \text{where } \Gamma := \bigcup_{T>0} \Gamma_T, \quad \text{and}$$

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where $\varepsilon > 0$ and $x_1, x_2 \in \mathbb{R}^d \setminus B_\varepsilon(0)$ are fixed.

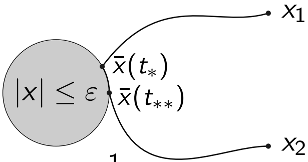
Properties of Bolza Constrained Minimizers

- If $\bar{x} = \bar{r}\bar{s} \in \Gamma_{\bar{T}}$ is a constrained minimizer and $\bar{r}(t) > \varepsilon$ for $t \in (a, b)$, then

$$\ddot{\bar{x}}(t) = \nabla V(\bar{x}(t)) \quad \text{and} \quad \frac{1}{2}|\dot{\bar{x}}(t)| = V(\bar{x}(t)), \quad \text{for every } t \in (a, b).$$

- If $\bar{x}$ achieves m and it does interact with the constraint, then there exist $t_* \leq t_{**}$ such that

$$\bar{r}(t) = \varepsilon \;\Leftrightarrow\; t \in [t_*, t_{**}];$$
$$t \in (-T, t_*) \;\Rightarrow\; \dot{\bar{r}}(t) < 0;$$
$$t \in (t_{**}, T) \;\Rightarrow\; \dot{\bar{r}}(t) > 0;$$



$$t \in (t_*, t_{**}) \Rightarrow \begin{cases} \ddot{\bar{x}}(t) = \nabla_T V(\bar{x}(t)) - \dfrac{1}{\varepsilon^2}|\dot{\bar{x}}(t)|^2 \bar{x}(t), \\ \dfrac{1}{2}|\dot{\bar{x}}(t)| = V(\bar{x}(t)). \end{cases}$$

$\Rightarrow$ If $\bar{x}$ achieves m then it may be not regular only in t_* and t_{**}.

Interaction with the Constraint

Proposition

If $\bar{x}$ achieves m, then one of the following situations occurs:

(A) $t_* < t_{**}$ and $\bar{x} \in \mathcal{C}^1(-\bar{T}, \bar{T})$;

(B) $t_* = t_{**}$ and $\bar{x} \in \mathcal{C}^1(-\bar{T}, \bar{T})$;

(C) $t_* = t_{**}$ and $\dot{\bar{x}}(t_*^-) \neq \dot{\bar{x}}(t_*^+)$; in such a case $\bar{x}$ undergoes a radial reflection, that is $\dot{\bar{r}}(t_*^-) = -\dot{\bar{r}}(t_*^+) \neq 0$ and $\dot{\bar{s}}(t_*^-) = \dot{\bar{s}}(t_*^+)$.

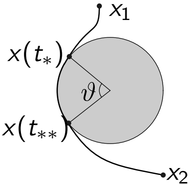
$\Rightarrow$ We can classify Bolza minimizers with respect to the discontinuity of the quantities x and $\dot{x}$ on the constraint.

Definition

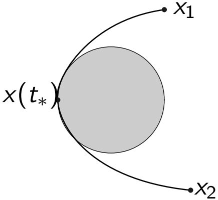
Given a constrained Bolza minimizer $x = rs$ we define :

$$\Delta_{\mathrm{pos}}(x) := |s(t_{**}) - s(t_*)|\,, \qquad \Delta_{\mathrm{vel}}(x) := \varepsilon^{\alpha/2}\left[\dot{r}(t_{**}^+) - \dot{r}(t_*^-)\right]$$

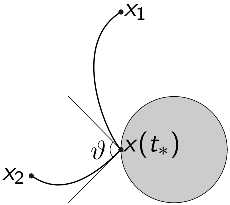
respectively as the normalized position-jump and velocity-jump of x.



(A) $t_* < t_{**}$, $\bar{x} \in \mathcal{C}^1$,
$\Delta_{\text{pos}} > 0$, $\Delta_{\text{vel}} = 0$,
x is position-jumping



(B) $t_* = t_{**}$, $\bar{x} \in \mathcal{C}^1$,
$\Delta_{\text{pos}} = 0$, $\Delta_{\text{vel}} = 0$,
x is parabolic



(C) $t_* = t_{**}$, $\dot{\bar{x}}(t_*^-) \neq \dot{\bar{x}}(t_*^+)$,
$\Delta_{\text{pos}} = 0$, $\Delta_{\text{vel}} > 0$,
x is velocity-jumping

From Bolza to Morse Minimizers

Definition

We say that $x \in H^1_{\mathrm{loc}}(\mathbb{R})$ is an *ε-constrained Morse minimizer* if

- $\min_t |x(t)| = \varepsilon$;
- $|x(t)| \to +\infty$ and $\dfrac{x(t)}{|x(t)|} \to \xi^{\pm}$, as $t \to \pm\infty$;
- for every $a < b$ and $T > 0$, and for every $z \in H^1(-T, T)$, with $\min_{t\in[-T,T]} |z(t)| = \min_{t\in[a,b]} |x(t)|$, there holds

$$z(-T) = x(a),\ z(T) = x(b) \quad \Longrightarrow \quad \mathcal{A}([a,b];x) \leq \mathcal{A}([-T,T];z).$$

Proposition

$\mathcal{M} = \{\varepsilon\text{-constrained Morse minimizer}\} \neq \emptyset$.

We argue by approximation, solving the Bolza problem with $x_1 = R\xi^-$ and $x_2 = R\xi^+$ and then letting $R \to +\infty$.

Constrained Minimizers have the Same Jumps

- Since by definition any restriction of a Morse minimizer is indeed a Bolza one (with the appropriate constraint), we have that also Morse minimizers can be classified according to their jumps.
- In general, for any fixed potential V, there is no reason to expect uniqueness for the Morse minimizers.

Nevertheless, it is possible to show that

Proposition

Let $V \in \mathcal{V}$ be fixed and let $x, \hat{x} \in \mathcal{M}$. Then

$$\Delta_{\mathrm{pos}}(x) = \Delta_{\mathrm{pos}}(\hat{x}) \quad \textit{and} \quad \Delta_{\mathrm{vel}}(x) = \Delta_{\mathrm{vel}}(\hat{x}).$$

We can then define

$$\Delta_{\mathrm{pos}}(V) := \Delta_{\mathrm{pos}}(x), \quad \forall x \in \mathcal{M}$$

$$\Delta_{\mathrm{vel}}(V) := \Delta_{\mathrm{vel}}(x), \quad \forall x \in \mathcal{M}$$

Looking for Free Parabolic Minimizers

Constrained Morse minimizers for potentials in the set

$$\{V : \Delta_{\mathrm{pos}}(V) = \Delta_{\mathrm{vel}}(V) = 0\}$$

are indeed solutions for $\ddot{x} = \nabla V(x)$ on the whole real line. Such potentials are then good candidates to belong to the set Π.
At this moment we do not know:

- if such set is not empty;
- whether it contains free parabolic minimizers.

These questions will find an answer after an investigation on the relation between the set In/Out and the classification of potentials with respect to their jumps.
We recall that In and Out were defined in terms of the Bolza level

$$c(V) := \inf \left\{ \mathcal{A}\left([a,b];x\right) : a < b,\ x \in H^1(a,b),\ x(a) = \xi^-,\ x(b) = \xi^+ \right\}.$$

Conclusion of the proof

Key Equivalences

$V \in \mathrm{In} \iff \Delta_{\mathrm{vel}}(V) = 0, \qquad V \in \mathrm{Out} \iff \Delta_{\mathrm{vel}}(V) > 0.$

We can then deduce that:

$\alpha_1 < \alpha_2$ implies:

- $\Delta_{\mathrm{pos}}(V, \alpha_2) > 0 \quad \Longrightarrow \quad \Delta_{\mathrm{pos}}(V, \alpha_1) > 0;$
- $\Delta_{\mathrm{vel}}(V, \alpha_1) > 0 \quad \Longrightarrow \quad \Delta_{\mathrm{vel}}(V, \alpha_2) > 0;$
- $\Delta_{\mathrm{pos}}(V, \alpha_1) > 0$ and $\Delta_{\mathrm{vel}}(V, \alpha_2) > 0$

$$\Longrightarrow \Delta_{\mathrm{pos}}(V, \bar{\alpha}) = \Delta_{\mathrm{vel}}(V, \bar{\alpha}) = 0,$$

  for a unique $\bar{\alpha} \in (\alpha_1, \alpha_2)$.

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