FLOWS

Typical theory assumptions	N-body Practice:
Complete	Incomplete (Singularities: $r_{ij} \to 0$)
Compact phase space	Non-compact phase space ($r_{ij} \to \infty$)
All periodic orbits non-degenerate	All periodic orbits degenerate (Symmetries)

Goal: Make the N-body flow

(A) Complete: remove collision singularities!
Regularize binaries. Blow up triples (& higher?)

(B) Symmetry-free: remove symmetries by symplectic reduction.

(C) Live on a compact space:
add boundaries at the ends corresponding to escape.

(A) and (B): done! for the planar 3 body problem.
Partial progress: spatial 3 body problem
& planar 4 body problem

(C): open

Joint w Rick Moeckel. U of Minn.

Partial History

(d, N) = (dim. of ambient space, Number of bodies)

(B): Reduction. Lagrange [1772] d arbitrary, N = 3.

(B): Symplectic Reduction. Meyer; Marsden-Weinstein [1974]

(A): Regularization method d=2. Levi-Civita [1921]

for binary collisions. Initially: perturbed Kepler

(A) and (B)! Lemaitre[1954]: d=2, N=3;

& d= 3, N=3 but w/ coord. singularities at collinearity

(A): Regularization method d=3. Kuustanheimo-Steifel [1965]

(B): Regularization, N arbitrary (& democratic). Heggie [1970]. d=2 & 3

(A): Blow-up Method. McGehee [1974]

for triple and higher collisions.

(C): Partial compactification of infinity: C. Robinson [1984]

Partial Results:

(d, N) = (dim. of ambient space, Number of bodies)

(2, 3): planar 3-body problem: (A) and (B) Done. (arXiv: (RM)^2)

$$\text{regularized reduced phase space} = \mathrm{T}^*(\mathbb{CP}^1) \times T^*([0,\infty))$$

regularized shape sphere

size r

symp. form: `twisted' when ang. mom. nonzero

$$r^2 = I = \frac{\sum m_i m_j r_{ij}^2}{\sum m_i}$$

(2, 4): planar 4-body problem: (A) and (B) in progress (looks good)

$$\text{regularized reduced phase space} = \mathrm{T}^*(K3) \times T^*([0,\infty))$$

regularized shape space

size r

symp. form: `twisted' when ang. mom. nonzero

(3, 3): spatial 3-body problem: in progress (problematic)

$$\text{regularized reduced phase space} = \mathrm{T}^*(\mathbb{CP}^2) \times T^*([0,\infty)) \times_f \mathcal{O}$$

regularized shape space

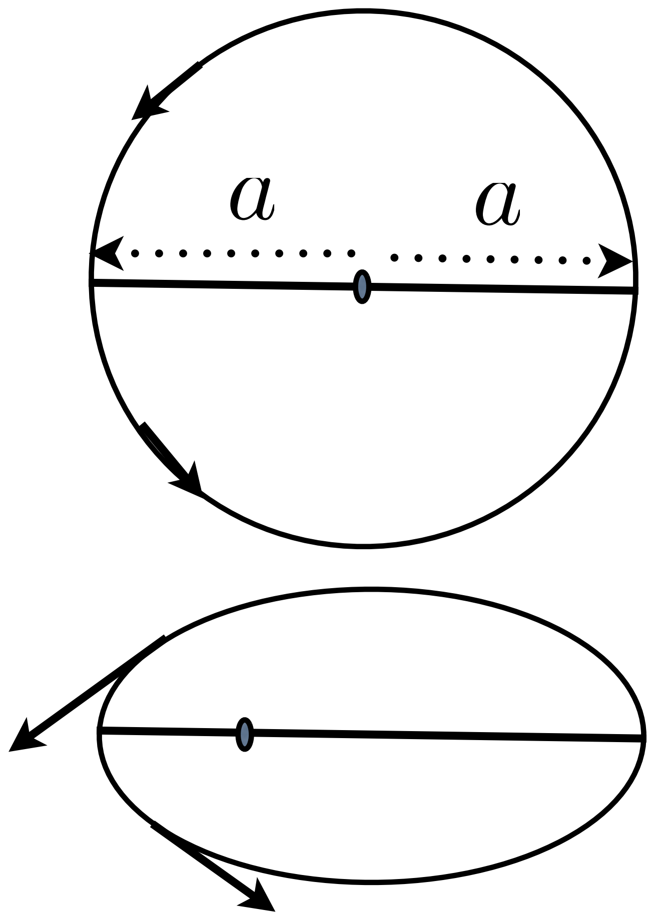
size r

fiber product over reg. shape space

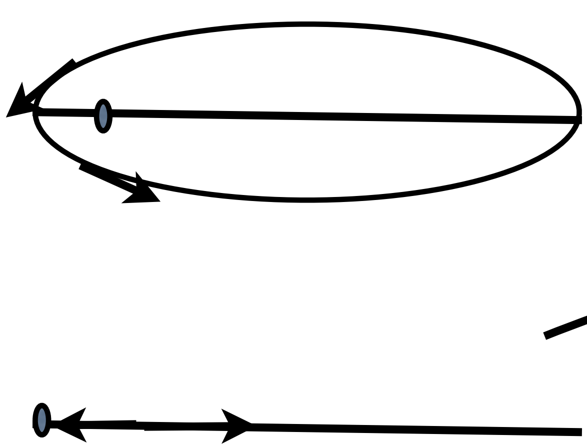
two-sphere bundle over reg. shape space fiber:instantaneous Euler rigid body

(A):Levi-Civita Regularization. d = 2

$$H_K = \frac{1}{2}|\dot{q}|^2 - \frac{1}{|q|} = \frac{-1}{2a} \qquad ; q \in \mathbb{C} = \mathbb{R}^2$$



Kepler: $\ddot{q} = \dfrac{-\beta q}{|q|^3}$



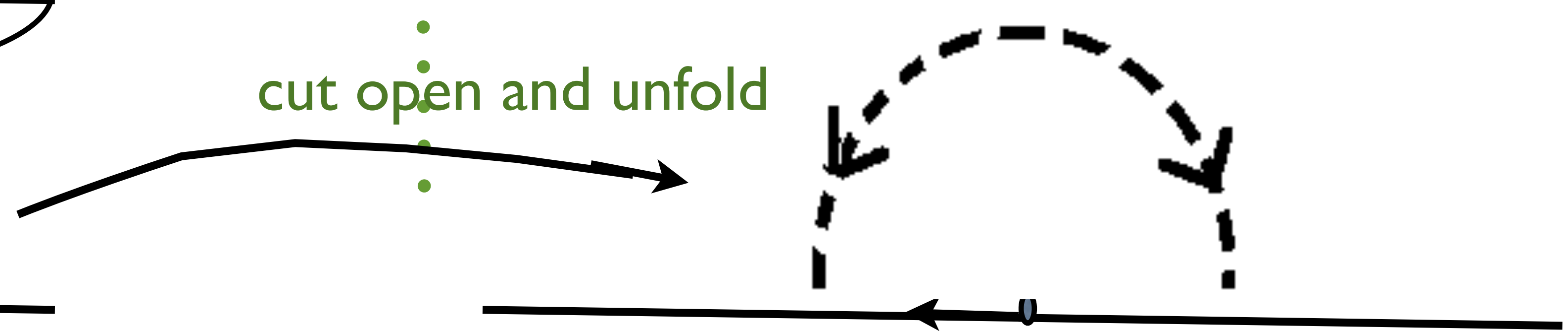
$$z^2 = q$$

2:1 branched cover

$$\frac{d}{d\tau} = |q|\frac{d}{dt}$$

slow time down as approach collision q=0

$$\implies \frac{d^2 z}{d\tau^2} = H_K z$$ harm. oscillator!



Derivation of L-C using Jacobi-Maupertuis [J-M]

$$ds^2_{JM} = 2(H - V)ds^2_{Kin} \iff \text{ solutions to Newton's eq. w Energy } H$$

so:

$$ds^2_{JM,Kep} = 2(\frac{-1}{2a} + \frac{1}{|q|})|dq|^2 \qquad \iff \text{ Kepler. with Energy } \tfrac{-1}{2a}$$

$q = z^2 \implies dq = 2zdz$:L.C. var. change

$$= 2(\frac{-1}{2a} + \frac{1}{|z|^2})4|z|^2|dz|^2$$

$$= 2(\frac{-4|z|^2}{2a} + 4)|dz|^2$$

$$E = 4, \qquad \omega = \frac{2}{\sqrt{a}}$$

$$= 2(E - \frac{\omega^2|z|^2}{2})|dz|^2$$

$$= ds^2_{JM,Harm}$$

(2,3) CASE

three Levi-Civita squaring maps:

$$z_{ij}^2 = Q_{ij}$$

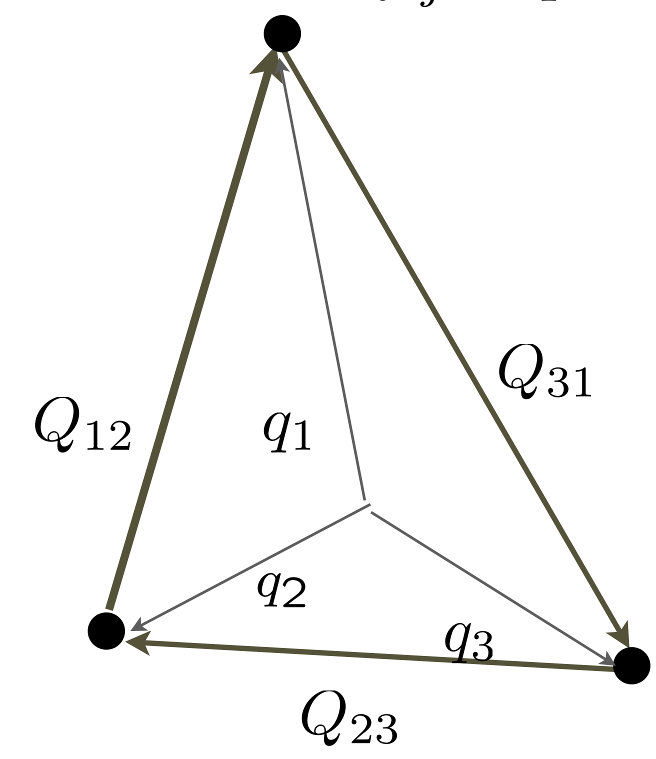
& the time change:

$$\frac{d}{d\tau} = r_{12} r_{23} r_{31} \frac{d}{dt}$$

(A): reg.

regularizes all binary collisions !

$$Q_{ij} = q_i - q_j = -Q_{ji}$$



$$Q_{12} + Q_{23} + Q_{31} = 0$$

$$\Longrightarrow \quad z_{12}^2 + z_{23}^2 + z_{31}^2 = 0$$

(B): reduce

Regularized shape space= $\{[z_{12}, z_{23}, z_{31}] \in \mathbb{CP}^2 : z_{12}^2 + z_{23}^2 + z_{31}^2 = 0\}$

a conic in the complex projective plane

$Q_{ij} = z_{ij}^2$

1

$S^2 = \mathbb{CP}^1 =$ standard shape sphere= $\{[Q_{12}, Q_{23}, Q_{31}] \in \mathbb{CP}^2 : Q_{12} + Q_{23} + Q_{31} = 0\}$

a complex projective line in the complex projective plane

(*) recall: homogeneous coordinates on complex projective n-space

$$[Z_0, Z_1, \ldots, Z_n] = [\lambda Z_0, \lambda Z_1, \ldots, \lambda Z_n],$$
$$\lambda \in \mathbb{C}, \lambda \neq 0, (Z_0, Z_1, \ldots, Z_n) \neq (0, 0, \ldots, 0)$$

(*) vector→ homogeneous coordinates implements reduc. by rotation & scaling

Well-known: conic in $\mathbb{CP}^2 \cong \mathbb{CP}^1$
Explicit map: $\mathbb{CP}^1 = \{[x_1, x_2]\} \to$ Our Conic $= \{z_{12}^2 + z_{31}^2 + z_{23}^2 = 0\}$ by:

$$z_{12} = 2ix_1x_2 \qquad z_{31} = x_1^2 + x_2^2 \qquad z_{23} = i(x_1^2 - x_2^2)$$

So $\mathbb{CP}^1 =$ regularized shape sphere. To visualize...

Combine w/

Affine coordinates: $v = \frac{x_2}{x_1} \in \mathbb{C}\{\infty\}$

or Stereo. projection: $c = stereo(x_1, x_2) \in S^2 \subset \mathbb{R}^3$

Use binary collisions as landmarks:

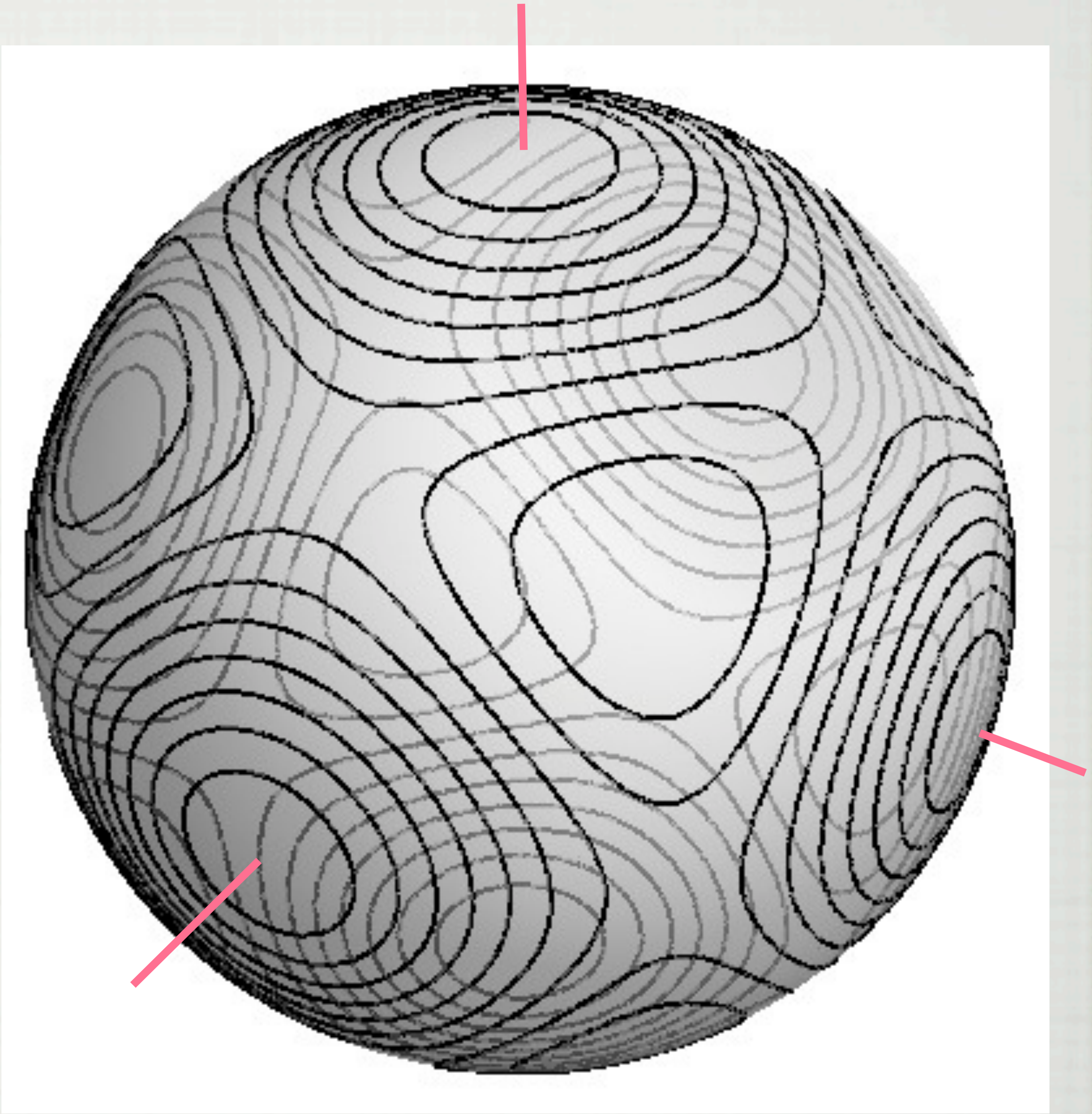
$$0 = r_{12} = |Q_{12}| = |z_{12}^2| = |2x_1x_2|^2 \implies [x_1, x_2] = [1, 0] \text{ or } [0, 1]$$
$$0 = r_{31} = |Q_{31}| = |z_{31}^2| = |x_1^2 + x_2^2|^2 \implies [x_1, x_2] = [1, i] \text{ or } [0, -i]$$
$$0 = r_{23} = |Q_{23}| = |z_{23}^2| = |x_1^2 - x_2^2|^2 \implies [x_1, x_2] = [1, 1] \text{ or } [0, -1]$$

Regularized Shape Sphere -- round version after stereographic projection

Coordinates $(c_1, c_2, c_3) \in \mathbb{R}^3$.
Can choose projection so

$$\rho_{12} = c_1^2 + c_2^2$$
$$\rho_{31} = c_3^2 + c_1^2$$
$$\rho_{23} = c_2^2 + c_3^2$$

$$\rho_{ij} = r_{ij}/\sqrt{I}$$



Binary collisions are on coordinate axes.

$$\rho_{12} = 0 \implies c_1 = c_2 = 0.$$

Collinear shapes on coordinate planes.

$$\rho_{12} = \rho_{31} + \rho_{23} \implies c_3 = 0.$$

Octahedral symmetry -- imagine an octahedron inflated to become round.

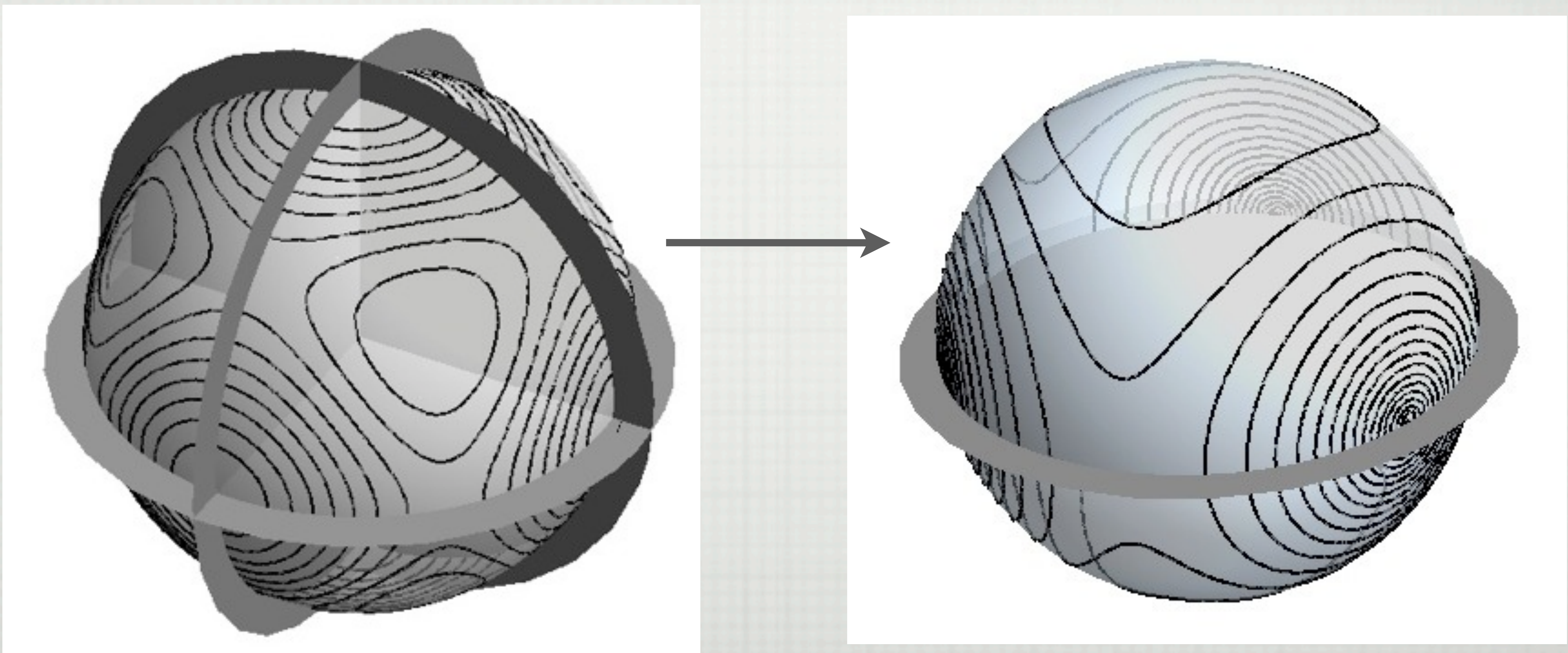
Lemaitre's Conformal Map: $\phi : \mathbb{C}^3 \to \mathbb{C}^3 \qquad X_{ij} = z_{ij}^2$

induces

$$\phi_{pr} : P(\mathcal{C}) \to P(\mathcal{W})$$

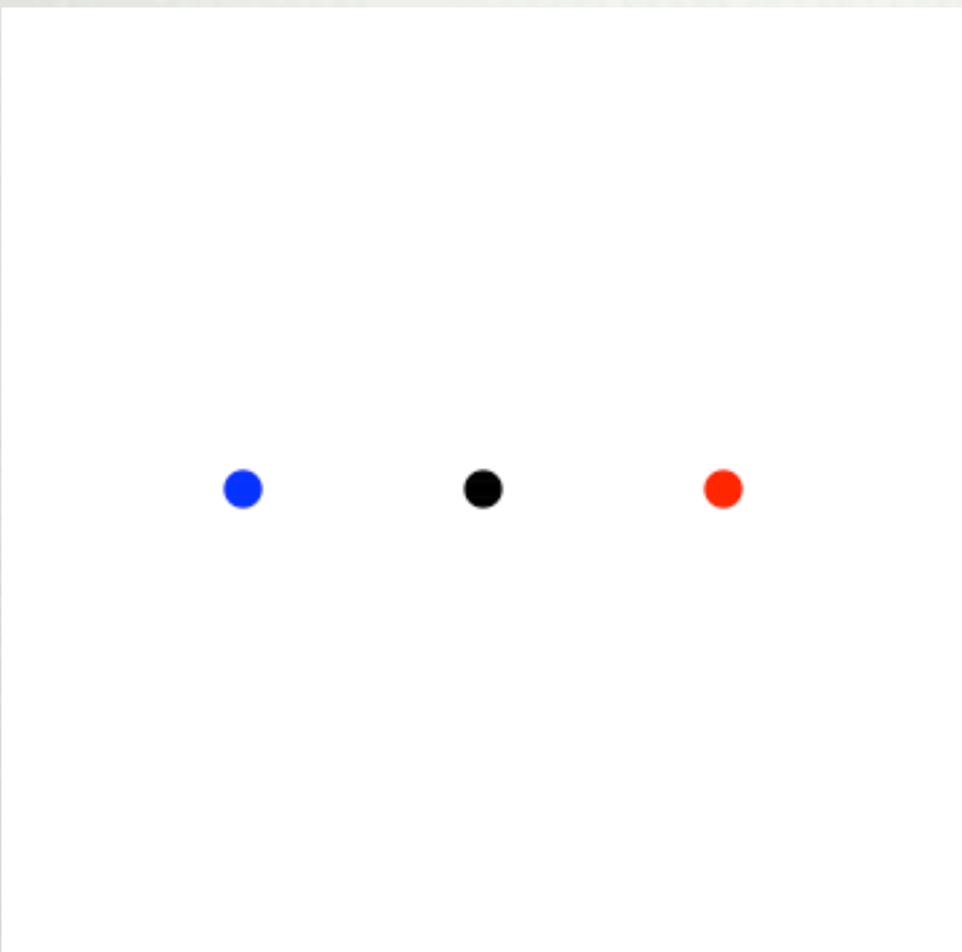
between regularized to unregularized shape spheres.

- **four-to-one cover branched over the binary collisions**
- **each octant of regularized sphere maps to a hemisphere**
- **behaves like the squaring map near the six regularized binary collision points**



Some Three-Body Orbits in the Regularized Reduced Configuration

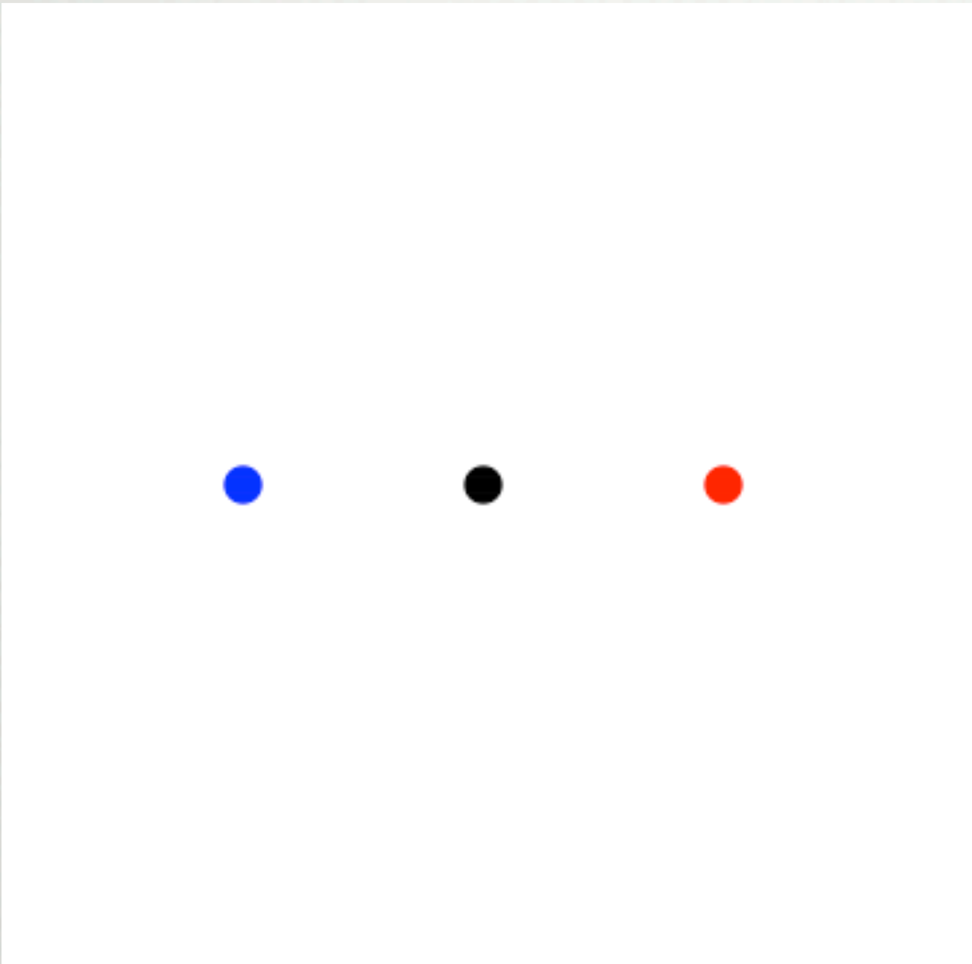
Figure-eight orbit

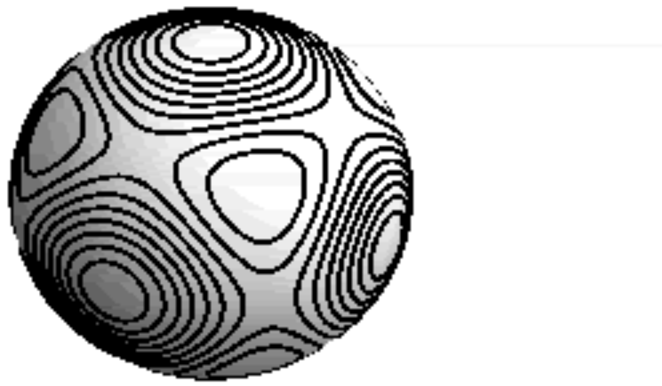


The orbit in regularized shape space is remarkably simple!

Some Three-Body Orbits in the Regularized Reduced Configuration

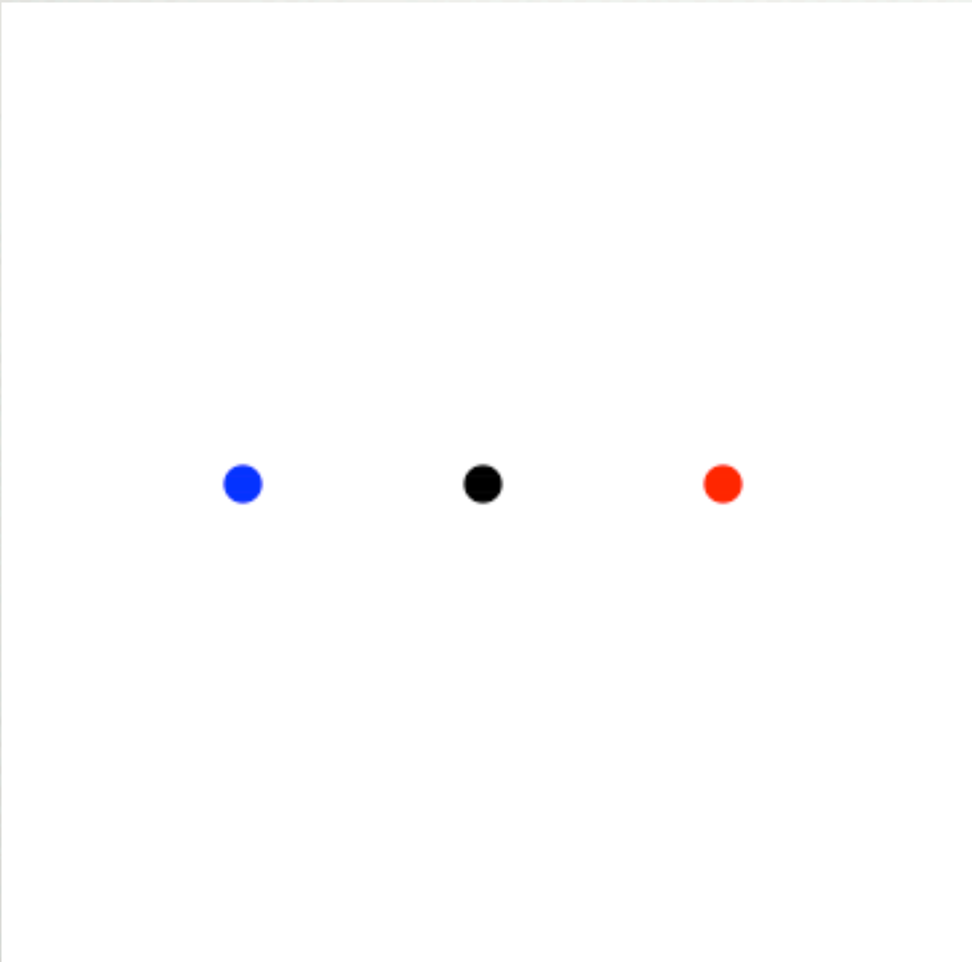
Figure-eight orbit





The orbit in regularized shape space is remarkably simple!

vs orbits plotted in usual reduced (shape) space



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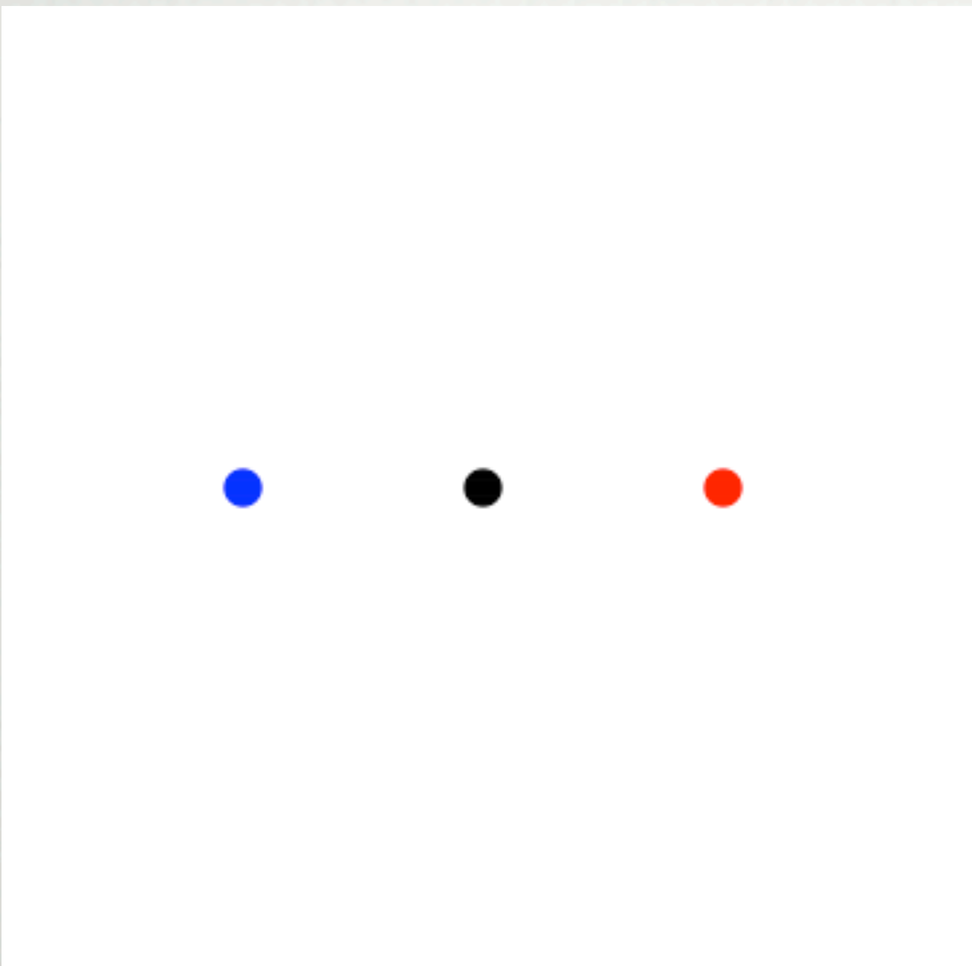
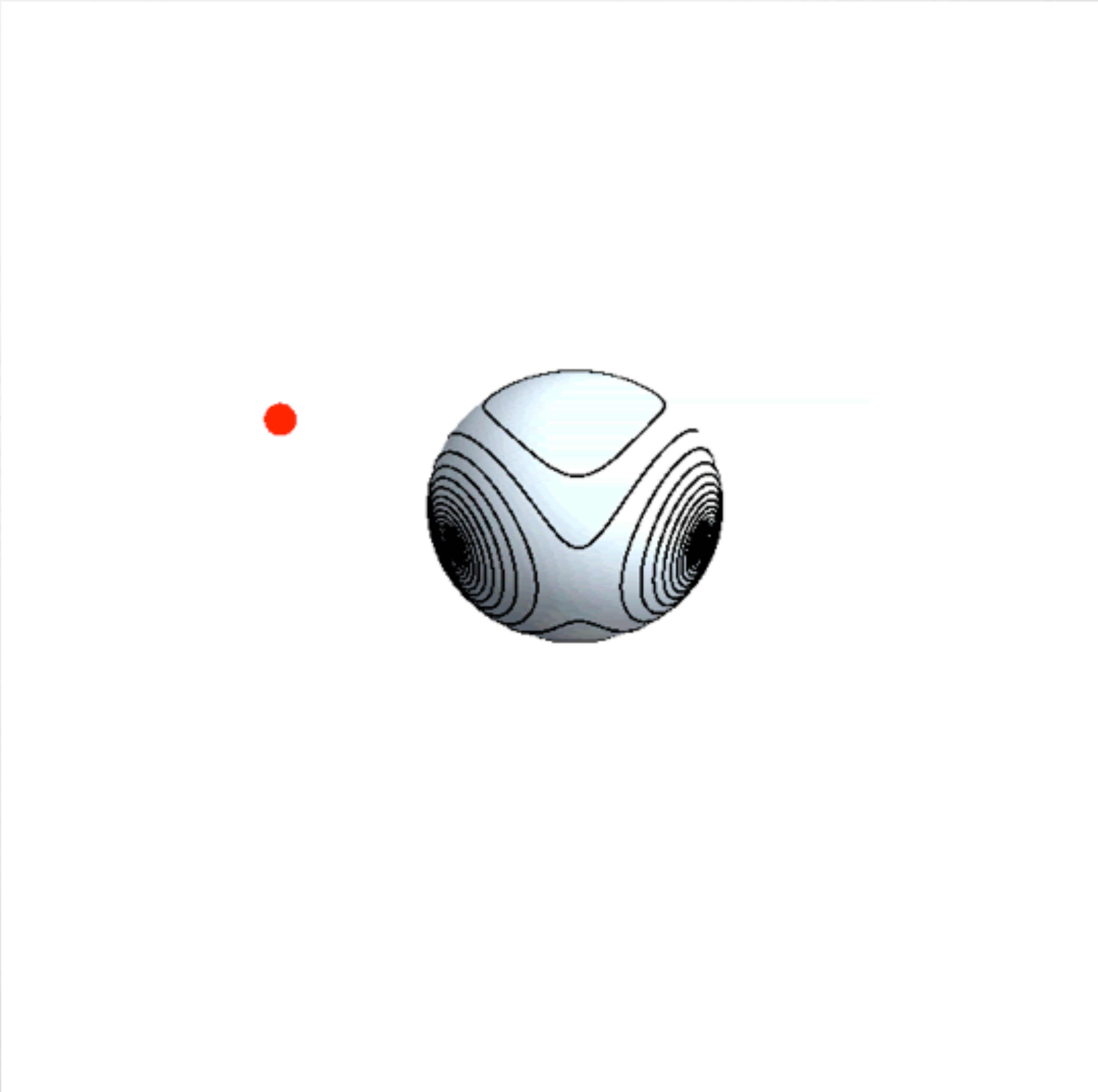




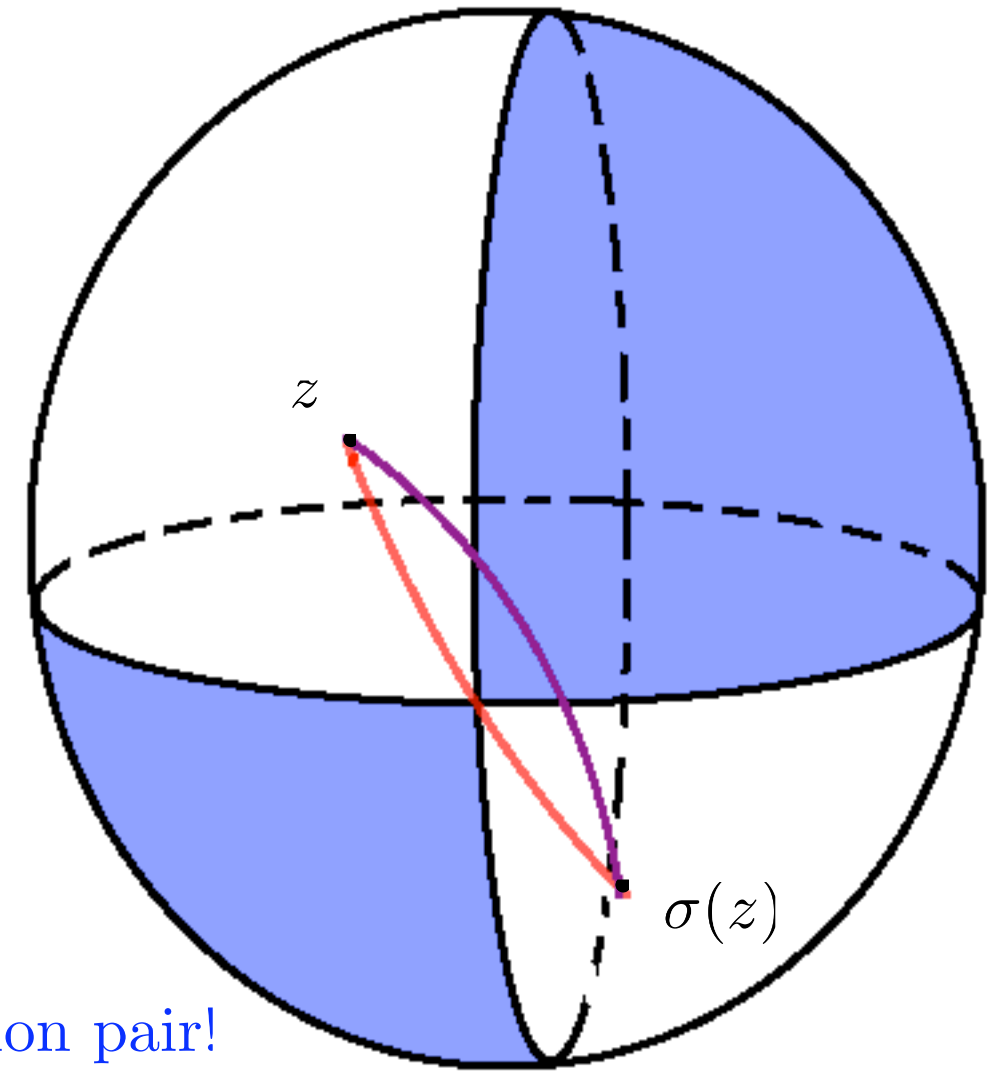
Regularizing map induces $4 = 2^2$ new symmetries

$$\sigma : z_{ij} \mapsto \pm z_{ij}$$

which all cover the identity on original space, since $z_{ij}^2 = q_{ij}$

$\Longrightarrow$ variants of brake or italian symmetry: $\sigma(z(-t)) = z(t)$

Interior binary collisions
no longer excluded
for (J-M) minimizers !



fixed points of σ: binary collision pair!

Guide to computing the reduced, regularize, blown-up dynamical equations

(*) (1) Reduce by translation

(*) (2) Separate size and shape

(3) Reduce by rotation : symp. reduction + homogeneous coordinates
requires fixing of the total angular momentum

(4) Compute the (co)metric [kinetic energy]in new coordinates

*** hardest work here***

(*) (5)L-C regularize (squaring map) these homogeneous coordinates
requires fixing of the the total energy

(6) McGehee blow-up

(1) Reduce by translation **[d = 2]**

Relative position coordinates

$$Q_{ij} = q_i - q_j = -Q_{ji} :$$

components of linear map $L : \mathbb{C}^N \to \mathbb{C}^{\binom{N}{2}}$, $L(q) = Q$.

with image(L) $\cong$(config. space)/(translations). $\cong \mathbb{C}^{N-1}$

dual map L^*: components $p_i = \sum_j P_{ij}$

$$H(q,p) = K(p) - U(q) = \left(\frac{|p_1|^2}{2m_1} + \dots\right) - \left(\frac{m_1 m_2}{|q_1 - q_2|} + \dots\right)$$

$$= K(P) - U(Q) = \left(\frac{|\sum P_{1j}|^2}{2m_1} + \dots\right) - \left(\frac{m_1 m_2}{|Q_{12}|} + \dots\right)$$

---

(2): Separate size and shape:

size $r = |Q|$ with $r^2 = I = \langle Q, Q \rangle = \frac{\sum m_i m_j |Q_{ij}|^2}{\sum m_i}$

Shape: $[Q] = [Q_{12}, \dots, Q_{N-1,N}] \in \mathbb{CP}^{N-2} \subset \mathbb{CP}^{\binom{N}{2}}$

$Q_{ij} + Q_{jk} + Q_{ki} = 0$

(3): Reduce by rotation

[d = 2]

scaling and rotation:

$$Q_{ij} \mapsto kQ_{ij}, P_{ij} \mapsto \frac{1}{\bar{k}} P_{ij}, \qquad k \in \mathbb{C} \setminus 0.$$

has momentum map:

$$\sum \bar{P}_{ij} Q_{ij} = p_r + i\,\mu \qquad = \Phi(Q, P)$$

SIZE MOMENTUM (p_r)

ANGULAR MOMENTUM (μ)

Momentum shift trick:
Take particular solution $P_{ij} = \Gamma_{ij}(Q)$ to $\Phi(Q, P) = 1$

CONNECTION ($\Gamma_{ij}(Q)$)

Substitute

$$P_{ij} = (p_r - i\mu)\Gamma_{ij} + Y_{ij}, \qquad \Phi(Q, Y) = 0$$

Yields general solution P to

$$\Phi(Q, P) = p_r + i\mu$$

Reduce by rotation... ct'd..

[d = 2]

Defines map $(Q, Y) \to (Q, (p_r - i\mu)\Gamma(Q) + Y)$

from 0-level of momentum map to level $p_r + i\mu$

inducing isomorphism

$$(\mathcal{P} \setminus \{0\})/S^1 \cong \mathbb{R}^+ \times \mathbb{R} \times T^*\mathbb{CP}^{N-2} \times \mathbb{R}$$

r $\quad$ p_r $\quad$ [Q; Y] $\quad$ μ

where $\mathcal{P} = (Q, P)$ phase space $(\cong T^*\mathbb{C}^{N-1})$

ANG. MOMENTUM

(4): Compute kinetic energy in new coord (hard work)

& so the total energy

$$K_\mu = \frac{1}{2}(p_r^2 + \frac{1}{r^2} K_{shape}([Q, Y]) + \frac{\mu^2}{r^2})$$

$$U = \frac{1}{r} \sum \frac{m_i m_j}{\rho_{ij}}$$

FUBINI-STUDY

$\rho_{ij} = \frac{|Q_{ij}|}{r}$ = normalized distance

$$H_\mu = K_\mu - U$$

WARNING: Eqns NOT canonical. Curvature term: $\Omega = d\Gamma$

)

(5): Apply Levi-Civita squaring transformation: **[d = 2]**

$$z_{ij}^2 = Q_{ij} \;,\quad \frac{d}{d\tau} = f\frac{d}{dt} \quad,\quad f = \prod_{i<j} \rho_{ij}$$

, OR $f = \prod_{i<j} \rho_{ij}/(\sum \rho_{ij})^{\binom{N}{2}} = \prod_{i<j} r_{ij}/(\sum r_{ij})^{\binom{N}{2}}$ **, OR...**

Use Poincaré trick for time reparam. by factor f at const. energy $H = E$

$$\tilde{H}_\mu = f(H_\mu - E)$$

Key to non-singularity at binary collisons:

$$fU = \frac{1}{r} \sum_{ij} m_i m_j \prod_{k\ell \neq ij} \rho_{k\ell}$$

not singular at simple binary collisons: $\rho_{ij} = 0$

$(r, p_r, [Z, \eta]) \mapsto (r, p_r, [Q, P])$ induced by $[Z] \mapsto [Q]; Q_{ij} = Z_{ij}^2$

$$\mathrm{K} \subset \mathbb{CP}^{\binom{N}{2}} \to \mathbb{CP}^{N-2} \subset \mathbb{CP}^{\binom{N}{2}}$$

pulled back triangle constraints (→ K) triangle constraints (→ $\mathbb{CP}^{N-2}$)

(6): McGehee blow-up: planar 3 body; eg

McGehee time τ. $\frac{d}{d\tau} = {}' = r^{\frac{3}{2}} \frac{d}{dt}$ (*)
Rescaled size momenta $v = \frac{p_r}{r^{1/2}}$.
Rescaled reg. shape momenta $\alpha = r^{1/2} Y$.
Reg. shape variables $[Z]$ unchanged. $z = x_2/x_1$ affine shape coord.

$$r' = \lambda(z) v r$$

$$v' = -\frac{1}{2}\lambda(z) v^2 + 2\tilde{K} - W(z)$$

$$\tilde{\mu}' = -\frac{1}{2}\lambda(z) v \tilde{\mu}$$ ← Normalized ang. mom. $\tilde{\mu} := \frac{f(r)}{r^2}\mu$ (1)

$$z' = (1 + |z|^2)^2 \alpha$$

$$\alpha' = \lambda(z) v\alpha - \tilde{K}_z + W_z + rh\lambda_z(z) + 2i\tilde{\mu}\lambda(z)\alpha$$ **MAG. TERM**

$\lambda = \frac{4Mm_1m_2m_3r_{12}r_{31}r_{23}(r_{12}+r_{31}+r_{23})}{I^2}$: conformal factor

kinetic : $2\tilde{K} = \lambda v^2 + \lambda \tilde{\mu}^2 + \frac{1}{2}(1 + |z|^2)^2 |\alpha|^2$.

potential $W(z) = \frac{\tilde{r}}{(1+|z|^2)^6}(m_1m_2\tilde{\rho}_{31}\tilde{\rho}_{23} + m_1m_3\tilde{\rho}_{12}\tilde{\rho}_{23} + m_2m_3\tilde{\rho}_{12}\tilde{\rho}_{31})$

normalized distances: $\rho_{ij} = r_{ij}/r = \tilde{\rho}_{ij}/\tilde{r}$,

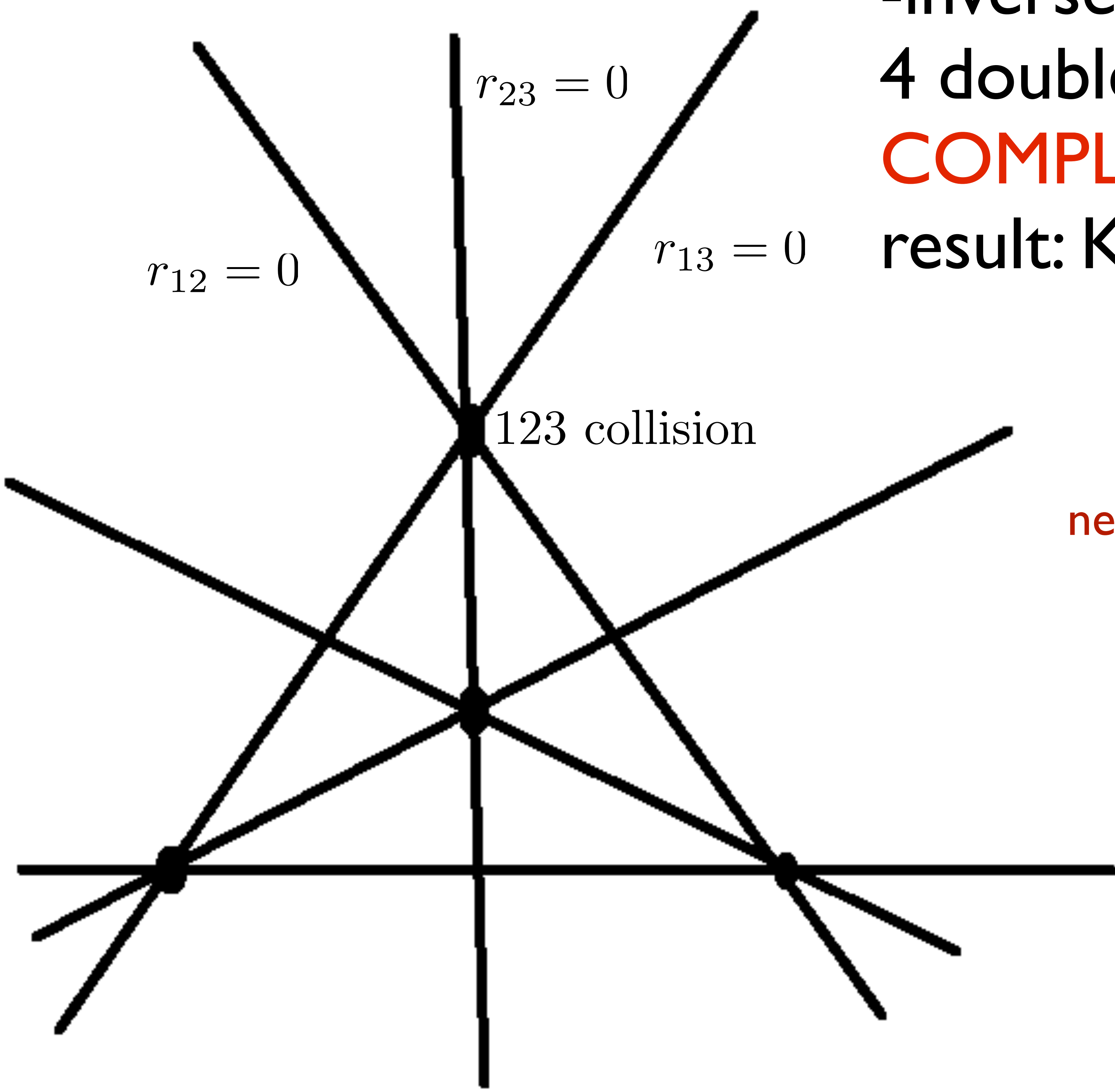
$$\tilde{\rho}_{12} = 4|z|^2, \quad \tilde{\rho}_{31} = |1 + z^2|^2, \quad \tilde{\rho}_{23} = |1 - z^2|^2$$

$$\tilde{r}^2 = \tilde{I} = \frac{m_1m_2\tilde{\rho}_{12}^2 + m_1m_3\tilde{\rho}_{31}^2 + m_2m_3\tilde{\rho}_{23}^2}{m_1 + m_2 + m_3}$$

(*) alternative time scaling: $f(r) = \left(\frac{r}{r+1}\right)^{\frac{3}{2}}$, better behavior for large r

Monday, January 14, 2013

planar 4-body



6 double collision lines.
2:1 branched cover each.
result: 32:1 branched cover of $\mathbb{CP}^2$
-inverse image of each of the
4 double collision points: a cone point
COMPLEX blow up (a la alg. geom).
result: K3 = reg. reduced shape space

next talk: complex blowup for triple collisions

J-M remarks.

A. $d = 2, N = 3, J = 0, H = -h < 0, m_1 = m_2 = m_3$:

$\implies$ J-M formulation takes form (roughly):

$$ds^2_{JM,reg.} = 2(-h(\hat{z}^2_{12}|\hat{z}_{23}|^2|\hat{z}_{31}|^2) + \tfrac{1}{Mr^2}(|\hat{z}_{23}|^2|\hat{z}_{13}|^2 + |\hat{z}_{12}|^2|\hat{z}^2_{32} + |\hat{z}_{21}|^2|\hat{z}_{31}|^2 ...))ds^2$$

with $\hat{z}_{ij} = z_{ij}/\sqrt{|z_{12}|^2 + |z_{23}}^2 + |z_{13}|^2$

B. Amusing toy case to see how a
regularized J-M solution can minimize
while its unregularized projection does not

Kepler: 0-energy: i.e PARABOLIC

$$ds^2_{JM} = \frac{1}{r}|dq|^2 \qquad , q = z^2 \implies dq = 2zdq, r = |q|^2$$

$$ds^2_{JM,reg} = 4|dz|^2 : \text{ EUCLIDEAN!}$$

Polar coordinates: $q = re^{i\theta}$

$$ds^2_{JM} = \frac{1}{r}(dr^2 + r^2 d\theta^2) = (\frac{dr}{\sqrt{r}})^2 + r d\theta^2$$

Change Variables: $u = \frac{1}{2} r^{1/2}$

$$\implies ds^2_{JM} = du^2 + 4u^2 d\theta^2$$

Again locally Euclidean, but origin a cone point!
Opening cone angle : 4π

Partial Results:

OVERFLOW

(d, N) = (dim. of ambient space, Number of bodies)

$\mathcal{P}(d,n) =$ regularized, reduced, blown-up phase space

$$= T^*(X(d,n)) \times T^*([0,\infty)) \times_f \mathcal{O}$$

$X(d,n) =$ regularized shape space; maybe blown up

$[0,\infty) =$ size space parameter $\sqrt{I}$ where $; I = \frac{\sum m_i m_j r_{ij}^2}{\sum m_i}$

$$d = 2 \implies \mathcal{O} = \emptyset$$

$$X(2,3) = \mathbb{CP}^1$$

$$X(2,4) = K3$$

(d, N) = (3,3): partial progress

$$\mathcal{P}(3,3) = T^*(\mathbb{CP}^2) \times T^*([0,\infty)) \times_f \mathcal{O}$$

with $\mathcal{O} \to^{\mathbb{CP}^1} \mathbb{CP}^2$