A symbolic dynamics approach to Kirchberg algebras.

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BIRS Workshop "Graph algebras: Bridges between graph C*-algebras and Leavitt path algebras"
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Outline

1. Why?
2. Who?
3. How?
4. What give us?
5. What's next?

Joint work with Ruy Exel (Departamento de Matemática, Universidade Federal de Santa Catarina, Florianópolis, Brazil),

> R. EXEL, E. PARDO, *Representing Kirchberg algebras as inverse semigroup crossed products, arXiv:1303.6268v1 (2013),*

submited to *Indiana University Mathematical Journal*.

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GOAL: Classify a class of purely infinite simple algebras as large as possible by using algebraic/combinatorial methods.

INSPIRACY:

1. Elliott's Classification Program: classify separable nuclear C^*-algebras via K-theoretic invariants.
2. Kirchberg-Phillips Theorem: separable nuclear purely infinite simple C^*-algebras satisfying the UCT are classifiable using K_0 and K_1 as invariants.

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PROBLEM: K-P Theorem needs a large amount of analytical technology.

A more combinatorial approach is possible on a restricted subclass:

1. Cuntz-Krieger algebras $\mathcal{O}_A$ (where $A \in M_n(\mathbb{Z}^+)$): basic model of purely infinite simple C^*-algebras.
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Expanded subclass:

1. Kumjian, Pask, Raeburn and Renault: graph C^*-algebra $C^*(E)$ for graph E.
2. Rørdam's Theorem applies, via Bates-Pask (graph moves).
3. Sørensen: combinatorial classification result for simple graph C^*-algebras of graphs with finitely many vertices and countably many edges.

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In the purely algebraic context, we have Leavitt path algebras.

Results using graph moves:

1. Abrams, Louly, P., Smith: Partial classification for purely infinite simple LPAs on finite graphs.
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HANDICAP: According to Rørdam's result, for any pair (G_0, G_1) of countable abelian groups there exists a Kirchberg algebra A such that $K_i(A) \cong G_i$ for $i = 0, 1$.

The above results only cover a piece of a combinatorial, purely algebraic version of Kirchberg-Phillips Theorem.

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WE NEED A COMBINATORIAL MODEL RELATED TO A SYMBOLIC DYNAMICAL SYSTEM!

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CANDIDATE: Katsura constructed a suitable combinatorial model for Kirchberg algebras.

Definition

Let $N \in \mathbb{N} \cup \{\infty\}$, let $A \in M_N(\mathbb{Z}^+)$ and $B \in M_N(\mathbb{Z})$ be row-finite matrices. Define a set Ω_A by

$$\Omega_A := \{(i,j) \in \{1,2,\ldots,N\} \times \{1,2,\ldots,N\} \mid A_{i,j} \geq 1\}.$$

For each $i \in \{1,2,\ldots,N\}$, define a set $\Omega_A(i) \subset \{1,2,\ldots,N\}$ by

$$\Omega_A(i) := \{j \in \{1,2,\ldots,N\} \mid (i,j) \in \Omega_A\}.$$

Notice that, by definition, $\Omega_A(i)$ is finite for all i. Finally, fix the following condition:

(0) $\Omega_A(i) \neq \emptyset$ for all i, and $B_{i,j} = 0$ for $(i,j) \notin \Omega_A$.

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Definition

Define $\mathcal{O}_{A,B}$ to be the universal C^*-algebra generated by mutually orthogonal projections $\{q_i\}_{i=1}^N$, partial unitaries $\{u_i\}_{i=1}^N$ with $u_i u_i^* = u_i^* u_i = q_i$, and partial isometries $\{s_{i,j,n}\}_{(i,j)\in\Omega_A, n\in\mathbb{Z}}$ satisfying the relations:

(i) $s_{i,j,n} u_j = s_{i,j,n+A_{i,j}}$ and $u_i s_{i,j,n} = s_{i,j,n+B_{i,j}}$ for all $(i,j) \in \Omega_A$ and $n \in \mathbb{Z}$.

(ii) $s_{i,j,n}^* s_{i,j,n} = q_j$ for all $(i,j) \in \Omega_A$ and $n \in \mathbb{Z}$.

(iii) $q_i = \sum_{j\in\Omega_A(i)} \sum_{n=1}^{A_{i,j}} s_{i,j,n} s_{i,j,n}^*$ for all i.

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When $B = (0)$, $\mathcal{O}_{A,(0)}$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_A$ (the Exel-Laca algebra if $N = \infty$). To be precise, $\mathcal{O}_A = C^*(E_A)$.

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Now, the following facts holds:

1. $\mathcal{O}_{A,B}$ is separable, nuclear and in the UCT class.
2. If the matrices A, B satisfy:
   (i) A is irreducible.
   (ii) $A_{i,i} \geq 2$ and $B_{i,i} = 1$ for every $1 \leq i \leq N$.

   then $\mathcal{O}_{A,B}$ is a Kirchberg algebra.
3. Every Kirchberg algebra can be represented, up to isomorphism, by an algebra $\mathcal{O}_{A,B}$ for matrices A, B satisfying the conditions $(2)(a\&b)$.
4. For any matrix B, $\mathcal{O}_A \hookrightarrow \mathcal{O}_{A,B}$.

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THUS, IT SEEMS THAT THIS IS THE RIGHT CLASS.

The natural injective $*$-homomorphism $\mathcal{O}_A \hookrightarrow \mathcal{O}_{A,B}$, suggest to deal with graph moves, to get some sort of classification stuff.

PROBLEM: Changes on A cannot be independent of suitable changes on B. Moreover, results associated to classical moves on A are unclear.

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WE NEED AN ASSOCIATED SYMBOLIC DYNAMICAL SYSTEM!

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<u>KATSURA'S PICTURE</u>: $\mathcal{O}_{A,B}$ is associated to a <u>topological graph E</u>. Hence, edges and vertices are locally compact spaces, and range and source are continuous maps. Thus:

1. No combinatorial nature object associated.

   $\mathcal{O}_{A,B}$ is seen as a Cuntz-Pimsner algebra associated to a full C^*-correspondence $X_{A,B}$ over $C_0(\{1,\dots,N\}\times\mathbb{T})$.

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IDEA: Mimic Exel-Laca picture of $\mathcal{O}_A \cong C_0(X_A) \rtimes_\alpha \mathbb{F}$, where X_A is the space of one-sided infinite paths on E_A, while α is a partial action of the free group (with generators the edges of E_A) on $C_0(X_A)$. This gives a symbolic dynamical picture of $\mathcal{O}_A$.

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EXEL-LACA'S STRATEGY:

1. Fix $\mathbb{F}$, and pick $C^*_{\text{par}}(\mathbb{F}) \cong C_0(\Omega_A) \rtimes_\alpha \mathbb{F}$.
2. Prove that the representation $\pi : \mathbb{F} \to \mathcal{O}_A$ is semi-saturated and tight.
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   (i) $\mathcal{O}_A \cong C^*_{\text{par}}(\mathbb{F})/J$.
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PROBLEM: For $\mathcal{O}_{A,B}$, the group acting must be

$$G = \mathbb{F} * \mathbb{F}'/\mathcal{R},$$

where $\mathbb{F}'$ is the free group with generators the partial unitaries, and $\mathcal{R}$ the normal subgroup generated by the relations (i) in $\mathcal{O}_{A,B}$ definition. And:

1. The natural representation is not semi-saturated.
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<u>WE NEED A DIFFERENT STRATEGY</u>.

SOLUTION: We construct the symbolic dynamics system from scratch, using Exel's techniques.

1. Use A, B to define a semigroupoid $\Lambda_{A,B}$.
2. Prove it satisfies the right properties:
   (i) $\Lambda_{A,B}$ is left cancellative.
   (ii) Every pair of intersecting elements have a unique lcm.
   (iii) $\Lambda_{A,B}$ has no springs and is categorical.
3. Construct an associated inverse semigroup with zero $\mathcal{S}(\Lambda_{A,B})$, whose semilatice of idempotents is denoted E.
4. Construct the (tight) groupoid of germs $\mathcal{G}_{\Lambda_{A,B}}$ associated to the action of $\mathcal{S}(\Lambda_{A,B})$ on the space $\widehat{E}_{\text{tight}}$ of tight characters defined over E.

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Under the above properties, the universal C^*-algebra $\mathcal{O}_{\Lambda_{A,B}}$ of tight representations of $\Lambda_{A,B}$ is $*$-isomorphic to $C^*(\mathcal{G}_{\Lambda_{A,B}})$.

Then, we prove that the $\mathcal{O}_{\Lambda_{A,B}}$ is $*$-isomorphic to $\mathcal{O}_{A,B}$. Thus:

- $\mathcal{O}_{A,B} \cong C^*(\mathcal{G}_{\Lambda_{A,B}})$

  The image of $S(\Lambda_{A,B})$ into $\mathcal{O}_{\Lambda_{A,B}}$ goes to $S^{A,B}$ (the inverse semigroup of $\mathcal{O}_{A,B}$ generated by the $s_{i,j}$'s and the v_i's).

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Under this identification we have

Theorem

The action $\alpha : \mathcal{S}(\Lambda_{A,B}) \to \widehat{E}_{\text{tight}}$ *becomes the action* $\alpha : \mathcal{S}^{A,B} \to X_A$ *given by multiplication of elements of* X_A *on the left by elements of* $\mathcal{S}^{A,B}$.

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The C^*-algebra $\mathcal{O}_{A,B}$ is isomorphic to the inverse semigroup crossed product $C_0(X_A) \rtimes_\alpha \mathcal{S}^{A,B}$.

Notice that, when $B = (0)$, the previous corollary recover the picture of the Exel-Laca algebra $\mathcal{O}_A$. Also, we get the desired picture of $\mathcal{O}_{A,B}$ in terms of symbolic dynamics.

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The addition of an extra condition on the matrix B produces interesting consequences.

Definition

We say that the matrix B satisfies Condition (E) when $B_{i,j} = 0$ if and only if $(i, j) \notin \Omega_A$.

Lemma

$\Lambda_{A,B}$ is right cancellative if and only if B satisfies Condition (E).

Remark

If $\Lambda_{A,B}$ is right cancellative then $\mathcal{S}(\Lambda_{A,B})$ is a E^*-unitary inverse semigroup, whence $\mathcal{G}_{\Lambda_{A,B}}$ is Hausdorff.

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Outline

1. Why?
2. Who?
3. How?
4. What give us?
5. What's next?

The dynamical approach lets us to deal with some questions in a more intuitive form. For example, when looking for characterize simplicity, we need to get ride of when $\mathcal{G}_{\Lambda_{A,B}}$ is minimal and essentially principal.

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With respect to minimal, we have

Definition

A groupoid $\mathcal{G}$ is said to be minimal if the only invariant open subsets of $\mathcal{G}^{(0)}$ are the empty set and $\mathcal{G}^{(0)}$ itself.

Definition

If S is an inverse semigroup, and τ is an action by (partial) homeomorphisms on a topological space X, then:

1. We say that a subset W of X is invariant if for every $s \in S$ we have that $\tau_s(W) \subseteq W$.
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For the groupoid of germs $\mathcal{G}$ of the action of an inverse semigroup S on a locally compact Hausdorff space X, it is easy to see that irreducibility of X is equivalent to minimality of $\mathcal{G}$. Then we have

Theorem

Given the action α of $\mathcal{S}^{A,B}$ on X_A, the following are equivalent:

1. *The matrix A is irreducible.*
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With respect to essentially principal, we have

Definition

Let $\mathcal{G}$ be a locally compact, Hausdorff, étale groupoid. Then:

1. For any $x \in \mathcal{G}^{(0)}$, the isotropy group at x is

$$\mathcal{G}(x) = \{\gamma \in \mathcal{G} \mid d(\gamma) = t(\gamma) = x\}.$$

2. $\mathcal{G}$ is essentially principal if the interior of the isotropy group bundle

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We will connect essentially principal groupoids with the topological freeness of the action.

Definition

Let S be an E^*-unitary inverse semigroup, and let τ be an action of S on a topological space X.

1. Given $s \in S$ and $x \in X_{s^*s}$, we say x is a fixed point for s if $\tau_s(x) = x$.
2. We say that the action is topologically free if, for every $s \in S \setminus E(S)$, the interior of the set of fixed points for s is empty.

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2. We say that the action is topologically free if, for every $s \in S \setminus E(S)$, the interior of the set of fixed points for s is empty.

Now, we have the following result, connecting both notions.

Theorem

Let S be an E^*-unitary inverse semigroup, let τ be an action of S on a locally compact, Hausdorff space X, and let $\mathcal{G}$ be the corresponding groupoid of germs. Then $\mathcal{G}$ is essentially principal if and only if τ is topologically free.

Thus, we can deal with the problem from the point of view of topological freeness.

We get Exel-Laca's result when we act with elements of $\mathcal{S}^A$ (the inverse semigroup of $\mathcal{O}_A$ generated by the $s_{i,j,n}$'s).

Lemma

When restricted to elements $s \in \mathcal{S}^A \setminus E(\mathcal{S}^A)$, TFAE:

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The action of partial unitaries give us

Lemma

Given an element $\omega = s_{i_1,i_2,n_1} s_{i_2,i_3,n_2} \cdots s_{i_k,i_{k+1},n_k} \cdots$ *of* X_A*, the following are equivalent:*

1. ω *is fixed under the action of* $u_{i_1}^l$ $(l \in \mathbb{Z})$.
2. *For every* $j \geq 1$ *the element* $K_j := l \cdot \prod_{t=1}^{j} \frac{B_{i_t,i_{t+1}}}{A_{i_t,i_{t+1}}}$ *belongs to* $\mathbb{Z}$.

Thus, combining all the information we get

Theorem

Let α be the action of $\mathcal{S}^{A,B}$ on X_A, and let $\mathcal{G}_{\Lambda_{A,B}}$ the associated groupoid. The following are equivalent:

1. (i) *The graph E_A satisfies Condition (L).*
   (ii) *The matrix B satisfies Condition (E).*
   (iii) *For any fixed point $\omega = s_{i_1,i_2,n_1} s_{i_2,i_3,n_2} \cdots s_{i_k,i_{k+1},n_k} \cdots$ and every $n \geq 1$ there exist $m \geq n$ and j_{m+1} with:*
      (a) $(i_m, j_{m+1}) \in \Omega_A$.
      (b) $K_{m+1} = K_m \cdot \dfrac{B_{i_m,j_{m+1}}}{A_{i_m,j_{m+1}}} \notin \mathbb{Z}$.
2. *The groupoid $\mathcal{G}_{\Lambda_{A,B}}$ is essentially principal.*

And as a practical consequence:

Proposition

Let α be the action of $\mathcal{S}^{A,B}$ on X_A, and let $\mathcal{G}_{\Lambda_{A,B}}$ the associated groupoid. If

1. *The graph E_A satisfies Condition (L).*
2. *The matrix B satisfies Condition (E).*
3. *For any fixed point $\omega = s_{i_1,i_2,n_1} s_{i_2,i_3,n_2} \cdots s_{i_k,i_{k+1},n_k} \cdots$ and for every $n, r \geq 1$ there exist a sequence $j_{n+1}, j_{n+2}, \ldots, j_{n+r}$ with:*
   - (i) $(j_t, j_{t+1}) \in \Omega_A$ *for all* t.
   - (ii) $\lim_{r\to\infty} \prod_{t=1}^{r} \left(\frac{B_{j_{n+t},j_{n+t+1}}}{A_{j_{n+t},j_{n+t+1}}} \right) = 0.$

then the groupoid $\mathcal{G}_{\Lambda_{A,B}}$ is essentially principal.

Notice that this proposition includes Katsura's conditions for purely infinite simple. Now, we are ready to characterize simplicity, using a result of Clark et al. characterizing simplicity of groupoid C^*-algebras of Hausdorff groupoids.

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Theorem

Consider the initial matrices A, B. If the matrix B satisfies Condition (E), then the following are equivalent:

1. (i) *The matrix A is irreducible.*
   (ii) *The graph E_A satisfies Condition (L).*
   (iii) *For any fixed point $\omega = s_{i_1,i_2,n_1} s_{i_2,i_3,n_2} \cdots s_{i_k,i_{k+1},n_k} \cdots$ and every $n \geq 1$ there exist $m \geq n$ and j_{m+1} with:*
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2. $\mathcal{O}_{A,B}$ *is simple.*

Corollary

Consider the initial matrices A, B. If they satisfy Katsura's conditions for purely infinite simple and B satisfies Condition (E), then $\mathcal{O}_{A,B}$ is simple.

We need an extra property –Condition (E)– to characterize simplicity of $\mathcal{O}_{A,B}$. But we describe simplicity of $\mathcal{O}_{A,B}$ for a broad collection of algebras, including the ones given by Katsura.

Results are obtained in a more natural way, by linking this property to dynamical properties of X_A.

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Results are obtained in a more natural way, by linking this property to dynamical properties of X_A.

With respect to pure infiniteness, we use the notion of local contractiveness of groupoids, due to Anantharaman-Delaroche

Definition

We say that a second countable étale groupoid $\mathcal{G}$ is locally contracting in for every nonempty open subset U of $\mathcal{G}^{(0)}$ there exists an open subset V in U and an slice S such that $\overline{V} \subset S^{-1}S$ and $S\overline{V}S^{-1}$ is properly contained in V.

Under our picture, what we obtain is

Proposition

If every finite path in the graph E_A can be enlarged to a cycle and E_A satisfies Condition (L), then $\mathcal{G}_{\Lambda_{A,B}}$ is locally contracting.

So, we can prove

Theorem

Consider the initial matrices A, B. If

1. *The matrix A is irreducible.*
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then $\mathcal{O}_{A,B}$ is purely infinite simple.

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Finally, we will show that, under Condition (E), it is possible to show a partial version of Katsura's result.

Theorem

Let G_0, G_1 be finitely generated abelian groups. Then, there exist $N \in \mathbb{N}$, $A \in M_N(\mathbb{Z}^+), B \in M_N(\mathbb{Z})$ satisfying Condition (E), such that:

1. *$\mathcal{O}_{A,B}$ is unital Kirchberg algebra.*
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So, we can represent any unital Kirchberg algebra (up to isomorphism) with finitelly generated K-Theory as a Katsura algebra $\mathcal{O}_{A,B}$ such that the matrix B satisfies Condition (E), and thus as the groupoid C^*-algebra of a minimal essentially principal locally contracting groupoid $\mathcal{G}_{\Lambda_{A,B}}$.

Outline

We have some conclusion remarks, which could open new lines of research

Having the dynamical model, it's time to look at its "classification" (i.e., moves preserving something) to advance in the classification problem. In particular:

1. Pay attention to what kind of effect produces the classical moves on A to this model.
2. Consider the possibility of dealing with the $\mathcal{O}_2 \cong \mathcal{O}_{2^-}$ isomorphism in a combinatorial way, using this model.

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Extend the strategy to LPAs world:

1. Use Steinberg discret groupoid algebra construction for representing ${\mathcal{O}_{A,B}}^{\text{alg}}(K)$ as $K\mathcal{G}_{\Lambda_{A,B}}$.
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