Classification of Leavitt path algebras:

How to use tools from the classification of C*-algebras in the Algebra setting

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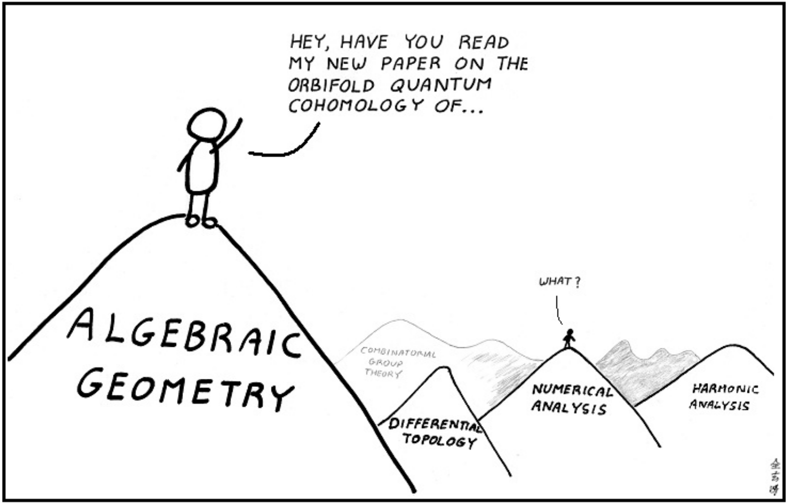

The Landscape of Modern Mathematics

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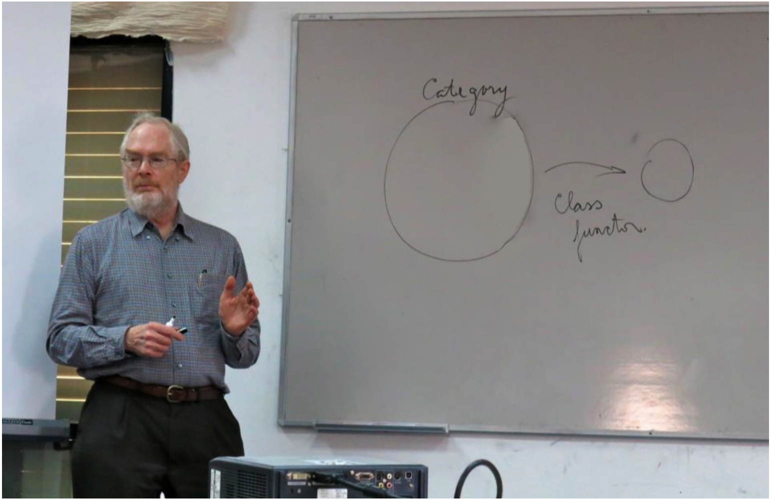

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Morita equivalence in the category of C^*-algebras is often called "strong Morita equivalence" to distinguish it from Morita equivalence of rings. Also, strong Morita equivalence for C^*-algebras is the same as being stably isomorphic.

$$A \sim_{SME} B \iff A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$$

There are many important results from the classification program, but two major accomplishments are:

Theorem (Elliott's Theorem)

If A and B are C^*-algebras that are AF (i.e., direct limits of finite-dimensional algebras), then $A \sim_{SME} B$ if and only if

$$(K_0^{\text{top}}(A), K_0^{\text{top, }+}(A)) \cong (K_0^{\text{top}}(B), K_0^{\text{top, }+}(B)).$$

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Theorem (Kirchberg-Phillips Classification Theorem)

If A and B are purely infinite, simple, separable, nuclear C^*-algebras that are in the bootstrap class to which the UCT applies, then $A \sim_{SME} B$ if and only if

$$K_0^{\text{top}}(A) \cong K_0^{\text{top}}(B) \textit{ and } K_1^{\text{top}}(A) \cong K_1^{\text{top}}(B).$$

Note: Many purely infinite, simple C^*-algebras fall into this class.

Can a similar classification be done for algebras?

The proof of Elliott's Theorem works for ultramatricial algebras over a field K (i.e., algebraic direct limits of finite-dimensional K-algebras), and can be used to show the ordered K_0-group is a complete Morita equivalence invariant for ultramatricial algebras. (Indeed, Elliott showed this in his original paper.)

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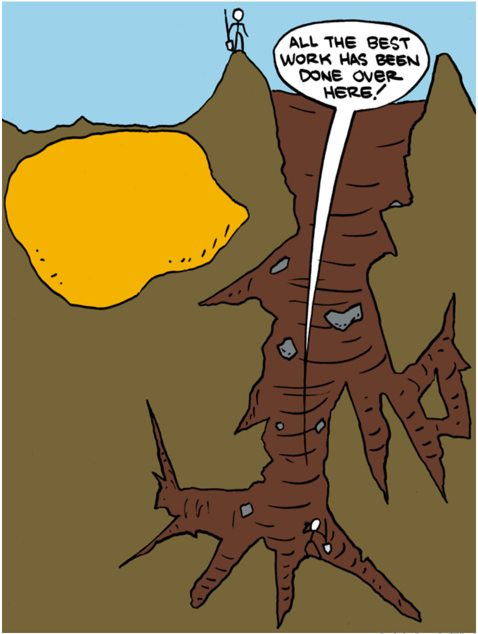
What about the Kirchberg-Phillips Classification Theorem? Can a similar result be obtained for purely infinite algebras? Can we use algebraic K-theory in place of topological K-theory? (We may need the higher algebraic K-groups . . . these may be harder to compute . . .)

Definition: A ring R is *purely infinite* if every left ideal of R contains an infinite idempotent.


ALL THE BEST WORK HAS BEEN DONE OVER HERE!

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Answer: Cuntz-Krieger C^*-algebras are simple, purely infinite, and were originally classified by techniques different from those used in the Kirchberg-Phillips theorem. Also, the K-groups are easy to compute.

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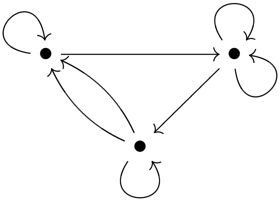
Cuntz-Krieger algebras were originally associated to finite square matrices, but the modern approach is to formulate them in terms of graphs.

A (directed) graph $E = (E^0, E^1, r, s)$ consists of a set of vertices E^0, a set of edges E^1, and maps $r : E^1 \to E^0$ and $s : E^1 \to E^0$ identifying the range and source of each edge. (We'll allow infinite graphs, but assume the vertex set and edge set are countable.)



Definition (Graph C^*-algebras)

If E is a graph, the *graph C^*-algebra* $C^*(E)$ is the universal C^*-algebra generated by a *Cuntz-Krieger E-family*, which consists of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries with mutually orthogonal ranges $\{s_e : e \in E^1\}$ satisfying

1. $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$
2. $p_v = \sum_{\{e \in E^1 : s(e) = v\}} s_e s_e^*$ for all $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$
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If E has no sinks and no infinite emitters, then

$$K_0^{\text{top}}(C^*(E)) \cong \operatorname{coker}\left(I - A_E^t : \bigoplus_{E^0} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z} \right)$$

and

$$K_1^{\text{top}}(C^*(E)) \cong \ker\left(I - A_E^t : \bigoplus_{E^0} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z} \right)$$

If E is a finite graph, put $I - A_E^t$ into Smith Normal Form.

$$I - A_E^t \quad \longleftrightarrow \quad \begin{pmatrix} d_1 & & & & & \\ & \ddots & & & & \\ & & d_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

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$$K_0^{\text{top}}(C^*(E)) \cong \text{coker}(I - A_E^t) \cong \mathbb{Z}_{d_1} \oplus \ldots \oplus \mathbb{Z}_{d_k} \oplus \mathbb{Z}^m$$

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Note: $K_1^{\text{top}}(C^*(E))$ is the free part of $K_0^{\text{top}}(C^*(E))$.

We expect that if E is a finite graph with no sinks or sources (and not a single cycle), then $C^*(E)$ is determined up to strong Morita equivalence by $K_0^{\text{top}}(C^*(E)) \cong \operatorname{coker}(I - A_E^t)$.

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This was proved by Cuntz and Krieger (and also relied on some work of Eilliott and of Rørdam) almost two decades before the Kirchberg-Phillips classification theorem. How was this accomplished?

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A large component of the proof used Symbolic Dynamics.

If E is a finite graph, the (two-sided) shift space X_E is the set

$$X_E := \{\dots e_{-2}e_{-1}e_0e_1e_2\dots \mid e_i \in E^1 \text{ and } r(e_i) = s(e_{i+1}) \text{ for all } i \in \mathbb{Z}\}$$

with the shift map $\sigma_E : X_E \to X_E$ given by $\sigma_E(x)_i = x_{i+1}$.

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We give the finite set of edges E^1 the discrete topology, so the infinite product

$$\prod_{\mathbb{Z}} E^1 = \dots E^1 \times E^1 \times E^1 \times \dots$$

is compact by Tychonoff's theorem. We then give $X_E \subseteq \prod_{\mathbb{Z}} E^1$ the subspace topology. The space X_E is closed (and hence compact).

The pair (X_E, σ_E) is a dynamical system.

Definition

The shift spaces (X_E, σ_E) and (X_F, σ_F) are conjugate if there exists a homeomorphism $\phi : X_E \to X_F$ with

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Definition

If X_E is a shift space, the *suspension flow* is the quotient space

$$SX_E := (X_E \times \mathbb{R})/\{(x,t) \sim (\sigma_E(x), t-1)\}.$$

There is a flow on SX_E induced by the flow ϕ_t on $X_E \times \mathbb{R}$ given by $\phi_t(x,s) = (x, s+t)$. The shift spaces (X_E, σ_E) and (X_F, σ_F) are said to be flow equivalent if there is a homeomorphism $h : SX_E \to SX_F$ carrying orbits of the flow on SX_E to orbits of the flow on SX_F and preserving the orientation.

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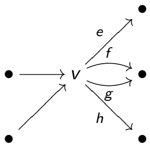
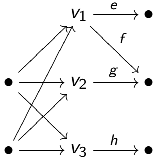
Conjugacy and Flow Equivalence are related to moves on the graphs.

Move (O): Outsplitting



Outsplitting
$\Longrightarrow$



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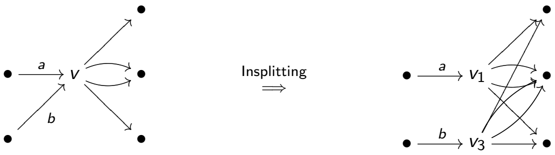
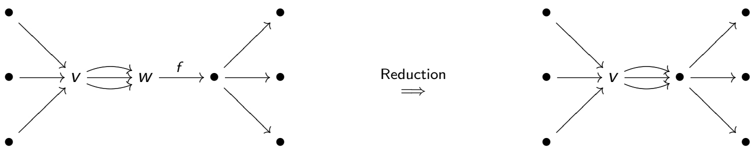
Move (I): Insplitting



$$r^{-1}(v) = \{a\} \cup \{b\}$$

Move (R): Reduction



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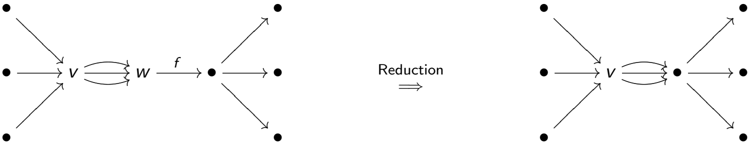

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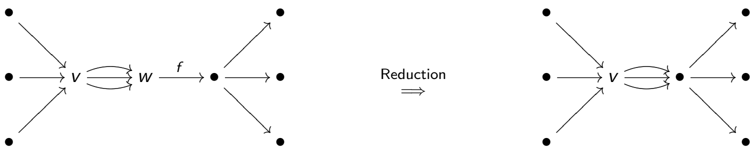

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Move (R) is also sometimes called the “Parry-Sullivan Move”.

For each move there is also an inverse move.

Inverse of Outsplitting is called Outamalgamation.
Inverse of Insplitting is called Inamalgamation.
Inverse of Reduction is called Delay.

Suppose E and F are finite, strongly connected graphs and neither is a single cycle.

Williams proved:

$$X_E \text{ is conjugate to } X_F \iff E \text{ can be transformed into } F \text{ via Moves (O), (I), and their inverses}$$

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Work of Parry and Sullivan together with work of Franks shows

$$X_E \text{ is flow equivalent to } X_F \overset{\text{Parry-Sullivan}}{\iff} E \text{ can be transformed into } F \text{ via Moves (O), (I), (R), and their inverses}$$

$$\overset{\text{Franks}}{\iff} \operatorname{coker}(I - A_E) \cong \operatorname{coker}(I - A_F) \text{ and } \operatorname{sgn}(\det(I - A_E)) = \operatorname{sgn}(\det(I - A_F))$$

Since $I - A_E$ and $I - A_F$ are finite matrices, their transposes have the same cokernels and determinants. Thus Franks' result can be restated as

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\begin{aligned}
& E \text{ can be transformed into } F \text{ via Moves (O), (I), (R), and their inverses} \\
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- Cuntz and Krieger showed the moves preserve strong Morita equivalence of the associated C^*-algebra. (This can be seen easily from the graphs, using the universal property.)

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The Cuntz Splice

Move (CS): Cuntz Splice

v is the base of two cycles

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Let E be a graph, and perform the Cuntz splice to obtain F.

$$A_F = \left(\begin{array}{cc|ccc} 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ \hline 0 & 1 & & & \\ 0 & 0 & & A_E & \\ \vdots & \vdots & & & \end{array}\right)$$

Then $K_0^{\text{top}}(C^*(E)) \cong K_0^{\text{top}}(C^*(F))$, but $\det(I - A_F^t) = -\det(I - A_E^t)$.

Work of Elliott together with work of Rørdam shows that the Cuntz splice preserves Morita equivalence of the associated C^*-algebra. However, unlike the other moves this cannot be shown explicitly, and relies on some "C^*-algebra magic".

Theorem (Cuntz and Krieger)

Suppose E and F are finite, strongly connected graphs and neither is a single cycle. Then $C^*(E)$ is strongly Morita equivalent to $C^*(F)$ if and only if $K_0^{\text{top}}(C^*(E)) \cong K_0^{\text{top}}(C^*(F))$.

Moreover, in this case one can transform E into F using moves (O), (I), (R), (CS), and their inverses.

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Proof:

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$\implies$ $C^*(E)$ strongly Morita equiv. to $C^*(F)$

□

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- The K-theory of $L_K(E)$ can be computed. In fact, if E is a finite graph with no sinks, then for any field K we have

$$K_0^{\mathrm{alg}}(L_K(E)) \cong \mathrm{coker}\left(I - A_E^t : \bigoplus_{E^0} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z} \right)$$

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Thus the best that can be accomplished is the following . . .

Theorem (Abrams, Louly, Pardo, and Smith)

Suppose E and F are finite, strongly connected graphs and neither is a single cycle. Also let K be any field. If

$$K_0^{\mathrm{alg}}(L_K(E)) \cong K_0^{\mathrm{alg}}(L_K(F)) \text{ and } \operatorname{sgn}(\det(I - A_E^t)) = \operatorname{sgn}(\det(I - A_F^t)),$$

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The hypotheses show

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Franks' theorem implies E can be turned into F via moves (O), (I), (R), and their inverses. These moves preserve Morita equivalence, so $L_K(E)$ is Morita equivalent to $L_K(F)$. □

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- $K_0^{\mathrm{alg}}(L_K(E))$ and $\mathrm{sgn}(\det(I - A_E^t))$ completely determine the Morita equivalence class of $L_K(E)$, and hence determine $K_n^{\mathrm{alg}}(L_K(E))$ for $n \in \mathbb{Z}$.

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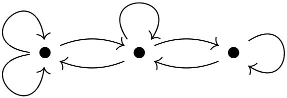
We cannot even answer this in the simplest case:

E_2

E_2^-



Is $L_K(E_2)$ Morita equivalent to $L_K(E_2^-)$? No one knows.

What about other unital, purely infinite graph C^*-algebras?

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Remember $C^*(E)$ is unital if and only if E has a finite number of vertices. We could consider infinite graphs with a finite number of vertices that are strongly connected. Their C^*-algebras are purely infinite and simple.

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We can't consider shift spaces here (the number of edges is infinite), but that's okay. Franks' result for finite, strongly connected graphs:

$$\begin{aligned} & E \text{ can be transformed into } F \text{ via Moves (O), (I), (R) and their inverses} \\ \iff & \operatorname{coker}(I - A_E^t) \cong \operatorname{coker}(I - A_F^t) \text{ and } \operatorname{sgn}(\det(I - A_E^t)) = \operatorname{sgn}(\det(I - A_F^t)) \end{aligned}$$

is a purely algebraic statement that does not rely on the notion of flow equivalence to state or prove.

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$$E^0_{\text{reg}} = \text{vertices of } E \text{ that emit a finite and nonzero number of edges}$$

$$E^0_{\text{sing}} = \text{vertices that emit infinitely many edges or no edges}$$

With respect to $E^0 = E^0_{\text{reg}} \cup E^0_{\text{sing}}$ we have

$$A_E = \begin{pmatrix} B_E & C_E \\ * & * \end{pmatrix}$$

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$$A_E = \begin{pmatrix} B_E & C_E \\ * & * \end{pmatrix}$$

where B_E and C_E have finite entries. Then

$$K_0^{\text{top}}(C^*(E)) \cong \text{coker}\left(\begin{pmatrix} I - B_E^t \\ -C_E^t \end{pmatrix} : \bigoplus_{E^0_{\text{reg}}} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z} \right)$$

and

$$K_1^{\text{top}}(C^*(E)) \cong \ker\left(\begin{pmatrix} I - B_E^t \\ -C_E^t \end{pmatrix} : \bigoplus_{E^0_{\text{reg}}} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z} \right)$$

If E is has a finite number of vertices, we can still put the matrix in Smith Normal Form.

$$\begin{pmatrix} I - B_E^t \\ -C_E^t \end{pmatrix} \qquad \longleftrightarrow \qquad \begin{pmatrix} d_1 & & & & & \\ & \ddots & & & & \\ & & d_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \\ 0 & \cdots & & & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & & & \cdots & 0 \end{pmatrix}$$

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Then

$$K_0^{\text{top}}(C^*(E)) \cong \mathbb{Z}_{d_1} \oplus \ldots \oplus \mathbb{Z}_{d_k} \oplus \mathbb{Z}^m \quad \text{and} \quad K_1^{\text{top}}(C^*(E)) \cong \mathbb{Z}^n.$$

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Also, the number of singular vertices is a Morita equivalence invariant:

$$|E^0_{\text{sing}}| = \text{rank}\, K_0^{\text{top}}(C^*(E)) - \text{rank}\, K_1^{\text{top}}(C^*(E)).$$

What about the moves?

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We have to make a few specifications about what is allowed for infinite graphs:

Move (O) can be performed at an infinite emitter, but when we partition outgoing edges, only one piece of the partition is allowed to have an infinite number of edges.

Move (I) can only be performed at a regular vertex.

Move (R) can only be performed at a regular vertex.

With these specifications, the moves still preserve Morita equivalence of the associated C^*-algebra.

Theorem (Sørensen)

Suppose E and F are strongly connected graphs that each have a finite number of vertices and an infinite number of edges. Then

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$$E \text{ can be transformed into } F \text{ via Moves (O), (I), (R) and their inverses}$$
$$\iff K_0^{\text{top}}(C^*(E)) \cong K_0^{\text{top}}(C^*(F)) \text{ and } K_1^{\text{top}}(C^*(E)) \cong K_1^{\text{top}}(C^*(F))$$

Note:

- The K_1^{top}-group is needed (as we would expect).
- The "sign of the determinant condition" disappears! (The matrices involved are not square, so we can't even take determinants.)
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In fact, this result implies . . .

Corollary

Suppose E is a strongly connected graph that has a finite number of vertices and an infinite number of edges, and if F is the graph obtained by performing the Cuntz splice to E, then F may be ontained by performing Moves (O), (I), (R), and their inverses to E.

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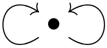
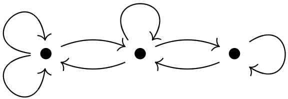
E_2 

E_2^- 

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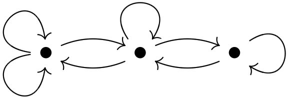
E_2  E_2^- 

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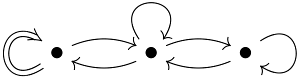
E_∞  E_∞^- 

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Suppose E and F are strongly connected graphs that each have a finite number of vertices and an infinite number of edges. Then the following are equivalent:

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(2) $K_0^{\text{top}}(C^*(E)) \cong K_0^{\text{top}}(C^*(F))$ *and* $K_1^{\text{top}}(C^*(E)) \cong K_1^{\text{top}}(C^*(F))$.

(3) $K_0^{\text{top}}(C^*(E)) \cong K_0^{\text{top}}(C^*(F))$ *and* $|E^0_{\text{sing}}| = |F^0_{\text{sing}}|$.

Moreover, in this case one can transform E into F using moves (O), (I), (R), and their inverses.

Efren Ruiz (University of Hawai'i at Hilo) and I have considered how we can use Sørensen's result

$$\begin{aligned} & E \text{ can be transformed into } F \text{ via Moves (O), (I), (R) and their inverses} \\ \Longleftrightarrow\ & K_0^{\mathrm{top}}(C^*(E)) \cong K_0^{\mathrm{top}}(C^*(F)) \text{ and } K_1^{\mathrm{top}}(C^*(E)) \cong K_1^{\mathrm{top}}(C^*(F)) \end{aligned}$$

to classify Leavitt path algebras.

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to classify Leavitt path algebras.

Problem: Algebraic K-theory not the same as the topological K-theory.

With respect to $E^0 = E^0_{\text{reg}} \cup E^0_{\text{sing}}$ we have

$$A_E = \begin{pmatrix} B_E & C_E \\ * & * \end{pmatrix}$$

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where B_E and C_E have finite entries. If K is any field, then

$$K_0^{\text{alg}}(L_K(E)) \cong \text{coker}\left(\begin{pmatrix} I - B_E^t \\ -C_E^t \end{pmatrix} : \bigoplus_{E^0_{\text{reg}}} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z} \right)$$

and

$$K_1^{\text{alg}}(L_K(E)) \cong \ker\left(\begin{pmatrix} I - B_E^t \\ -C_E^t \end{pmatrix} : \bigoplus_{E^0_{\text{reg}}} \mathbb{Z} \to \bigoplus_{E^0} \mathbb{Z} \right)$$

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We see the field matters in $K_1^{\text{alg}}(L_K(E))$.

Efren Ruiz and I considered a certain property of fields.

Definition

An abelian group has no free quotients if no nonzero quotient of the group is a free abelian group.

Theorem

The following are equivalent:

(1) *G has no free quotients.*

(2) *G is not a direct sum of a free abelian group and another group.*

(3) $\mathrm{Hom}_{\mathbb{Z}}(G, F) = \{0\}$ *for every free abelian group F.*

Definition

A field K has no free quotients if the abelian group $(K^{\times}, \cdot)$ has no free quotients.

Example

The following are examples of fields with no free quotients:

- $\mathbb{C}$
- $\mathbb{R}$
- All finite fields.
- All algebraically closed fields.
- All fields that are perfect with characteristic $p > 0$.
- All fields K such that $(K^\times, \cdot)$ is a torsion group.

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Example

$\mathbb{Q}$ is an example of a field with free quotients:

$$\mathbb{Q}^\times \cong \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots.$$

Theorem (Ruiz and T)

Let E and F be graphs, and let K be a field with no free quotients.

(1) *If* $K_0^{\text{alg}}(L_K(E)) \cong K_0^{\text{alg}}(L_K(F))$, *then* $K_0^{\text{top}}(C^*(E)) \cong K_0^{\text{top}}(C^*(F))$.

(2) *If* $K_1^{\text{alg}}(L_K(E)) \cong K_1^{\text{alg}}(L_K(F))$, *then* $K_1^{\text{top}}(C^*(E)) \cong K_1^{\text{top}}(C^*(F))$.

(3) *If* $K_0^{\text{alg}}(L_K(E)) \cong K_0^{\text{alg}}(L_K(F))$ *and* $K_1^{\text{alg}}(L_K(E)) \cong K_1^{\text{alg}}(L_K(F))$, *then* $|E^0_{\text{sing}}| = |F^0_{\text{sing}}|$.

These implications do **not** hold if the hypothesis that K has no free quotients is dropped.

We can put this together with Sørensen's result to obtain a classification for unital Leavitt path algebras of infinite graphs.

Theorem (Ruiz and T)

Let E and F be strongly connected graphs with a finite number of vertices and an infinite number of edges. If K is a field with no free quotients, then the following are equivalent:

(1) $L_K(E)$ *is Morita equivalent to* $L_K(F)$.

(2) $K_0^{\mathrm{alg}}(L_K(E)) \cong K_0^{\mathrm{alg}}(L_K(F))$ *and* $K_1^{\mathrm{alg}}(L_K(E)) \cong K_1^{\mathrm{alg}}(L_K(F))$.

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Moreover, in this case E can be transformed into F via the moves (O), (I), (R), and their inverses.

This implies that for simple unital Leavitt path algebras of infinite graphs over a field with no free quotients, all algebraic K-theory information is contained in the K_0^{alg}-group and K_1^{alg}-group.

Corollary

If E and F are strongly connected graphs with a finite number of vertices and an infinite number of edges, K is a field with no free quotients, and

$$K_0^{\text{alg}}(L_K(E)) \cong K_0^{\text{alg}}(L_K(F)) \ \text{ and } \ K_1^{\text{alg}}(L_K(E)) \cong K_1^{\text{alg}}(L_K(F)),$$

then

$$K_n^{\text{alg}}(L_K(E)) \cong K_n^{\text{alg}}(L_K(F)) \text{ for all } n \in \mathbb{Z}.$$

What happens when the underlying field has free quotients?

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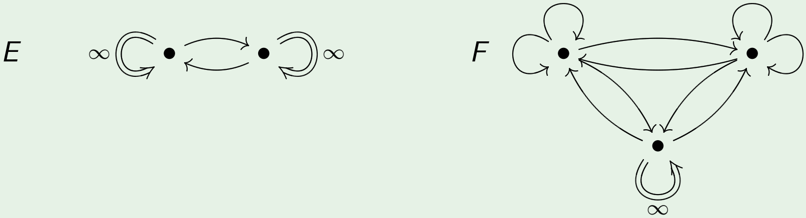
Example (An interesting (counter)example)

Let E and F be the following graphs and let $K = \mathbb{Q}$.



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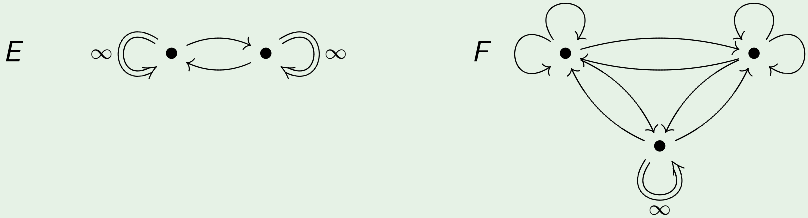

Then

$$K_0^{\text{alg}}(L_{\mathbb{Q}}(E)) \cong K_0^{\text{alg}}(L_{\mathbb{Q}}(F)) \cong \mathbb{Z} \oplus \mathbb{Z}$$

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but . . .

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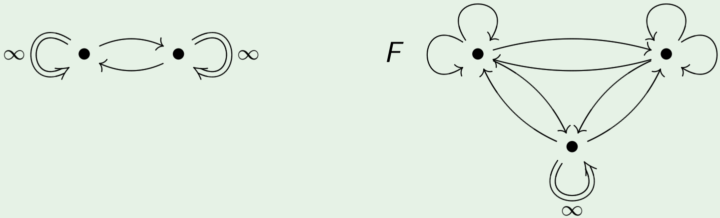
Example

E F



Observations: For general fields

- $K_n^{\text{alg}}(L_K(E)) \cong K_n^{\text{alg}}(L_K(F))$ for $n = 0, 1$ does not imply that $L_K(E)$ and $L_K(F)$ are Morita equivalent.

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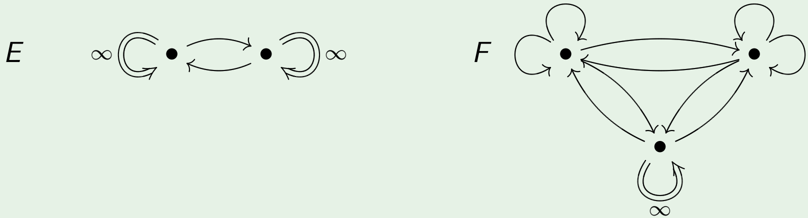

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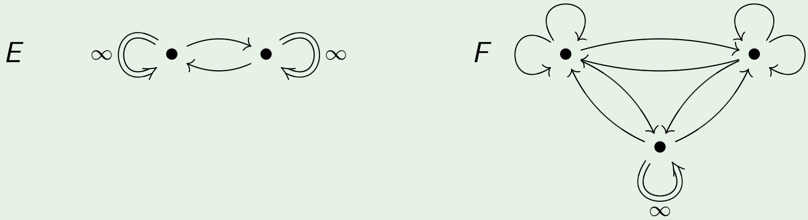

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- The number of singular vertices in E cannot be determined from the two groups $K_0^{\text{alg}}(L_K(E))$ and $K_1^{\text{alg}}(L_K(E))$.

Theorem (Ruiz and T)

Let E and F be strongly connected graphs with a finite number of vertices and an infinite number of edges. If K has no free quotients, TFAE:

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(3) *$K_0^{\text{alg}}(L_K(E)) \cong K_0^{\text{alg}}(L_K(F))$ and $|E^0_{\text{sing}}| = |F^0_{\text{sing}}|$.*

But our example shows in general (2) $\not\Rightarrow$ (1). Remarkably, we can prove the following . . .

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Theorem (Ruiz and T)

Let E and F be strongly connected graphs with a finite number of vertices and an infinite number of edges. Let K be any field. Then $L_K(E)$ is Morita equivalent to $L_K(F)$ if and only if $K_0^{\text{alg}}(L_K(E)) \cong K_0^{\text{alg}}(L_K(F))$ and $|E^0_{\text{sing}}| = |F^0_{\text{sing}}|$.

So, in general, (1) $\Longleftrightarrow$ (3) $\Longrightarrow$ (2).

So the proper invariant for $L_K(E)$ when E has an infinite number of edges is

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Combining the theorem of Abrams, Louly, Pardo, and Smith with the theorem of Ruiz and Tomforde gives a nearly complete classification of unital simple Leavit path algebras.

Theorem (Classification of simple Unital Leavitt Path Algebras)

Let $L_K(E)$ and $L_K(F)$ be simple unital Leavitt path algebras.

(1) *If E and F both have a finite number of edges, and if*

$$K_0^{\text{alg}}(L_K(E) \cong K_0^{\text{alg}}(L_K(F)) \text{ and } \operatorname{sgn}(\det(I - A_E^t)) = \operatorname{sgn}(\det(I - A_F^t)),$$

then $L_K(E)$ is Morita equivalent to $L_K(F)$.

(2) *If E and F both have an infinite number of edges, then $L_K(E)$ is Morita equivalent to $L_K(F)$ if and only if*

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The only missing part is to determine if the "sign of the determinant condition" is necessary in (1). When K has no free quotients, we can replace $|E^0_{\text{sing}}| = |F^0_{\text{sing}}|$ in (2) with $K_1^{\text{alg}}(L_K(E)) \cong K_1^{\text{alg}}(L_K(E))$.

Thank you!