K-theory of twisted C^*-algebras associated to higher-rank graphs

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Joint work with Alex Kumjian and David Pask.

Higher-rank graphs

Definition (Kumjian-Pask, 2000)

For $k \in \mathbb{N}$, a *k-graph* is a countable category Λ with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the factorisation property: whenever $d(\lambda) = m + n$ there are unique $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$.

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- Λ^n denotes $d^{-1}(n)$.
- Factorisation property gives $\Lambda^0 = \{\mathrm{id}_o : o \in \mathrm{Obj}(\Lambda)\}$.
- The domain and codomain maps determine maps $s, r : \Lambda \to \Lambda^0$; and then $r(\lambda)\lambda = \lambda = \lambda s(\lambda)$ for all λ.
- Write, for example, $v\Lambda^n$ for $r^{-1}(v) \cap \Lambda^n$.
- *row-finite* means $v\Lambda^n$ is always finite; *no sources* means it's always nonempty.

Cohomology

- For an abelian group G, a *G-valued 2-cocycle* on Λ is a function

$$c : \Lambda^{*2} := \{(\mu, \nu) \in \Lambda \times \Lambda : s(\mu) = r(\nu)\} \to G$$

such that $c(r(\lambda), \lambda) = c(\lambda, s(\lambda)) = 0$ and

$$c(\lambda, \mu) + c(\lambda\mu, \nu) = c(\mu, \nu) + c(\lambda, \mu\nu).$$

Group of cocycles is $Z^2(\Lambda, \mathbb{T})$.

- Standard example: $k = 2$, and $c(\alpha, \beta) = d(\alpha)_2 d(\beta)_1 g$ for some $g \in G$.

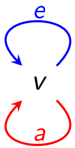
C^*-algebras

- If Λ is row-finite with no sources, and $c \in Z^2(\Lambda, \mathbb{T})$, then $C^*(\Lambda, c)$ is universal for partial isometries $\{s_\lambda : \lambda \in \Lambda\}$ such that

(CK1) $\{s_v : v \in \Lambda^0\}$ are mutually orthogonal projections
(CK2) $s_\mu s_\nu = c(\mu, \nu) s_{\mu\nu}$ when $s(\mu) = r(\nu)$;
(CK3) $s_\mu^* s_\mu = s_{s(\mu)}$ for every μ; and
(CK4) $s_v = \sum_{\mu \in v\Lambda^n} s_\mu s_\mu^*$ for all $v \in \Lambda^0$, $n \in \mathbb{N}^k$.

- Technical adjustment to (CK4) needed when Λ has sources.

Example



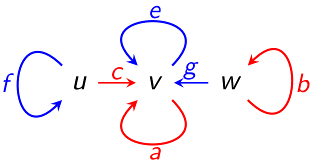
$ea = ae$

- let $z \in \mathbb{T}$, and put $c(\mu,\nu) = z^{d(\mu)_2 d(\nu)_1}$.
- Relation (CK2) implies that $s_v = 1_{C^*(\Lambda)}$ and $C^*(\Lambda)$ is generated by elements s_e and s_a such that $s_a s_e = z s_e s_a$:

(CK3) $s_e^* s_e = s_a^* s_a = 1$; and
(CK4) $1 = \sum_{\alpha \in v\Lambda^{e_1}} s_\alpha s_\alpha^* = s_e s_e^*$, and similarly for s_f.

- So $C^*(\Lambda, c_z)$ is universal for unitaries U, V such that $UV = zVU$: the noncommutative torus A_z.
- up to cohomology, these are the only cocycles, so the only twisted algebras for this graph.

Example



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- Adjustment to (CK4): impose only when $v\Lambda^n$ is nonempty.
- For $\theta \in [0, 1)$, $c_\theta(\mu, \nu) = e^{2\pi i d(\mu)_2 d(\nu)_1 \theta}$ gives a cocycle.

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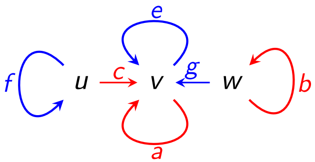

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- $U := s_e + s_f + s_g$ and $V := s_a + s_b + s_c$ generate $C^*(\Lambda, c_z)$ and satisfy:
  - $U^*U = V^*V = 1$;
  - $UV = e^{2\pi i\theta} VU$ and $U^*V = e^{-2\pi i\theta} VU^*$; and
  - $(1 - UU^*)(1 - VV^*) = 0$.
- $C^*(\Lambda, c)$ is universal for these relations.
- A theorem of Baum-Hajac-Matthes-Szymański says $C^*(\Lambda, c) \cong C(S^3_{00\theta})$.

Main theorem

Theorem (Kumjian-Pask-S)

Suppose that Λ is a row-finite k-graph with no sources, and that $c \in Z^2(\Lambda, \mathbb{R})$. For each $t \in \mathbb{R}$, there is an isomorphism

$$K_*(C^*(\Lambda), e^{itc}) \cong K_*(C^*(\Lambda))$$

which preserves the classes of the s_v.

Structure of $C^*(\Lambda, c)$

- If $d(\lambda) = d(\mu) + q$, then $\lambda = \alpha\beta$ with $d(\alpha) = d(\mu)$, and then

$$s_\mu^* s_\lambda = \overline{c(\alpha,\beta)} s_\mu^* s_\alpha s_\beta = \begin{cases} \overline{c(\alpha,\beta)} s_\beta & \text{if } \alpha = \mu \\ 0 & \text{otherwise.} \end{cases}$$

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- So for $\mu, \nu \in \Lambda$ and $p \geq d(\mu), d(\nu)$,

$$s_\mu^* s_\nu = \sum_{\lambda \in r(\mu)\Lambda^p} s_\mu^* s_\lambda s_\lambda^* s_\nu = \sum_{\mu\mu' = \nu\nu' \in \Lambda^p} \overline{c(\mu,\mu')} c(\nu,\nu') s_{\mu'} s_{\nu'}^*.$$

- So $C^*(\Lambda, c) = \overline{\operatorname{span}}\{s_\mu s_\nu^*\}$.

K-theory

- We are interested in the K-theory of $C^*(\Lambda, c)$.
- In many cases of interest, $K_*(C^*(\Lambda))$ is known or computable.
- Our approach follows Elliott's computation of K-theory of noncommutative tori.
- Outline: start with $h \in Z^2(\Lambda, \mathbb{R})$, and put $c = e^{ih}$.
  - Construct continuous field A of C^*-algebras over $[0, 1]$ with $A_0 = C^*(\Lambda)$ and $A_1 = C^*(\Lambda, c)$;
  - Demonstrate A as a full corner of a crossed-product $(B \otimes C([0, 1])) \rtimes \mathbb{Z}^k$.
  - Apply Elliott's inductive argument using Pimsner-Voiculescu.

Central-extension algebras and continuous fields

- Let G be a locally compact abelian group, Λ a row-finite k-graph with no sources and c a G-valued 2-cocycle on Λ.
- A c-representation (ϕ, π) of (Λ, G) on B is
  - a map $\phi : \Lambda \to M(B)$ and a homomorphism $\pi : C^*(G) \to M(B)$ such that
  - $\pi(f)\phi(\lambda) = \phi(\lambda)\pi(f)$ for all λ, f.
  - the $\phi(\lambda)$ satisfy (CK1), (CK3) and (CK4).
  - $\phi(\mu)\phi(\nu) = \pi(c(\mu, \nu))\phi(\mu\nu)$.
- the image of π is central in $M(C^*(\Lambda, G, c))$.

Spanning elements

- Suppose that (ϕ, π) is a c-representation of (Λ, G).
- For $p \geq d(\mu), d(\nu)$, familiar calculations (which work because the $\pi(f)$ are central) give

$$\phi(\mu)^*\phi(\nu) = \sum_{\mu\mu'=\nu\nu'\in\Lambda^p} \pi(c(\nu,\nu') - c(\mu,\mu'))\phi(\mu')\phi(\nu')^*.$$

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- So $C^*(\pi, \phi) = \overline{\text{span}}\{\phi(\mu)\pi(f)\phi(\nu)^* : \mu, \nu \in \Lambda, f \in C^*(G)\}$.
- The $\phi(\mu)$ are partial isometries, so $\| \sum a_{\mu,\nu}\phi(\mu)\pi(f_{\mu,\nu})\phi(\nu)^* \| \leq \sum \|f_{\mu,\nu}\|_\infty$.
- So there is a universal C^*-algebra $C^*(\Lambda, G, c)$ generated by products $i_\Lambda(\lambda)i_G(f)$ where (i_Λ, i_G) is a universal c-representation.

Central-extension algebras and continuous fields

- $C^*(\Lambda, G, c)$ is a $C(\widehat{G})$-algebra.
- General theory says it is the algebra of sections of an upper semicontinuous bundle of C^*-algebras.
- The fibre $C^*(\Lambda, G, c)_\chi$ over $\chi \in \widehat{G}$ is the quotient by $\langle \pi(g) - \chi(g)1 : g \in G \rangle$;
- The universal property of $C^*(\Lambda, G, c)$ gives $\rho_\chi : C^*(\Lambda, G, c) \to C^*(\Lambda, \chi \circ c)$ with
$$\rho_\chi(\phi(\lambda)) = s_\lambda \text{ and } \rho_\chi(\pi(f)) = f(\chi).$$

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$$\rho_\chi(\phi(\lambda)) = s_\lambda \text{ and } \rho_\chi(\pi(f)) = f(\chi).$$
- $\ker(\rho_\chi) \supseteq \langle \pi(g) - \chi(g)1 : g \in G \rangle$, so $\tilde{\rho}_\chi : C^*(\Lambda, G, c)_\chi \to C^*(\Lambda, \chi \circ c)$.
- Universal property of $C^*(\Lambda, \chi \circ c)$ gives inverse to $\tilde{\rho}_\chi$.
- So each $C^*(\Lambda, G, c)_\chi \cong C^*(\Lambda, \chi \circ c)$.

Continuity of the bundle

Lower semicontinuity via an argument due to Rieffel ('89)

- (Kumjian-Pask, '00) gives groupoid $\mathcal{G}_\Lambda$ with $C^*(\Lambda) \cong C^*(\mathcal{G}_\Lambda)$.
- (Kumjian-Pask-S, '11) for each $\chi \in \widehat{G}$ there is $\sigma_\chi \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$ with $C^*(\Lambda, \chi \circ c) \cong C^*(\mathcal{G}_\Lambda, \sigma_\chi)$.

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- $\langle a, b \rangle_\chi := (a^* *_{\sigma_\chi} b)|_{\mathcal{G}_\Lambda^{(0)}}$ gives rise to Hilbert module X_χ, with left action $L_\chi : C^*(\Lambda, \chi \circ c) \to \mathcal{L}(X_\chi)$.

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- Regard the L_χ as adjointable actions on the same module X.

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- Each $\chi \mapsto L_\chi(s_\mu s_\nu^*)$ is strongly continuous; so $\chi \mapsto L_\chi(\rho_\chi(a))$ is strongly continuous for a dense family of $a \in C^*(\Lambda, G, c)$.
- Now if $\chi_n \to \chi$, fix $\|x\| = 1$ such that $\|L_\chi(\rho_\chi(a))x\| > \|L_\chi(\rho_\chi(a))\| - \varepsilon/2$; then

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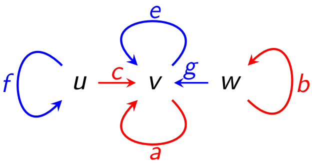
Example: fields of Heegaard-type 3-spheres



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- Consider $c \in Z^2(\Lambda, \mathbb{Z})$ given by $c(\mu, \nu) := d(\mu)_2 d(\nu)_1$.
- $C^*(\Lambda, \mathbb{Z}, c)$ is generated by U, V, W s.t.
  - W is a central unitary;
  - $U^*U = V^*V = 1$;
  - $UV = WVU$ and $U^*V = W^*VU^*$; and
  - $(1 - UU^*)(1 - VV^*) = 0$.
- Each $C^*(\Lambda, \mathbb{Z}, c)_{e^{2\pi i\theta}} \cong C(S^3_{00\theta})$.
- Note: Λ has sources. But a technique due to Farthing ('08) sidesteps the issue.

Trivial AF bundles

- Universal property of $C^*(\Lambda, c)$ gives an action γ of $\mathbb{T}^k$ such that $\gamma_z(s_\mu) = z^{d(\mu)} s_\mu$.
- $C^*(\Lambda, c) \times_\gamma \mathbb{T}^k$ is an AF algebra and there is a k-graph $\Lambda \times_d \mathbb{Z}^k$ and cocycle $\tilde{c}$ such that $C^*(\Lambda, c) \times_\gamma \mathbb{T}^k \cong C^*(\Lambda \times_d \mathbb{Z}^k, \tilde{c})$.
- So $K_*(C^*(\Lambda, c)) \cong K_*(C^*(\Lambda \times_d \mathbb{Z}^k, \tilde{c}) \times_{\hat{\gamma}} \mathbb{Z}^k)$.

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- So $K_*(C^*(\Lambda, c)) \cong K_*(C^*(\Lambda \times_d \mathbb{Z}^k, \tilde{c}) \times_{\hat{\gamma}} \mathbb{Z}^k)$.
- A neat argument due to Ben Whitehead shows that each $C^*(\Lambda \times_d \mathbb{Z}^k, G, \tilde{c}) \cong C^*(\Lambda \times_d \mathbb{Z}^k) \otimes C^*(G)$.
- For $G = \mathbb{R}$, can restrict to $[0, t] \subseteq \mathbb{R}$: $C^*(\Lambda \times_d \mathbb{Z}^k, \mathbb{R}, \tilde{c})_{[0,t]} \cong C^*(\Lambda \times_d \mathbb{Z}^k) \otimes C([0, t])$.
- The $\rho_u : C^*(\Lambda \times_d \mathbb{Z}^k, \mathbb{R}, \tilde{c})_{[0,t]} \to C^*(\Lambda \times_d \mathbb{Z}^k, \mathbb{R}, c)_u$ induce isomorphisms in K-theory (which preserve the class of the identity).

Elliott's argument ('80)

If $\psi : (B, \beta, \mathbb{Z}) \to (C, \gamma, \mathbb{Z})$ and $\psi_* : K_*(B) \to K_*(C)$ is an isomorphism, then $\tilde{\psi}_* : K_*(B \times_\beta \mathbb{Z}) \to K_*(C \times_\gamma \mathbb{Z})$ is an isomorphism.

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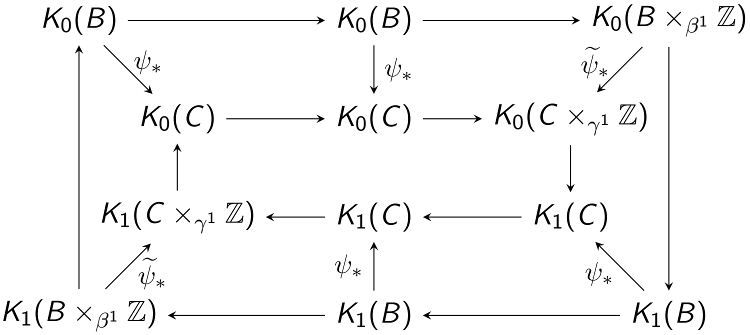
- Naturality of Pimsner-Voiculescu gives a diagram:



- Now the Five Lemma applies.

K-theory of twisted k-graph algebras

Theorem (Kumjian-Pask-S)

Suppose that Λ is a row-finite k-graph with no sources, and that $c \in Z^2(\Lambda, \mathbb{R})$. For each $t \in \mathbb{R}$, there is an isomorphism

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which preserves the classes of the s_v.

Proof.

We proved that $\tilde{\rho}_u : C^*(\Lambda \times_d \mathbb{Z}^k, \mathbb{R}, c)_{[0,t]} \to C^*(\Lambda \times_d \mathbb{Z}^k, e^{iuc})$ induces isomorphism on K-theory.

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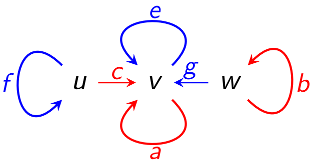
Takai duality and some book-keeping does the rest. □

K-theory of quantum 3-spheres



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- Hajac-Matthes-Szymański ('06):
  $C^*(\Lambda) \cong C(H^3_{000}) := (\mathcal{T} \otimes \mathcal{T})/\mathcal{K} \otimes \mathcal{K}$.
- The inclusion $\mathcal{K} \hookrightarrow \mathcal{T}$ induces the zero map on K-theory.
- The Künneth theorem and the 6-term sequence for
  $0 \to \mathcal{K} \otimes \mathcal{K} \to \mathcal{T} \otimes \mathcal{T} \to C(H^3_{000})$ give $K_*(C(H^3_{000})) \cong (\mathbb{Z}, \mathbb{Z})$.
- Plugging into the main result, $K_*(C(H^3_{00\theta})) \cong (\mathbb{Z}, \mathbb{Z})$,
  recovering a theorem of Baum-Hajac-Matthes-Szymański.

Kirchberg algebras

- If Λ is aperiodic and cofinal, and every vertex can be reached from a cycle with an entrance, then $C^*(\Lambda)$ is simple purely infinite.
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- The same conditions imply that $C^*(\Lambda, c)$ is a Kirchberg algebra, for any $c \in Z^2(\Lambda, \mathbb{T})$.

Corollary (Kumjian-Pask-S): Suppose that Λ is cofinal and aperiodic and every vertex can be reached from a cycle with an entrance. If $c \in Z^2(\Lambda, \mathbb{R})$ then $C^*(\Lambda, e^{itc}) \cong C^*(\Lambda)$ for all $t \in \mathbb{R}$.

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