Simple Cuntz-Pimsner Rings

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(joint work with T.M. Carlsen and E. Pardo)

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Overview

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Section Cuntz-Pimsner rings

R-systems

- R is an associative ring.
- P, Q are R-bimodules.
- $\psi : P \otimes Q \longrightarrow R$ is an R-bimodule homomorphism.
- The triple (P, Q, ψ) is called an R-system.
- I, J are two-sided ideals of R.
- An ideal I of R is a Ψ-invariant if

$$\Psi(PI \otimes Q) = \Psi(P \otimes IQ) \subseteq I\,.$$

Covariant representations

Let (P, Q, ψ) be an R-system, then a covariant representation is a quadruple (S', T', σ', B) satisfying:

(1) B is a ring,

(2) $S' : P \to B$ and $T' : Q \to B$ are additive maps,

(3) $\sigma' : R \to B$ is a ring homomorphism,

(4) Given $p \in P$, $q \in Q$ and $r \in R$,

$$S'(pr) = S'(p)\sigma'(r)\,, \qquad S'(rp) = \sigma'(r)S'(p)\,,$$

$$T'(qr) = T'(q)\sigma'(r) \qquad \text{and} \qquad T'(rq) = \sigma'(r)T'(q)\,.$$

(5) $\sigma'(\psi(p \otimes q)) = S'(p)T'(q)$ for $p \in P$ and $q \in Q$.

Condition (FS)

For $p \in P$ and $q \in Q$ let us define $\theta_{q,p} \in \mathrm{End}_R(Q_R)$ given by

$$\theta_{q,p}(x) = q\psi(p \otimes x)$$

for $x \in Q$, and $\theta_{p,q} \in \mathrm{End}_R({}_R P)$ given by

$$\theta_{p,q}(y) = \psi(y \otimes q)p$$

for $y \in P$.

$$\mathcal{F}_P(Q) = \mathrm{span}\{\theta_{q,p} : p \in P, q \in Q\} \text{ and } \mathcal{F}_Q(P) = \mathrm{span}\{\theta_{p,q} : p \in P, q \in Q\}$$

Definition 1

(P, Q, ψ) satisfies **condition (FS)** if for any finite set $\{q_1, \ldots, q_n\} \subseteq Q$ and any finite set $\{p_1, \ldots, p_m\} \subseteq P$ exist $\Theta \in \mathcal{F}_P(Q)$ and $\Psi \in \mathcal{F}_Q(P)$ such that $\Theta(q_i) = q_i$ and $\Psi(p_j) = p_j$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$

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Cuntz-Pimsner Covariant representations

$\Delta : R \longrightarrow \mathrm{End}_R(Q_R)$ given by $\Delta(r)(q) = rq$ for $r \in R, q \in Q$.

Definition 2

A two-sided ideal I of R is **ψ-compatible** if $I \subseteq \Delta^{-1}(\mathcal{F}_P(Q))$, and **faithful** if $I \cap \ker \Delta = \{0\}$.

J will denote a fixed faithful and ψ-compatible ideal in R.

Definition 3

A covariant representation (S', T', σ', B) is said to be **Cuntz-Pimsner invariant relative to J** if

$$\pi_{T',S'}(\Delta(x)) = \sigma'(x) \text{ for all } x \in J,$$

where $\pi_{T',S'} : \mathcal{F}_P(Q) \to B$ satisfies $\pi_{T',S'}(\theta_{q,p}) = T'(q)S'(p)$ for all $p \in P$ and $q \in Q$.

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Theorem 4

There is a covariant representation $(S, T, \sigma, \mathcal{O}_{(P,Q,\psi)}(J))$ which is Cuntz-Pimsner invariant relative to J and **universal** *in the sense that every covariant representation which is Cuntz-Pimsner invariant relative to J factors through it.*

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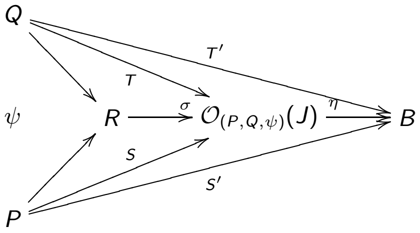

$\mathbb{Z}$-graduation

Given $n \in \mathbb{N}$ exist unique additive maps

$$T^n : Q^{\otimes n} \to \mathcal{O}_{(P,Q,\psi)}(J) \qquad \text{and} \qquad S^n : P^{\otimes n} \to \mathcal{O}_{(P,Q,\psi)}(J)$$

such that for $q_1, q_2, \dots, q_n \in Q$ and $p_1, p_2, \dots, p_n \in P$

$$T^n(q_1 \otimes q_2 \otimes \cdots \otimes q_n) = T(q_1)T(q_2)\dots T(q_n)$$
$$S^n(p_1 \otimes p_2 \otimes \cdots \otimes p_n) = S(p_1)S(p_2)\dots S(p_n)\,.$$

Then $\mathcal{O}_{(P,Q,\psi)}(J)$ is a $\mathbb{Z}$-graded ring with grading

$$\mathcal{O}_{(P,Q,\psi)}(J)^{(n)} = \operatorname{span}(\{T^{k+n}(q)S^k(p) \mid k \in \mathbb{N},\ q \in Q^{\otimes k+n},\ p \in P^{\otimes k}\} \cup \{T^n(q) \mid q \in Q^{\otimes n}\})$$

$$\mathcal{O}_{(P,Q,\psi)}(J)^{(-n)} = \operatorname{span}(\{T^{k}(q)S^{k+n}(p) \mid k \in \mathbb{N},\ q \in Q^{\otimes k},\ p \in P^{\otimes k+n}\} \cup \{S^n(p) \mid p \in P^{\otimes n}\})$$

$$\mathcal{O}_{(P,Q,\psi)}(J)^{(0)} = \operatorname{span}(\{T^{k}(q)S^{k}(p) \mid k \in \mathbb{N},\ q \in Q^{\otimes k},\ p \in P^{\otimes k}\} \cup \{\sigma(r) \mid r \in R\})\,.$$

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Let $E=(E^0,E^1,r,s)$ be a directed graph and let F be any field. We define the ring $R_E:=\oplus_{v\in E^0}R_v$ where each R_v is a copy of F. We define the R_E-bimodules $Q_E:=\oplus_{e\in E^1}Q_e$ and $P_E:=\oplus_{e\in E^1}P_{\bar{e}}$ where each $Q_e,P_{\bar{e}}$ is a copy of F. The left and the right multiplication are defined by

$$r_v\cdot q_e\cdot s_w=\delta_{v,s(e)}\delta_{w,r(e)}r_vs_wq_e$$

$$r_v\cdot p_{\bar{e}}\cdot s_w=\delta_{w,s(e)}\delta_{v,r(e)}r_vs_wp_{\bar{e}}\,.$$

Finally we define $\psi_E:P_E\otimes_{R_E}Q_E\to R_E$ the R_E-bimodule homomorphism given by

$$\psi_E(p_{\bar{f}}\otimes q_e)=\delta_{s(e),s(f)}p_{\bar{f}}q_e\,.$$

Then

$$\ker\Delta=\mathrm{span}_F\{\mathbf{1}_v\mid v\in E^0\text{ and }vE^1=\emptyset\}\,,$$
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Leavitt path algebras

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Let $(S, T, \sigma, \mathcal{O}_{(P_E,Q_E,\psi_E)}(J_E))$ be the universal covariant representation of (P_E, Q_E, ψ_E). Then if for each $v \in E^0$ and $e \in E^1$ define

$$p_v = \sigma(\mathbf{1}_v)\,, \qquad x_e = T(\mathbf{1}_e) \qquad \text{and} \qquad y_e = S(\mathbf{1}_{\overline{e}})\,.$$

$\mathcal{O}_{(P_E,Q_E,\psi_E)}(J_E)$ is generated by

$$\{p_v \mid v \in E^0\} \cup \{x_e \mid e \in E^1\} \cup \{y_e \mid e \in E^1\}$$

and these elements satisfy:

(i) $p_{s(e)}x_e = x_e = x_e p_{r(e)}$ for $e \in E^1$,

(ii) $p_{r(e)}y_e = y_e = y_e p_{s(e)}$ for $e \in E^1$,

(iii) $y_e x_f = \delta_{e,f}\, p_{r(e)}$ for $e, f \in E^1$,

(iv) $p_v = \sum_{e \in vE^1} x_e y_e$ for $v \in E^0$ with $0 < |vE^1| < \infty$.

In fact, $\mathcal{O}_{(P_E,Q_E,\psi_E)}(J_E)$ is isomorphic to the Leavitt path algebra $L_F(E)$ of E.

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In fact, $\mathcal{O}_{(P_E,Q_E,\psi_E)}(J_E)$ is isomorphic to the Leavitt path algebra $L_F(E)$ of E.

Definition 5

For an ideal I in R, let $\psi^{-1}(I)$ be the ideal

$$\{x \in R \mid \psi(px \otimes q) \in I \text{ for all } q \in Q \text{ and all } p \in P\},$$

and let $I^{[\infty]}$ be the ideal

$$\bigcap_{k=1}^{\infty} I^{[k]}$$

where $I^{[k]}$ is defined recursively by $I^{[1]} = I$ and $I^{[k]} = \psi^{-1}(I^{[k-1]}) \cap I$ for $k > 1$.

Example 6

Let (P_E, Q_E, ψ_E) and let I be an ideal of R_E and let $H = \{v \in E^0 \mid \mathbf{1}_v \in I\}$. Then $I = \text{span}_F\{\mathbf{1}_v \mid v \in H\}$ and it follows that

$$I^{[k]} = \text{span}_F \left\{ \mathbf{1}_v \mid v \in H \text{ and } r(e) \in H \text{ for all } e \in \bigcup_{i=1}^{k-1} vE^i \right\}$$

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The ideal intersection property

Definition 7

A subring A of $\mathcal{O}_{(P,Q,\psi)}(J)$ has the **ideal intersection property** if the implication $K \cap A = \{0\} \implies K = \{0\}$ holds for every ideal K in $\mathcal{O}_{(P,Q,\psi)}(J)$.

Proposition 8

The following 3 conditions are equivalent:

1. *The subring $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ does not have the ideal intersection property.*
2. *There is a non-zero graded ideal $\bigoplus_{k\in\mathbb{Z}} H^{(k)}$ in $\mathcal{O}_{(P,Q,\psi)}(J)$, an $n \in \mathbb{N}$ and a family $(\phi_k)_{k\in\mathbb{Z}}$ of injective $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$-bimodule homomorphisms $\phi_k : H^{(k)} \to \mathcal{O}_{(P,Q,\psi)}(J)^{(k+n)}$ such that $x\phi_k(y) = \phi_{k+j}(xy)$ and $\phi_k(y)x = \phi_{k+j}(yx)$ for $k, j \in \mathbb{Z}$, $x \in \mathcal{O}_{(P,Q,\psi)}(J)^{(j)}$ and $y \in H^{(k)}$.*
3. *There is a non-zero ψ-invariant ideal I_0 of R, an $n \in \mathbb{N}$ and an injective R-bimodule homomorphism $\eta : I_0 \to Q^{\otimes n}$ such that $S_p(T_{\eta(x)}(q)) = \eta(\psi(px \otimes q))$ for $p \in P$, $x \in I_0$ and $q \in Q$, and such that $I_0 \subseteq J^{[\infty]}$.*

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Condition (L)

Definition 9

We say that a ψ-invariant ideal I in R is a ψ-**invariant cycle** if there exist $n \in \mathbb{N}$ and an injective R-bimodule homomorphism $\eta : I \to Q^{\otimes n}$ such that $S_p(T_{\eta(x)}(q)) = \eta(\psi(px \otimes q))$ for $p \in P$, $x \in I$ and $q \in Q$, and we say that J satisfies **condition (L)** with respect to the R-system (P, Q, ψ) if there are no non-zero ψ-invariant cycles I in R such that $I \subseteq J^{[\infty]}$.

Define $J_{(S',T',\sigma',B)} = \{x \in R | \sigma'(x) \in \pi_{T',S'}(\mathcal{F}_P(Q))\}$.

Theorem 10

The following 4 conditions are equivalent:

1. *The ideal J satisfies condition (L).*
2. *The subring $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ has the ideal intersection property.*
3. *Every non-zero ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ contains a non-zero graded ideal.*
4. *If (S', T', σ', B) is an injective covariant representation of (P, Q, ψ) and $J = J_{(S',T',\sigma',B)}$, then the ring homomorphism $\eta^J_{(S',T',\sigma',B)} : \mathcal{O}_{(P,Q,\psi)}(J) \to B$ is injective.*

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3. *Every non-zero ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ contains a non-zero graded ideal.*
4. *If (S', T', σ', B) is an injective covariant representation of (P, Q, ψ) and $J = J_{(S',T',\sigma',B)}$, then the ring homomorphism $\eta^J_{(S',T',\sigma',B)} : \mathcal{O}_{(P,Q,\psi)}(J) \to B$ is injective.*

Definition 9

We say that a ψ-invariant ideal I in R is a ψ-**invariant cycle** if there exist $n \in \mathbb{N}$ and an injective R-bimodule homomorphism $\eta : I \to Q^{\otimes n}$ such that $S_p(T_{\eta(x)}(q)) = \eta(\psi(px \otimes q))$ for $p \in P$, $x \in I$ and $q \in Q$, and we say that J satisfies **condition (L)** with respect to the R-system (P, Q, ψ) if there are no non-zero ψ-invariant cycles I in R such that $I \subseteq J^{[\infty]}$.

Define $J_{(S',T',\sigma',B)} = \{x \in R | \sigma'(x) \in \pi_{T',S'}(\mathcal{F}_P(Q))\}$.

Theorem 10

The following 4 conditions are equivalent:

1. *The ideal J satisfies condition (L).*
2. *The subring $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ has the ideal intersection property.*
3. *Every non-zero ideal in $\mathcal{O}_{(P,Q,\psi)}(J)$ contains a non-zero graded ideal.*
4. *If (S', T', σ', B) is an injective covariant representation of (P, Q, ψ) and $J = J_{(S',T',\sigma',B)}$, then the ring homomorphism $\eta^J_{(S',T',\sigma',B)} : \mathcal{O}_{(P,Q,\psi)}(J) \to B$ is injective.*

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Example 11

Let (P_E, Q_E, ψ_E) and let J_E. Then

$$J_E^{[k]} = \text{span}_F\{\mathbf{1}_v \mid v \in E^0 \text{ and } 0 < |vE^i| < \infty \text{ for } i = 1, 2, \ldots, k\}$$

for each $k \in \mathbb{N}$ and that

$$J_E^{[\infty]} = \text{span}_F\{\mathbf{1}_v \mid v \in E^0 \text{ and } 0 < |vE^i| < \infty \text{ for all } i \in \mathbb{N}\}.$$

A non-zero ideal I_H of R_E is a ψ_E-invariant cycle if and only if H is the union of cycles without exit.

Thus J_E satisfies condition (L) with respect to the R_E-system (P_E, Q_E, ψ_E) if and only every closed path in (E^0, E^1, r, s) has an exit.

Definition 12

We say that the ideal J has the **Cuntz-Krieger uniqueness property** with respect to the R-system (P, Q, ψ) if the following holds:

If $(S_1, T_1, \sigma_1, B_1)$ and $(S_2, T_2, \sigma_2, B_2)$ are two injective covariant representations of (P, Q, ψ) and they are both Cuntz-Pimsner invariant relative to J, then there is a ring isomorphism ϕ between $\mathcal{R}\langle S_1, T_1, \sigma_1\rangle$ and $\mathcal{R}\langle S_2, T_2, \sigma_2\rangle$ such that $\phi \circ \sigma_1 = \sigma_2$, $\phi \circ S_1 = S_2$ and $\phi \circ T_1 = T_2$.

Cuntz-Krieger uniqueness property

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Cuntz-Krieger uniqueness property

Theorem 13

The following 5 conditions are equivalent:

1. *The ideal J has the Cuntz-Krieger uniqueness property.*
2. *If (S', T', σ', B) is an injective covariant representation of (P, Q, ψ) which is Cuntz-Pimsner invariant relative to J, then the ring homomorphism*
$$\eta^{J}_{(S',T',\sigma',B)} : \mathcal{O}_{(P,Q,\psi)}(J) \to B$$
is injective.
3. *The subring $\sigma(R)$ has the ideal intersection property.*
4. *The subring $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ has the ideal intersection property, and J is a maximal faithful, ψ-compatible ideal.*
5. *The ideal J satisfies condition (L) and is a maximal faithful, ψ-compatible ideal.*

Graded ideals

If I is a ψ-invariant ideal in R, then $R_I = R/I$, $Q_I = Q/QI$ and ${}_IP = P/IP$, and $\wp_I$ denotes the corresponding quotient maps.

There is an R_I-bimodule homomorphism $\psi_I : {}_IP \otimes Q_I \to R_I$ given by $\psi_I(\wp_I(p) \otimes \wp_I(q)) = \wp_I(\psi(p \otimes q))$.

The triple $({}_IP, Q_I, \psi_I)$ is then an R_I-system satisfying condition (FS).

Definition 14

a **T-pair** is a pair (I, J') where I and J' are ideals in R such that $I \subseteq J$, I is ψ-invariant, and $J'_I := \wp_I(J')$ is a faithful, ψ_I-compatible ideal in R_I.

Theorem 15

There is a bijection between the T-pairs (I, J') with $J \subseteq J'$ and the graded ideals of $\mathcal{O}_{(P,Q,\psi)}(J)$.

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Theorem 15

There is a bijection between the T-pairs (I, J') with $J \subseteq J'$ and the graded ideals of $\mathcal{O}_{(P,Q,\psi)}(J)$.

Condition (K)

Definition 16

We say that the ideal J satisfies **condition (K)** with respect to the R-system (P, Q, ψ) if J_I' satisfies condition (L) with respect to the R_I-system $({}_IP, Q_I, \psi_I)$ whenever (I, J') is a T-pair of (P, Q, ψ) such that $J \subseteq J'$.

Theorem 17

The following 3 conditions are equivalent:

1. *Every ideal of $\mathcal{O}_{(P,Q,\psi)}(J)$ is graded.*
2. *The ideal J satisfies condition (K).*
3. *If (S', T', σ', B) is a covariant representation of (P, Q, ψ) which is Cuntz-Pimsner invariant relative to J, and $(I, J') = \omega_{(S',T',\sigma',B)}$, then the ring homomorphism*

$$\eta^{(I,J')}_{(S',T',\sigma',B)} : \mathcal{O}_{({}_IP,Q_I,\psi_I)}(J_I') \to B$$

is injective.

Condition (K)

Definition 16

We say that the ideal J satisfies **condition (K)** with respect to the R-system (P, Q, ψ) if J'_I satisfies condition (L) with respect to the R_I-system $({}_IP, Q_I, \psi_I)$ whenever (I, J') is a T-pair of (P, Q, ψ) such that $J \subseteq J'$.

Theorem 17

The following 3 conditions are equivalent:

1. *Every ideal of $\mathcal{O}_{(P,Q,\psi)}(J)$ is graded.*
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is injective.

Definition 18

We say that J is a **super maximal** ψ-compatible ideal if the only T-pairs (I, J') of (P, Q, ψ) which satisfies that $J \subseteq J'$, are $(0, J)$ and (R, R).

It follows that J is a super maximal ψ-compatible ideal if and only if the only graded ideals in $\mathcal{O}_{(P,Q,\psi)}(J)$ are $\{0\}$ and $\mathcal{O}_{(P,Q,\psi)}(J)$.

Example 19

Let (P_E, Q_E, ψ_E) and J_E. It follows that J_E is super maximal ψ_E-compatible ideal if and only if the only saturated hereditary subsets of E^0 are $\emptyset$ and E^0.

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Theorem 20

The following 5 conditions are equivalent:

1. *The ring $\mathcal{O}_{(P,Q,\psi)}(J)$ is simple.*
2. *The subring $\sigma(R)$ has the ideal intersection property and J is a super maximal ψ-compatible ideal.*
3. *The subring $\mathcal{O}_{(P,Q,\psi)}(J)^{(0)}$ has the ideal intersection property and J is a super maximal ψ-compatible ideal.*
4. *The ideal J satisfies condition (L) and is a super maximal ψ-compatible ideal.*
5. *If (S', T', σ', B) is a non-zero covariant representation of (P, Q, ψ) which is Cuntz-Pimsner invariant relative to J, then the ring homomorphism*

$$\eta^{J}_{(S',T',\sigma',B)} : \mathcal{O}_{(P,Q,\psi)}(J) \to B$$

is injective.

Section Examples

fractional skew monoid ring

Let R be a ring with local units and let $\alpha : R \to R$ be an injective ring homomorphism such that $\alpha(R)R\alpha(R) \subseteq \alpha(R)$.

Let

$$P = \text{span}\{r_1\alpha(r_2) \mid r_1, r_2 \in R\} \qquad \text{and} \qquad Q = \text{span}\{\alpha(r_1)r_2 \mid r_1, r_2 \in R\}$$

and

$$\psi : P \otimes Q \to R \qquad \text{given by} \qquad p \otimes q \mapsto pq\,,$$

then (P, Q, ψ) is an R-system.

Then R is a uniquely maximal, faithful, ψ-compatible ideal and that if:

1. α is an automorphism then $\mathcal{O}_{(P,Q,\psi)}(R) \cong R \times_\alpha \mathbb{Z}$.
2. R is unital and $\alpha(R) = \alpha(R)R\alpha(R) = \alpha(1)R\alpha(1)$ then $\mathcal{O}_{(P,Q,\psi)}(R) \cong R[t_+, t_-; \alpha]$

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We say that an ideal I of R is *strongly α-invariant* if $\alpha^{-1}(I) = I$.

Proposition 21

Let R be a ring with local units, $\alpha : R \to R$ an injective ring homomorphism satisfying $\alpha(R)R\alpha(R) \subseteq \alpha(R)$. Then there is a bijective correspondence between graded ideals of $R[t_+, t_-; \alpha]$ and strongly α-invariant ideals of R.

Corollary 22

Let R be a ring with local units, $\alpha : R \to R$ an injective ring homomorphism satisfying $\alpha(R)R\alpha(R) \subseteq \alpha(R)$. Then the following three conditions are equivalent:

1. *The ring R is a super maximal ψ-compatible ideal.*
2. *The only graded ideals in $R[t_+, t_-; \alpha]$ are $\{0\}$ and $R[t_+, t_-; \alpha]$.*
3. *The only strongly α-invariant ideals in R are $\{0\}$ and R.*

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fractional skew monoid ring

Definition 23

Let $n \in \mathbb{N}$ and let R be a ring with local units. A ring homomorphism $\alpha : R \to R$ is said to be **inner with periodicity** n if there exist $u, v \in \mathcal{M}(R)$ such that $vu = 1$ (where 1 denotes the unit of $\mathcal{M}(R)$), and $\alpha^n(r) = urv$ and $\alpha(ur) = u\alpha(r)$ for all $r \in R$. If α is not inner of any periodicity, then it is said to be **outer**.

Proposition 24

Let R be a ring with local units, $\alpha : R \to R$ an injective ring homomorphism satisfying $\alpha(R)R\alpha(R) \subseteq \alpha(R)$. Consider the following three conditions:

1. *There exists an $n \in \mathbb{N}$ such that the homomorphism α is inner with periodicity n.*
2. *The ring R is a ψ-invariant cycle.*
3. *The ring R does not satisfy condition (L) with respect to (P, Q, ψ).*

Then (1) implies (2), and (2) implies (3). If in addition R is a super maximal ψ-compatible ideal, and α^n is strict for every $n \in \mathbb{N}$, then (3) implies (1) and the three conditions are equivalent.

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fractional skew monoid ring

Corollary 25

Let R be a unital ring and let $\alpha : R \to R$ be an injective ring homomorphism such that $\alpha(R) = eRe$ for some idempotent $e \in R$. Then the following two statements are equivalent:

1. *The fractional skew monoid ring $R[t_+, t_-; \alpha]$ is simple.*
2. *The homomorphism α is outer and the only strongly α-invariant ideals in R are $\{0\}$ and R.*

Corollary 26

Let R be a ring with local units and let $\alpha : R \to R$ be a ring automorphism. Then the following two statements are equivalent:

1. *The crossed product $R \times_\alpha \mathbb{Z}$ is simple.*
2. *The automorphism α is outer and the only strongly α-invariant ideals in R are $\{0\}$ and R.*

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