Computable Model Theory

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1 Overview of the field

Computability theory formalizes the intuitive concepts of computation and information content. Pure computability studies these notions and how they relate to definability in arithmetic and set theory. Applied computability examines other areas of mathematics through the lens of effective processes. Tools from computability can be used to give precise definitions of concepts such as randomness. The workshop focussed on two closely related fields: computable algebra and computable model theory.

In computable algebra (and computable mathematics in general) we study the computational complexity of the constructions and objects that we use in mathematics. As mathematicians, we all know that certain objects or constructions we deal with are more complicated than others, or maybe equivalent in some sense to others. In computable algebra we use the tools of computability theory to give precise meaning to these intuitions. For instance, we can state and prove a theorem that says that building a maximal ideal in a commutative ring is more difficult than building a prime ideal. The motivations to measure the complexity of mathematics come from two sides. On one side is the foundations of mathematics, with the objective of understanding what type of computational assumptions do we make in regular mathematical practice. On the other side is the purely mathematical objective of getting a better and deeper understanding of the objects and constructions we deal with.

Model theory is the study of abstract classes of structures by analyzing their relationship with the mathematical language used to describe them. It can be thought of as a higher level of generalization compared to algebra. Computable model theory addresses similar kinds of questions as computable algebra but applies them to these more abstract settings.

2 Recent developments and main themes of the workshop

1. Information content of structures and isomorphisms

When dealing with sets of natural numbers there is a clear notion of what it means for one set to encode (or contain more information than) another. A set A is encoded into a set B if there is some program that can compute membership in A while referring (finitely often for each input) to an infinite tape containing B. This notion is known as *Turing reducibility*; informally, we say that B *computes* A. Equicomputability – encoding exactly the same information – is the equivalence relation induced by Turing reducibility; its equivalence classes are called the *Turing degrees*.

In the case of information encoded into a structure, the situation is less clear. Turing computability applies not to an isomorphism type, but rather to a particular presentation of the structure. A single structure can be presented in different ways by enumerating its elements in a different order. Isomorphic copies of the same structure may have different Turing degrees. This makes it hard to encode concrete information, such as a fixed Turing degree, into the isomorphism type of a structure. On the other hand, it is possible to encode other types of information into the isomorphism type of a structure, that cannot be encoded into a single set. For example, the property of "noncomputability" cannot be captured in a single set: there is no set of natural numbers that can be computed from every noncomputable set, and is not itself computable. However, Slaman and Wehner [8, 7] constructed algebraic structures such that a set A can compute presentations of these structures if and only if the set A is non-computable. The structure constructed by Wehner was a graph. By results of Hirschfeldt, Khoussainov, Slinko and Shore [5], examples of such structures also exist among rings, integral domains of arbitrary characteristic, partial orderings, lattices, commutative semi-groups, and 2-step nilpotent groups. Indeed, these authors show that graphs, as well as the classes of structures just mentioned, are sufficiently complicated that, roughly speaking, any interesting example of computable model theory can be realized within each class of structures.

Not all classes of structures have this property. For example, as a corollary of a result of Downey and Jockusch [3], there is no Boolean algebra that can capture the notion of non-computability in the way that Wehner's graph did. Whether or not there exists a linear ordering that captures the notion of non-computability is one of the long-standing open questions in computable model theory. To briefly indicate where the difficulty of coding into linear orderings comes from, we recall that particular computations are finite. This is why finite approximations to countable structures play a major role in the constructions. Whereas the class of finite graphs is rich, for every natural number n, there is only one linear ordering of n elements.

More recently, Greenberg, Montalbán and Slaman [4] constructed a structure whose copies are computable precisely in the non-hyperarithmetic degrees. So the notion of "non-hyperarithmeticity" can also be captured by a structure. Interestingly, they are able to exhibit a linear order with this property. Csima and Kalimullin [2] give another recent example in the same vein. They construct a structure that can be computed exactly by the hyperimmune sets, a notion defined by rates of growth of the functions computable from a set.

These results are closely related to the more general problem of characterising the form of *degree spectra* of structures. The degree spectrum of a structure is the collection of all Turing degrees of presentations of the structure. A basic problem of computable model theory is finding which collections of Turing degrees can be realized as degree spectra of structures. For example, an early and often utilized result of Knight's in this area [6] is that degree spectra of structures are upwards closed in the Turing degrees. However, not all upward-closed sets of Turing degrees are degree spectra. Partial results state, for example, that every cone is a degree spectrum, but no union of finitely many cones can be a degree spectrum of a structure.

Similar questions were raised about the complexity of isomorphisms between structures. There is much work concerning the concept of *computable categoricity*, which holds when any two computable copies of the given structure are isomorphic via a computable map. In general the questions is how complicated must be isomorphisms between structures.

2. Relative complexity of structures

The degree spectrum is a useful tool to measure the relative complexity of a structure. For instance, we can say that a structure $\mathcal{A}$ is more complicated than a structure $\mathcal{B}$ if every set that can compute a copy of $\mathcal{A}$ can also compute a copy of $\mathcal{B}$, or equivalently, if the degree spectrum of $\mathcal{A}$ is a subset of the degree spectrum of $\mathcal{B}$.

There are other ways of measuring the complexity of structures. For instance, one can use the notion of Σ-reducibility, which is is a computable version of the notion of interpretability. Kalimullin, from Kazan, showed that these measures of complexity are not equivalent, but they are still closely connected. This latter notion of complexity, which is mostly studied in Russia, is less computational and more structural that the degree-spectra notion.

This approach can also be used to study uncountable structures. Since countable objects are most suitable for the development of computability theory, most of the work in computable model theory concerns only countable structures. However, several notions of computability have been developed for uncountable objects, such as Turing operators on reals, alpha-recursion theory, Blum-Shub-Smale computability, and Borel reducibility. Each of these notions entails an approach to studying the effective content of uncountable structures. The general question of how information is coded into a structure is relevant also in the uncountable case.

3. Effective stability theory

Stability is a fundamental concept in model theory. It is involved in describing important "watersheds" in the study of abstract theories. These watersheds separate between theories whose models have common structure and theories where extra structure is absent. In one extreme end of the spectrum lie *uncountably categorical* and *strongly minimal* theories. These are theories which are closely related to algebraically closed fields or vector spaces.

A major question studied in recent years is what does the rigid structure of the models of these theories say about the complexity of the countable models. A theorem of Baldwin and Lachlan's [1] states that the countable models of such theories can be ordered according to an inherent notion of dimension. The question is how free one is to control the complexity of the structures as their dimension varies.

3 Presentation highlights

The quality of the talks during the meeting was very high; the organizers feel they were a great part of the success of the meeting. We discuss some of the progress communicated during these talks.

a. Models of strongly minimal theories.

Uri Andrews and Steffen Lempp discussed very recent results. Lempp talked on joint work with Andrews on possible spectra of computable models for strongly minimal disintegrated theories of bounded arity. The study of these spectra has been of intense interest to computable model theorists for the past thirty years, but progress has been slow, and the gap of our knowledge on what spectra could be achieved is very large. In prior work, Andrews and A. Medvedev had shown that for strongly minimal disintegrated theories in a finite language, only three spectra are possible, namely, $\emptyset$, $\{0\}$ and $\omega + 1$. Still concentrating on distintegrated theories but allowing the language to be infinite while bounding the arity of the relations in the language, Andrews and Lempp were able to obtain sharp results in the binary case, where there are exactly seven possible spectra, and are making progress toward their conjecture that for each bound n on the arity of the language, there is only a finite number of possible spectra, depending only on n; in the ternary case, they show that the number of possible spectra is between 9 and 18.

Andrews reported on work with Knight. They showed that if a strongly minimal theory has one computable model then every model is computable in $0^{(4)}$. In some special cases they can lower this bound to $0^{(3)}$ or even $0^{(2)}$. This is one of the most significant advances in recent years.

b. Interpretations into fields.

Russell Miller reported on joint work with Kirsten Eisentragger, Jennifer Park, Bjorn Poonen, and Sasha Shlapentokh, showing that graphs can be coded in fields in a way that allows effective decoding. As mentioned above, in the 2001 paper of Hirschfeldt, Khoussainov, Shore, and Slinko, there were general conditions allowing transfer of various properties from graphs to other classes (including groups, lattices, and rings). For example, the fact that there is a graph with computable dimension n yields a structure of computable dimension n in each of the other classes. HKSS was based on earlier work of Lavrov, Goncharov, and others. The problem of whether fields could be added to the list in HKSS had resisted solution. Now, fields are on the list. The proof represents an infusion of ideas from algebraic number theory into computable structure theory.

c. Degrees of categoricity

There is much known about degree spectra of structures, and also about the complexity of isomorphisms between structures. Recent work abstracts the latter. A degree **d** is a degree of categoricity if there is a computable structure A that is $\mathbf{d}$-categorical, and moreover, any **c** such that A is **c**-categorical is such that $\mathbf{c} \geq \mathbf{d}$. In contrast, a degree **d** is low for isomorphism if every pair of computable structures which are **d**-isomorphic are also computably isomorphic. Barbara Csima and David Reed Solomon gave talks discussing these concepts, giving both positive and negative results, and some examples.

d. Admissible fragments and Σ-definability

The smallest admissible set containing a structure $\mathcal{A}$ is HF$(\mathcal{A})$, the collection of hereditarily finite sets built up from using the elements of $\mathcal{A}$ as "ur-elements". A structure $\mathcal{A}$ is Σ-reducible to a structure $\mathcal{B}$ if $\mathcal{A}$ can be interpreted in HF$(\mathcal{B})$ using existential formulas. This notion of reducibility (relative complexity) makes sense even when the structures are uncountable. Andrei Morozov discussed what happens if one takes $\mathcal{B}$ to be the real numbers. Vadim Puzarenko talked about further generalisations of these notions, including countable categoricity in admissible fragments.

4 Informal discussions at the workshop

There were many informal discussions at the workshop. As expected most of the scientific progress during the workshop occurred during these discussions. We list a few.

1. Following Andrews' and Lempp's talks on new results on strongly minimal theories, there were discussions aimed at pushing these results further. In particular, Andrews and Knight met to try to improve their result saying that for a strongly minimal theory, if one model is computable, then all models have copies computable in $\emptyset^{(4)}$. Andrews and Lempp have new results on the "spectrum" for a strongly minimal theory, where this is the set of dimensions of computable models, and a large group met to try to extend these results.

2. During Montalbán's talk on the theme of statements holding on a cone he gave some computability-theoretic versions of Vaught's Conjecture of this kind. Knight and Montalbán discussed Vaught's Conjecture.

3. Puzarenko, Morozov, and Knight considered the problem of whether, for a computable structure of Scott rank ω_1^{CK}, the computable infinitary theory must be $\aleph_0$-categorical. This followed from Puzarenko's talk on a characterization of $\aleph_0$-categoricity for an infinitary theory.

4. Greenberg, Knight and Melnikov discussed a result characterizing the ω_1-computably categorical structures, and tried to find examples illustrating this result.

5. Following a question posed by Andrews, a large group met and pushed current techniques sufficiently to show that the collection of bi-hyper-hyper-immune Turing degrees form a degree spectrum.

5 Problem session

Julia Knight and Antonio Montalbán led an open problem session. Uri Andrews wrote a preliminary account of the problems.

- Bakhadyr Khoussainov stated two problems, due originally to Sergey Goncharov and Terry Millar.

  Problem 1. Suppose T is a decidable complete theory with only countably many countable models, up to isomorphism. Does the prime model have a decidable copy?

  As background, Khoussainov recalled the result of Harrington, and Goncharov-Nurtazin, saying that a theory has a decidable prime model iff the principal types are dense, and there is computable enumeration of these types. It follows that if the saturated model is decidable, so is the prime model. Julia Knight mentioned that Jessica Millar had attempted to give a positive answer to Problem 1, in her thesis. Uri Andrews noted that the answer is positive for ω-stable theories, but "small" does not imply ω-stable.

  Problem 2. Is there a saturated computable structure of computable dimension 2?

  Khoussainov noted that if we replace "saturated" by "atomic", the answer is positive.

- Uri Andrews stated the next two problems.

  Problem 3. Is there a strongly minimal modular group such that $SRM((G, +, R_1, \ldots, R_n)) = \{0\}$, where the R_i are subgroups of cartesian powers of G.

  Andrews said "This really has a lot to do with the word problem on a division ring. Someone who knows more about word problems and who is willing to dig in, might be able to do this a lot easier than I could. I've given up, but that's no indication that it's hard. It might just take some tools from word problems that I just don't know."

  Problem 4. Is it true that for any atomic theory T, there is a theory T^* such that the degree spectrum of T^* is the union of the degree spectra of the non-prime models of T.

  The problem arose in work of Andrews, Joe Miller, and others, comparing theory spectra and structure spectra. Let T be true arithmetic. Solovay showed that the degrees of non-prime models of T are the degrees of enumerations of Scott sets that contain the arithmetical sets. There are other theories T^*, in particular, other completions of PA, for which the theory spectrum is this same set.

- The next problem is due to Russell Miller.

  Problem 5. Is there a low differentially closed field (of characteristic 0) with no computable copy?

  Miller noted that the answer is positive if we replace low by low_2.

- The next two problems are due to Julia Knight. For a computable finitely generated group, there is a computable infinitary Σ_3 scott sentence.

  Problem 6. Is there a computable finitely generated group whose index set is m-complete Σ^0_3?

  Problem 7. Is there a computable finitely generated group with no computable infinitary $d-\Sigma_2$ scott sentence?

- The next two problems are due to Antonio Montalbán. If $\mathcal{K}$ is a class of countable structures closed under $\cong$, the *categoricity ordinal* of $\mathcal{K}$ is the least α such that $\mathcal{K}$ contains a Δ_α categorical structure. Montalbán noted that if $\mathcal{K}$ is axiomatized by a Π_2 sentence of $L_{\omega_1\omega}$ and $\mathcal{K}$ is Σ-small, then, on a cone, it has a computably categorical structure. Hence, on a cone, the categoricity ordinal is 1. Being axiomatized by a Π_2 sentence means that the class K is $\mathbf{\Pi_2}$ in the Borel hierarchy.

  Problem 8. What is the effectiveness of this observation?

  Problem 9. If $\mathcal{K}$ is not Σ-small, but Π_2, then can it have categoricity ordinal $\alpha > 1$?

- The next problem is due to Andrei Morozov.

  Problem 10. In $HF(\mathbb{C})$, does there exists a Σ-definable copy of $\mathbb{C}$ that is not Σ-definably isomorphic to the base copy of $\mathbb{C}$?

  By way of background, Morozov has shown that if we replace $\mathbb{C}$ by $\mathbb{R}$, the problem has a positive answer. Andrews said, "I'm pretty sure that if you can bound how far up the HF-heierarchy the copy is, you can get the Σ-isomorphism."

- The next problem is due to Iskander Kalimullin.

  Problem 11. If a structure $\mathcal{A}$ has a hyperarithmetical copy, is the degree spectrum of $\mathcal{A}$ Δ^1_1.

  As background, if $\mathcal{A}$ is hyper arithmetical, and does not have Δ^1_1 degree spectrum, then it must have high Scott rank.

- The next problem, due to Rod Downey, was suggested by Steffen Lempp, at the urging of Khoussainov.

  Problem 12. Suppose $\mathcal{A}$ is a linear order, and for every computable copy, there is a computable non-trivial self-embedding. Does $\mathcal{A}$ necessarily have a strongly η-like interval?

  As background, Lempp noted that an interval is *strongly η-like* if the equivalence relation of being finitely far apart has a finite bound on the size of classes.

6 Outcome of the Meeting

The workshop was very successful. Our main criterion in making up the small group (20 researchers) was a balance between having a variety of interests and having people who can effectively collaborate. We believe that this was achieved, as witnessed by both the variety of the talks and the informal discussions, several of which resulted in new results.

The field has seen some remarkable advances in the last year or two, among them Miller's work on interpretation into fields and the Andrews-Knight work on strongly minimal theories. The workshop was a good opportunity to disseminate these results to researchers in the community, and to observe how they can be improved.

We hope to organize a follow-up meeting within two or three years.

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