Bost-Connes Systems and Induction

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1. Motivation

$$e^{\pi\sqrt{163}} = 262537412640768743.99999999999925007259$$

Almost an integer, and not by accident!

Put $q(\tau) = e^{2\pi i\tau}$. There is an analytic function of τ, the *j-invariant*, with

$$j(\tau) = q(\tau)^{-1} + 744 + 196884q(\tau) + 21493860q(\tau)^2 + \ldots$$

For $\tau_0 = (1+\sqrt{-163})/2$, we have $q(\tau_0) = -e^{-\pi\sqrt{163}}$, and

$$j(\tau_0) = -262537412640768000 \approx q(\tau_0)^{-1} + 744$$

The 'miracle' has to do with the fact that $\mathbb{Z}[\tau]$ is a PID, so that the field $\mathbb{Q}(\sqrt{-163})$ has no everywhere unramified abelian extensions.

Kronecker's Jugendtraum/Hilbert's 12th Problem: in general, can we generate abelian extensions of number fields by special values of analytic functions?

In the known case, imaginary quadratic fields, the analytic functions come from certain concrete algebraic curves.

Connes' idea: consider functions on *non-commutative* geometric objects, ie elements in some C^*-dynamical system, and evaluate them at certain 'fabulous' KMS states.

Conjecture. (Connes et al.) For a number field K, there exists a C^*-dynamical system A_K (the 'Bost-Connes system') with a $\mathbb{Q}$-subalgebra A_K^0 satisfying:a) Extremal KMS$_\infty$ states of A_K form a principal homogenous space under G_K^{ab}, and

b) For each such state ϕ and every $a \in A_K^0$, $\phi(a)$ is an algebraic number. This correspondence is G_K^{ab}-equivariant:

$$\phi(\tau a) = \tau(\phi(a)), \text{ for all } \tau \in G_K$$

c) The set of all $\phi(a)$ generates K^{ab}/K.

d) The partition function of the system is $\zeta_K(\beta)$.

Candidates for A_K

- Bost-Connes original system is the Hecke algebra corresponding to the pair

$$\begin{bmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{bmatrix} \subset \begin{bmatrix} 1 & \mathbb{Q} \\ 0 & \mathbb{Q}^* \end{bmatrix}.$$

  They recover the well-known result, $\mathbb{Q}^{ab} = \mathbb{Q}(\sqrt[\infty]{1})$.
- Laca and Frankenhuijsen generalize this to an arbitrary number field by considering the Hecke pair

$$\begin{bmatrix} 1 & \mathcal{O} \\ 0 & \mathcal{O}^* \end{bmatrix} \subset \begin{bmatrix} 1 & K \\ 0 & K^* \end{bmatrix}$$

This has the right phase transition and spontaneous symmetry breaking only when $\mathcal{O}$ is a principal ideal domain.

- The 'right' definition is a semigroup crossed product, which explicitly puts in the correct G_K^{ab} action.

We establish relations between these.

Local Class Field Theory

- $\mathcal{K}$ non-archimedean:

$$\begin{array}{ccccccccc} 0 \to & \mathcal{O}^* & \to & \mathcal{K}^* & \xrightarrow{v} & \mathbb{Z} & \to 0 \\ & \downarrow\cong & & \downarrow & & \downarrow & \\ 0 \to & I_v^{ab} & \to & G_{\mathcal{K}}^{ab} & \xrightarrow{rec_{\mathcal{K}}} & \hat{\mathbb{Z}} & \to 0 \\ & & & Frob_v & \mapsto & 1 & \end{array}$$

- $\mathcal{K}$ archimedean, ie $\mathcal{K} = \mathbb{R}$ or $\mathbb{C}$: $\mathcal{K}^*/\mathcal{K}^o \xrightarrow[rec_{\mathcal{K}}]{\sim} G_{\mathcal{K}}^{ab}$.

Number Fields - Notation

Fix K a number field (finite extension of $\mathbb{Q}$).

- $\mathcal{O}$: ring of integers of K
- v a valuation:
  - $v \nmid \infty$: $v = v_{\mathfrak{P}}$, valuation at a prime $\mathfrak{P}$ of $\mathcal{O}$
  - $v \mid \infty$: $v : K \hookrightarrow \mathbb{R}$ or $K \hookrightarrow \mathbb{C}$
- K_v: completion of K at v
- $\mathcal{O}_v$: ring of integers of K_v (for $v \nmid \infty$)
- $K_\infty = \prod_{v|\infty} K_v$
- $K_\infty^* = \prod_{v|\infty} K_v^*$, K_∞^o - connected comp. of K_∞^*

- $K_+^* = \{x \in K^* : v(x) > 0, \quad \forall v \nmid \infty\}$,
  $\mathcal{O}_+^* = K_+^* \cap \mathcal{O}^*$: groups of totally positive elements
- Finite adèles: restricted product

$$\mathbb{A}_f = \prod_{\substack{v \nmid \infty \\ \mathcal{O}_v \subset K_v}}' K_v$$

- Adèles: locally compact topological ring

$$\mathbb{A} = \prod_{\substack{\text{all } v \\ \mathcal{O}_v \subset K_v}}' K_v = K_\infty \times \mathbb{A}_f$$

- Idèles:

$$\mathbb{A}^* = \prod_{\substack{\text{all } v \\ \mathcal{O}_v^* \subset K_v^*}}' K_v^* = K_\infty^* \times \mathbb{A}_f^*$$

- $\hat{\mathcal{O}} = \prod_{v \nmid \infty} \mathcal{O}_v$
- $J \cong \mathbb{A}_f^* / \hat{\mathcal{O}}^*$: the group of all ideals
- $P \cong K_+^* / \mathcal{O}_+^*$: the subgroup of J consisting of principal fractional ideals with a totally positive generator
- $\mathrm{Cl}_+ = J/P$: the narrow ideal class group - always finite

Global Class Field Theory

Global Reciprocity Law glues together local reciprocity maps:

$$\begin{array}{ccccccccc} 1 & \to & \overline{K_\infty^o K^*} & \to & \mathbb{A}^* & \xrightarrow{rec_K} & G_K^{ab} & \to & 1 \\ & & & & \uparrow & & \uparrow & & \\ & & & & K_v^* & \xrightarrow{rec_{K_v}} & G_{K_v}^{ab} & & \end{array}$$

Main point ('reciprocity') is $rec_K(K^*) = id$.

For us it's convenient to express the Global Reciprocity Law purely in terms of $\mathbb{A}_f^*$:

The restrictions of rec_K to $\mathbb{A}_f^* \supset \hat{\mathcal{O}}^*$ give isomorphisms

$$\mathbb{A}_f^*/\overline{K_+^*} \cong G_K^{ab}$$
$$\hat{\mathcal{O}}^*/\overline{\mathcal{O}_+^*} \cong G_{K^{ab}/H_+(K)}$$

Here $H_+(K)$ is the (finite) extension of K which is its 'universal cover' - maximal extension unramified at all $v \nmid \infty$.

Induction

$\rho : H \hookrightarrow G$: injective homomorphism of discrete abelian groups.

X: locally compact space with a left action of H by homeomorphisms. H acts diagonally on $G \times X$. Put

$$G \times_H X = H \backslash (G \times X).$$

The composition $i : X \to G \times X \to G \times_H X$ is H-equivariant and induces $H \backslash X \cong G \backslash (G \times_H X)$.

$i(X)$ is clopen in $G \times_H X$, the corresponding projection in the multiplier algebra of $C_0(G \times_H X) \rtimes_r G$ is full, and

$$C_0(X) \rtimes_r H \cong \mathbf{1}_{i(X)}(C_0(G \times_H X) \rtimes_r G)\mathbf{1}_{i(X)}.$$

The Bost-Connes System

$\mathbb{A}_f^* \times \mathbb{A}_f^*$ acts on $\mathbb{A}_f^* \times \mathbb{A}_f$ by

$$(g,h)(x,y) = (gxh^{-1}, hy)$$

1st coordinate acts on 1st component, 2nd acts diagonally.

Consider $\omega : \mathbb{A}_f^* \times \mathbb{A}_f \to \mathbb{A}_f^* \times \mathbb{A}_f$,

$$(x,y) \mapsto (x^{-1}, xy)$$

Then $\omega((g,h)(x,y)) = (h,g)\omega(x,y)$.

Restricting the actions, we see that ω intertwines with the flip homomorphsim $\overline{K_+^*} \times \hat{\mathcal{O}}^* \to \hat{\mathcal{O}}^* \times \overline{K_+^*}, (g,h) \mapsto (h,g)$, and induces

$$_{(\overline{K_+^*} \times \hat{\mathcal{O}}^*)}\backslash(\mathbb{A}_f^* \times \mathbb{A}_f) \cong {}_{(\hat{\mathcal{O}}^* \times \overline{K_+^*})}\backslash(\mathbb{A}_f^* \times \mathbb{A}_f)$$

Identify these two quotients:

$$
\begin{aligned}
{}_{(\overline{K_+^*}\times\hat{\mathcal{O}}^*)}\backslash(\mathbb{A}_f^* \times \mathbb{A}_f) &\cong \mathbb{A}_f^*/\overline{K_+^*} \times_{\hat{\mathcal{O}}^*} \mathbb{A}_f \\
&\cong G_K^{ab} \times_{\hat{\mathcal{O}}^*} \mathbb{A}_f \\
&\cong X
\end{aligned}
$$

$$
\begin{aligned}
{}_{(\hat{\mathcal{O}}^*\times\overline{K_+^*})}\backslash(\mathbb{A}_f^* \times \mathbb{A}_f) &\cong \mathbb{A}_f^*/\hat{\mathcal{O}}^* \times_{\overline{K_+^*}} \mathbb{A}_f \\
&\cong J \times_{(\overline{K_+^*}/\overline{\mathcal{O}_+^*})} \mathbb{A}_f/\overline{\mathcal{O}_+^*} \\
&\cong J \times_P \mathbb{A}_f/\overline{\mathcal{O}_+^*} = X'
\end{aligned}
$$

Note: Both X and X' come with a right action of $J = \mathbb{A}_f^*/\hat{\mathcal{O}}^*$, via $rec_K \times id$ and via and the 1st component, respectively.

Define two subsets

$$Y = G_K^{ab} \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}$$
$$Y' = \{(g, \omega) \in X^+ : g\omega \in \hat{\mathcal{O}}/\hat{\mathcal{O}}^*\}$$

Two isomorphic dynamical systems:

Bost-Connes:
$$\begin{cases} A_K = \mathbf{1}_Y (C_0(X) \rtimes J) \mathbf{1}_Y \\ \sigma_t^K (f u_g) = N_K(g)^{it} f u_g. \end{cases}$$

Twisted Bost-Connes:
$$\begin{cases} A'_K = \mathbf{1}_{Y'} (C_0(X') \rtimes J) \mathbf{1}_{Y'} \\ \sigma_t'^K (f u_g) = N_K(g)^{it} f u_g. \end{cases}$$

Connection with Hecke Algebras

The '$ax+b$' groups

$$H^+_{\mathcal{O}} = \begin{bmatrix} 1 & \mathcal{O} \\ 0 & \mathcal{O}^*_+ \end{bmatrix} \subset H^+_K = \begin{bmatrix} 1 & K \\ 0 & K^*_+ \end{bmatrix}$$

form a Hecke pair: any double coset is a finite union of cosets.

The Hecke algebra $C^*_r(H^+_{\mathcal{O}}, H^+_K)$ has a time evolution

$$\sigma_t\left(\begin{bmatrix} 1 & y \\ 0 & x \end{bmatrix}\right) = \mathrm{N}_K(x)^{it}.$$

Theorem 1. *Consider the inclusions* $\hat{\mathcal{O}}/\overline{\mathcal{O}_+^*} \subset \mathbb{A}_f/\overline{\mathcal{O}_+^*}$ *and* $Z_{H^+(K)} = G_{K^{ab}/H_+(K)} \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}} \subset X$, *and let* p_1, p_2 *be the corresponding projections. There are* C^**-algebra isomorphisms*

$$C_r^*(H_K^+, H_{\mathcal{O}}^+) \cong$$
$$p_1(C_0(\mathbb{A}_f/\overline{\mathcal{O}_+^*}) \rtimes P)p_1 \cong$$
$$p_2(A_K)p_2$$

The isomorphisms can be chosen so that the canonical time evolution $C_r^*(H_{\mathcal{O}}^+, H_K^+)$ *is compatible with the one on the cross products given by* $\sigma_t(fu_x) = N_K(x)^{it} fu_x$, *restricted to the corner.*

Proof. The first isomorphism is straightforward: use duality and the fact that $H_K^+ = K \rtimes K^*$, so we can project in two stages. For the second isomorphism, induce from P to J. Let $i : \mathbb{A}_f/\overline{\mathcal{O}_+^*} \hookrightarrow J \times_P \mathbb{A}_f/\overline{\mathcal{O}_+^*} = X'$ be the obvious inclusion. By the induction lemma,

$$C_0(\mathbb{A}_f/\overline{\mathcal{O}_+^*}) \rtimes P \cong \mathbf{1}_{i(\mathbb{A}_f/\overline{\mathcal{O}_+^*})}(C_0(X') \rtimes J)\mathbf{1}_{i(\mathbb{A}_f/\overline{\mathcal{O}_+^*})}.$$

Tracing through the various identifications, we get $i(p_1)\mathbf{1}_{i(\mathbb{A}_f/\overline{\mathcal{O}_+^*})} = p'\mathbf{1}_{Y'}$, where p' corresponds to p_2 under the isomorphism $A'_K \cong A_K$. Then

$$\begin{aligned}
&p_1(C_0(\mathbb{A}_f/\overline{\mathcal{O}_+^*}) \rtimes P)p_1 \cong \\
&i(p_1)\mathbf{1}_{i(\mathbb{A}_f/\overline{\mathcal{O}_+^*})}(C_0(X') \rtimes J)\mathbf{1}_{i(\mathbb{A}_f/\overline{\mathcal{O}_+^*})}i(p_1) \cong \\
&p'\mathbf{1}_{Y'}(C_0(X') \rtimes J)\mathbf{1}_{Y'}p' \cong p'(A'_K)p' \cong p_2(A_K)p_2 \square
\end{aligned}$$

Functoriality of Bost-Connes Systems

Let $\tau : K \hookrightarrow L$ be an inclusion of number fields. Put

$$X_\tau = J_L \times_{J_K} X_K$$

Compare actions of J_L on X_τ and X_L.

Lemma 1. *$\pi_\tau : X_\tau \to X_L, \pi_\tau(g, x) = gx$ is J_L-equivariant with dense image.*

π_τ defines a J_L-equivariant injective homomorphism $\mathbb{C}_0(X_L) \to C_b(X_\tau)$, hence $\pi_\tau^* : C_0(X_K) \rtimes J_L \to M(C_0(X_\tau) \rtimes J_L)$.

$\iota_{\mathcal{T}} : X_K \to X_{\mathcal{T}}, x \mapsto (\mathcal{O}_L, x)$ is a J_K-equivariant embedding which induces an isomorphism $\iota_{\mathcal{T}}^* : \mathbf{1}_{\iota_{\mathcal{T}}(X_K)}(C_0(X_{\mathcal{T}}) \rtimes J_L)\mathbf{1}_{\iota_{\mathcal{T}}(X_K)} \to C_0(X_K) \rtimes J_K$.

$\tilde{A}_{\mathcal{T}} = (C_0(X_{\mathcal{T}}) \rtimes J_L)\mathbf{1}_{\iota_{\mathcal{T}}(X_K)}$ is a $(C_0(X_L) \rtimes J_L)$-$(C_0(X_K) \rtimes J_K)$ correspondence:

- right Hilbert $C_0(X_K) \rtimes J_K$-module via $(\iota_{\mathcal{T}}^*)^{-1}$, and the $C_0(X_K) \rtimes J_K$-valued inner product by $\langle \xi, \eta \rangle = \iota_{\pi}^*(\xi^* \eta)$.
- right $C_0(X_L) \rtimes J_L$-module via $\pi_{\mathcal{T}}^*$

To get a correspondence of BC algebras, we need to pass to a corner: $A_\tau = \mathbf{1}_{Y_L} \tilde{A}_\tau \mathbf{1}_{Y_K}$.

Lemma 2. *Let $\tau : K \to L$, $\rho : L \to K$ field embeddings. Then $A_\rho \otimes_{A_L} A_\tau \cong A_{\rho\circ\tau}$, canonically.*

To make A_K's into compatible C^*-dynamical systems, we normalize the time evolution by

$$\sigma_t^K(fu_g) = N_K(g)^{it/[K:\mathbb{Q}]} fu_g$$

We have a 1-parameter family of isometries on $A_\tau \subset C_0(X_\tau) \rtimes J_L$ given by $U_t^\tau fu_g = N_L(g)^{it/[L:\mathbb{Q}]} fu_g$. This makes A_τ into an equivariant correspondence of C^*-dynamical systems (A_L, σ^L) and (A_K, σ^K) in the sense that

$$U_t^\tau(a\xi) = \sigma_t^L(a)U_t^\tau\xi$$
$$U_t^\tau(\xi b) = U_t^\tau\xi\sigma_t^K(b)$$
$$\langle U_t^\tau\xi, U_t^\tau\eta\rangle = \sigma_t^K(\langle\xi,\eta\rangle).$$

Theorem The maps $K \mapsto (A_K, \sigma_K), (\tau : K \to L) \mapsto (A_\tau, U_t^\tau)$ defines a functor from the category of number fields (with embeddings as morphisms) and C^*-dynamical systems (with isomorphism classes of equivariant correspondences as morphisms).