Piecewise Conjugacy as an Isomorphism Invariant for Operator Algebras

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A complex number a is called *algebraic* if there exists a nonzero polynomial $p(X) \in \mathbb{Q}[X]$ such that $p(a) = 0$. The polynomial is unique if we require that it be irreducible and monic. We say that a is an algebraic integer if the unique irreducible, monic polynomial which it satisfies has integer coefficients. We know that the set $\overline{\mathbb{Q}}$ of all algebraic numbers is a field, and the algebraic integers form a ring. For an algebraic number a, the set K of all $f(a)$, with $f(X) \in \mathbb{Q}[X]$ is a field, called an algebraic number field. If all the roots of the polynomial $p(X)$ are in K, then K is called Galois over $\mathbb{Q}$.

QUESTION. Which invariants of a number field characterize it up to isomorphism?

The absolute Galois group of a number field K is the group $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ consisting of all automorphisms σ of $\overline{\mathbb{Q}}$ such that $\sigma(a) = a$ for all $a \in K$. Let $f(X) \in K[X]$ be irreducible, and let Z_f be the set of its roots. The group of permutations of Z_f is a finite group, which is given the discrete topology. Then G_K acts on Z_f. We put a topology on G_K, so that the homomorphism of G_K to the group of permutations of Z_f is continuous for every such $f(X)$. Then G_K is a topological group; it is compact and totally disconnected.

THEOREM. [Uchida, 1976] Number fields E and F are isomorphic as fields if and only if G_E and G_F are isomorphic as topological groups.

The absolute Galois group is not well understood at all (it is considered an anabelian object). What we do understand well are abelian

Galois groups. For a number field K we denote by K^{ab} the maximal abelian extension of K. This is the maximal extension which is Galois (i.e., any irreducible polynomial which has a root in K^{ab} has all its roots in it), and such that the Galois group of K^{ab} over K is abelian. For example, the theorem of Kronecker and Weber says that $\mathbb{Q}^{\text{ab}}$ is the field generated by all the numbers $\exp(\frac{2\pi i}{n})$, i.e., by all roots of unity. Unfortunately,

EXAMPLE. The abelianized Galois groups of $\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\sqrt{-3})$ are isomorphic.

THEOREM. [Cornelissen and Marcolli, to appear] Let E and F be number fields. Then, E and F are isomorphic if and only if there exists an isomorphism of topological groups

$$\psi \colon G_E^{\mathrm{ab}} \to G_F^{\mathrm{ab}}$$

such that for every character χ of G_F^{ab} we have $L_{F,\chi} = L_{E,\psi\circ\chi}$, where $L_{F,\chi}$ denotes the L-function associated with ψ .

Cornelissen and Marcolli make use of our work on multivariable dynamics, i.e., the concept of piecewise conjugacy and the fact that piecewise conjugacy is an invariant for isomorphisms between certain operator algebras associated with multivariable dynamical systems.

(X, σ) a topological dynamical system, i.e.,

- X locally compact Hausdorff space
- $\sigma : X \to X$ proper continuous map.

Similarly

(A, α) a C*-dynamical system, i.e.,

- A is a C*-algebra
- $\sigma : A \to A$ non-degenerate $*$-endomorphism.

Multivariably...

(X, σ) is a multivariable dynamical system:
X locally compact Hausdorff
$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, where $\sigma_i : X \to X$, $1 \leq i \leq n$, are continuous (proper) maps.

and a similar definition for a multivariable C*-dynamical system (A, α).

We want an operator algebra $\mathcal{A}$ that encodes (X, σ):

$\mathcal{A}$ contains $C_0(X)$ and $S_1, \dots, S_n$ satisfying covariance relations:

$$fS_i = S_i(f \circ \sigma_i)$$

for $1 \leq i \leq n$ and $f \in C_0(X)$

$\mathbb{F}_n^+$ is the free semigroup on n letters.
For $w \in \mathbb{F}_n^+$, say $w = i_k \dots i_1$, write $S_w = S_{i_k} \dots S_{i_1}$.

The covariance algebra is

$$\mathcal{A}_0 = \Big\{ \sum_{\text{finite}} S_w f_w : f_w \in C_0(X) \Big\}.$$

This is an algebra since:

$$(S_v)(fS_w g) = S_{vw}(f \circ \sigma_w) g$$

where $\sigma_w = \sigma_{i_k} \circ \dots \circ \sigma_{i_1}$.

We need a norm condition in order to complete $\mathcal{A}_0$.
Given the choices:

(1) Contractive: $\|S_i\| \leq 1$ for $1 \leq i \leq n$

(2) Row Contractive: $\left\| \begin{bmatrix} S_1 & S_2 & \dots & S_n \end{bmatrix} \right\| \leq 1$.

we get:

Completing $\mathcal{A}_0$ using (1) yields the semicrossed product $C_0(X) \times_\sigma \mathbb{F}_n^+$.

Completing $\mathcal{A}_0$ using (2) yields the tensor algebra $\mathcal{T}_+(X, \sigma)$.

The semicrossed product $C_0(X) \times_\sigma \mathbb{Z}^+$.

It was introduced by Arveson (1967), Arveson and Josephson (1969) and formalized by Peters (1985).

For each $x \in X$, $f \in C_0(X)$ define

$\pi_x(f)\xi = (f(x)\xi_0, (f \circ \eta)(x)\xi_1, (f \circ \eta^{(2)})(x)\xi_2, \dots)$.

Let S_x be the forward shift

$$S_x\xi = (0, \xi_0, \xi_1, \xi_2, \dots).$$

It turns out that $C_0(X) \times_\sigma \mathbb{Z}^+$ the norm closed operator algebra acting on $\mathcal{H} \equiv \oplus_{x \in X} \mathcal{H}_x$ and generated by the operators $\pi(f) \equiv \oplus_{x \in X} \pi_x(f)$, $f \in C_0(X)$, and $S \equiv \oplus_{x \in X} S_x$.

The classification problem

Classify the semicrossed products $C_0(X) \times_\sigma \mathbb{Z}^+$ as algebras.

A sufficient condition: Assume that σ_1 and σ_2 are topologicaly conjugate, i.e., there exists a homeomorphism

$$\gamma : X_1 \to X_2$$

so that

$$\gamma \circ \sigma_1 = \sigma_2 \circ \gamma.$$

Then the semicrossed products $C_0(X_1) \times_{\sigma_1} \mathbb{Z}^+$ and $C_0(X_2) \times_{\sigma_2} \mathbb{Z}^+$ are isomorphic as algebras.

Necessity:

- Arveson and Josephson (1969). X_i compact, σ_i no fixed points, plus some extra conditions

- Peters (1985). X_i compact, σ_i no fixed points.

- Hadwin and Hoover (1988). X_i compact, the set

$$\{x \in X_i \mid \sigma_1(x) \neq x, \sigma_1^{(2)}(x) = \sigma_1(x)\}$$

has empty interior.

- Power (1992). X_i locally compact, σ_i homeomorphisms

THEOREM. (Davidson and Katsoulis, 2008) Let X_i be a locally compact Hausdorff space and let σ_i a proper continuous map on X_i, for $i = 1, 2$. Then the dynamical systems (X_1, σ_1) and (X_2, σ_2) are conjugate if and only if the semicrossed products $C_0(X_1) \times_{\sigma_1} \mathbb{Z}^+$ and $C_0(X_2) \times_{\sigma_2} \mathbb{Z}^+$ are isomorphic as algebras.

Piecewise conjugate multisystems

Two multivariable dynamical systems (X, σ) and (Y, τ) are said to be *conjugate* if there exists a homeomorphism γ of X onto Y and a permutation $\alpha \in S_n$ so that $\tau_i = \gamma \sigma_{\alpha(i)} \gamma^{-1}$ for $1 \le i \le n$.

DEFINITION. We say that two multivariable dynamical systems (X, σ) and (Y, τ) are *piecewise conjugate* if there is a homeomorphism γ of X onto Y and an open cover $\{\mathcal{U}_\alpha : \alpha \in S_n\}$ of X so that for each $\alpha \in S_n$,

$$\gamma^{-1} \tau_i \gamma|_{\mathcal{U}_\alpha} = \sigma_{\alpha(i)}|_{\mathcal{U}_\alpha}.$$

The difference in the two concepts of conjugacy lies on the fact that the permutations depend on the particular open set. As we shall see, a single permutation generally will not suffice.

PROPOSITION. Let (X, σ) and (Y, τ) be piecewise conjugate multivariable dynamical systems. Assume that X is connected and that

$$E := \{x \in X : \sigma_i(x) = \sigma_j(x), \text{for some } i \neq j\}$$

has empty interior. Then (X, σ) and (Y, τ) are conjugate.

For $n = 2$, we can be more definitive.

PROPOSITION. Let X be connected and let $\sigma = (\sigma_1, \sigma_2)$; and let E as above. Then piecewise conjugacy coincides with conjugacy if and only if $\overline{X \backslash E}$ is connected.

The multivariable classification problem.

THEOREM. Let (X, σ) and (Y, τ) be two multivariable dynamical systems. If $\mathcal{T}_+(X, \sigma)$ and $\mathcal{T}_+(Y, \tau)$ or $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ and $\mathrm{C}_0(Y) \times_\tau \mathbb{F}_n^+$ are isomorphic as algebras, then the dynamical systems (X, σ) and (Y, τ) are piecewise conjugate.

For the tensor algebras, sufficiency holds in the following cases:

(i) X has covering dimension 0 or 1

(ii) σ consists of no more than 3 maps. ($n \leq 3$.)

For instance:

THEOREM. Suppose that X is a compact subset of $\mathbb{R}$. Then for two multivariable dynamical systems (X, σ) and (Y, τ), the following are equivalent:

1. (X, σ) and (Y, τ) are piecewise topologically conjugate.
2. $\mathcal{T}_{+}(X, \sigma)$ and $\mathcal{T}_{+}(Y, \tau)$ are isomorphic.
3. $\mathcal{T}_{+}(X, \sigma)$ and $\mathcal{T}_{+}(Y, \tau)$ are completely isometrically isomorphic.

The analysis of the $n = 3$ case is the most demanding and required non-trivial topological information about the Lie group $SU(3)$. The conjectured converse reduces to a question about the unitary group $U(n)$.

CONJECTURE. Let Π_n be the $n!$-simplex with vertices indexed by S_n. Then there should be a continuous function u of Π_n into $U(n)$ so that:

1. each vertex is taken to the corresponding permutation matrix,

2. for every pair of partitions (A, B) of the form

   $$\{1, \ldots, n\} = A_1 \dot\cup \ldots \dot\cup A_m = B_1 \dot\cup \ldots \dot\cup B_m,$$

   where $|A_s| = |B_s|,\ 1 \le s \le m$, let

   $$\mathcal{P}(A, B) = \{\alpha \in S_n : \alpha(A_s) = B_s, 1 \le s \le m\}.$$

   If $x = \sum_{\alpha \in \mathcal{P}(A,B)} x_\alpha \alpha$, then the non-zero matrix coefficients of $u_{ij}(x)$ are supported on $\bigcup_{s=1}^m B_s \times A_s$. We call this the *block decomposition condition*.

We have established this conjecture for $n = 2$ and 3 and Chris Ramsey the cases $n = 4, 5$.

When one moves away from classical dynamical systems and commutative C*-algebras, there is very limited understanding for algebraic isomorphism invariants. Instead we investigate isomorphism invariants for isometric isomorphisms.

The C*-dynamical systems $(\mathcal{A}, \sigma)$ and $(\mathcal{B}, \tau)$ are called *outer conjugate* if there exists a $*$-isomorphism $\gamma : \mathcal{A} \to \mathcal{B}$ and a unitary $U \in M(\mathcal{A})$ so that

$$\sigma(A) = U^*(\gamma^{-1} \circ \beta \circ \gamma(A))U,$$

for all $A \in \mathcal{A}$.

THEOREM (Davidson andKatsoulis 2008). Let $(\mathcal{A}, \sigma)$ and $(\mathcal{B}, \tau)$ be unital C*-dynamical systems and assume that $\mathcal{A}$ is simple and both σ and τ are automorpisms. Then, the semicrossed products $\mathcal{A} \times_\sigma \mathbb{Z}^+$ and $\mathcal{B} \times_\tau \mathbb{Z}^+$ are isometrically isomorphic if and only if the dynamical systems $(\mathcal{A}, \sigma)$ and $(\mathcal{B}, \tau)$ are outer conjugate.

Recently, Davidson and Kakariadis removed the requirement that the C*-algebras be simple from the above result. Therefore semicrossed products of arbitrary unital C*-algebras by automorphism are classified **up to isometric isomorphism** by outer conjugacy.

With Ken Davidson we considered only classical dynamical systems (dynamical systems over commutative C*-algebras) and our notion of piecewise conjugacy applies exclusively to such systems. Motivated by the interaction between number theory and non-selfadjoint operator algebras, one wonders whether a useful analogue of piecewise conjugacy can be developed for multivariable systems over arbitrary C*-algebras. The goal here is to obtain a natural notion of piecewise conjugacy that generalizes that of Davidson and Katsoulis from the commutative case while remaining an invariant for isomorphisms between non-selfadjoint operator algebras associated with such systems.

DEFINITION. Let A be a unital C*-algebra and let $P(A)$ be its pure state space equipped with the w^*-topology. The *Fell spectrum* $\hat{A}$ of A is the space of unitary equivalence classes of non-zero irreducible representations of A. (The usual unitary equivalence of representations will be denoted as $\sim$.) The GNS construction provides a surjection $P(A) \rightarrow \hat{A}$ and $\hat{A}$ is given the quotient topology.

Let A be a unital C*-algebra A and $\alpha = (a_1, \alpha_2, \ldots, \alpha_n)$ be a multivariable system consisting of unital $*$-epimorphisms. Any such system (A, α) induces a multivariable dynamical system $(\hat{A}, \hat{\alpha})$ over its Fell spectrum $\hat{A}$.

DEFINITION. Two multivariable systems (A, α) and (B, β) are said to be *piecewise conjugate on their Fell spectra* if the induced systems $(\hat{A}, \hat{\alpha})$ and $(\hat{B}, \hat{\beta})$ are piecewise conjugate, in the sense of the definition above.

We have the following result with Kakariadis.

THEOREM. Let (A, α) and (B, β) be multivariable dynamical systems consisting of $*$-epimorphisms. Assume that either $\mathcal{T}_+(A, \alpha)$ and $\mathcal{T}_+(B, \beta)$ or $A \times_\alpha \mathbb{F}_{n_\alpha}^+$ and $B \times_\beta \mathbb{F}_{n_\beta}^+$ are isometrically isomorphic. Then the multivariable systems (A, α) and (B, β) are piecewise conjugate over their Fell spectra.

PROBLEM. Is there an analogous result for the Jacobson spectrum?

In particular this implies that when the associated operator algebras are isomorphic then both (A, α) and (B, β) have the same number of $*$-*epimorphisms*. (We call this property invariance of the dimension). In the commutative case, the invariance of the dimension holds for systems consisting of arbitrary endomorphisms. Is it true here?

THEOREM. There exist multivariable systems (A, α_1, α_2) and $(B, \beta_1, \beta_2, \beta_3)$ consisting of $*$-monomorphisms for which $\mathcal{T}_+(A, \alpha_1, \alpha_2)$ and $\mathcal{T}_+(B, \beta_1, \beta_2, \beta_3)$ are isometrically isomorphic.

PROBLEM. [Invariance of dimension for semi-crossed products] Let (A, α) and (B, β) be multivariable dynamical systems consisting of $*$-endomorphisms. Prove or disprove: if $A \times_\alpha \mathbb{F}^+_{n_\alpha}$ and $B \times_\beta \mathbb{F}^+_{n_\beta}$ are isometrically isomorphic then $n_\alpha = n_\beta$.

THEOREM. Let (A, α) and (B, β) be two automorphic multivariable C*-dynamical systems and assume that A is primitive. Then the following are equivalent:

1. $A \times_\alpha \mathbb{F}_{n_\alpha}^+$ and $B \times_\beta \mathbb{F}_{n_\beta}^+$ are isometrically isomorphic.

2. $\mathcal{T}^+(A, \alpha)$ and $\mathcal{T}^+(B, \beta)$ are isometrically isomorphic.

3. (A, α) and (B, β) are outer conjugate.

DEFINITION. We say that two multivariable C*-dynamical systems (A,α) and (B,β) are *outer conjugate* if they have the same dimension and there are $*$-isomorphism $\gamma : A \to B$, unitary operators $U_i \in B$ and $\pi \in S_n$ so that

$$\gamma^{-1}\alpha_i\gamma(b) = U_i^*\beta_{\pi(i)}(b)U_i.$$

for all $b \in B$ and i.

Assume now that (A, α) and (B, β) are two multivariable dynamical systems such that $\mathcal{T}^{+}(A, \alpha)$ and $\mathcal{T}^{+}(B, \beta)$ (or $A \times_{\alpha} \mathbb{F}_{n_{\alpha}}^{+}$ and $B \times_{\beta} \mathbb{F}_{n_{\beta}}^{+}$) are isometrically isomorphic via a mapping γ. Since γ is isometric, it follows that $\gamma|_{A}$ is a $*$-monomorphism that maps A onto B (This is the only point where we use that γ is isometric.) We will be denoting $\gamma|_{A}$ by γ as well.

Let S_i, $i = 1, \ldots, n_\alpha$, (resp. T_i, $i = 1, 2, \ldots, n_\beta$) be the generators in $\mathcal{T}^+(A, \alpha)$ (resp. $\mathcal{T}^+(B, \beta)$) and let b_{ij} be the T_i-Fourier coefficient of $\gamma(s_j)$, i.e.,

$$\gamma(S_j) = b_{0j} + T_1 b_{1j} + T_2 b_{2j} + \cdots + T_n b_{nj} + Y,$$

where Y involves Fourier terms of order 2 or higher.

Since γ is a homomorphism,

$$\gamma(a)\gamma(S_j) = \gamma(aS_j) = \gamma(S_j \alpha_j(a)) = \gamma(S_j)\gamma\alpha_j(a),$$

for all $a \in A$. Hence, $\beta_i \gamma(a) b_{ij} = b_{ij} \gamma \alpha_j(a)$, $a \in A$, and so

$$\beta_i(b) b_{ij} = b_{ij} \gamma \alpha_j \gamma^{-1}(b) = b_{ij} \tilde{\alpha}_j(b),$$

for all $b \in B$.

From the intertwining equation

$$\beta_i(b)b_{ij} = b_{ij}\tilde{\alpha}_j(b), b \in B \qquad (*)$$

we obtain.

- Since A is primitive, $b_{i,j}$ is either zero or invertible!

- If $b_{ij} \neq 0$ then $\beta_i \sim \tilde{\alpha}_j$.

Therefore each equivalence class $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is equivalent to exactly one class $\{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_m\}$.

Need to show that $m = n$. Bwoc let $m < n$.

Start with an "arbitrary" n-tuple $(y_1, y_2, \ldots, y_n)$.

From the equation

$$T_1 y_1 + T_2 y_2 + \cdots + T_n y_n = \lim_e \gamma(x_e),$$

where x_e are non-commutative polynomials in $S_1, S_2, \ldots, S_m$ and remembering that

$$\gamma(S_j) = b_{0j} + T_1 b_{1j} + T_2 b_{2j} + \cdots + T_n b_{nj} + Y,$$

we obtain

$$\begin{aligned}
y_1 &= \lim_e b_{11}x_e^1 + b_{12}x_e^2 + \cdots + b_{1m}x_e^m, \\
y_2 &= \lim_e b_{21}x_e^1 + b_{22}x_e^2 + \cdots + b_{2m}x_e^m, \\
&\vdots \\
y_n &= \lim_e b_{n1}x_e^1 + b_{n2}x_e^2 + \cdots + b_{nm}x_e^m.
\end{aligned}$$

Perform Gaussian elimination to reduce this system to

$$\begin{aligned}
\bar{y}_2 &= \lim_e \bar{b}_{22}x_e^2 + \bar{b}_{23}x_e^3 + \cdots + \bar{b}_{2m}x_e^m, \\
\bar{y}_3 &= \lim_e \bar{b}_{32}x_e^2 + \bar{b}_{33}x_e^3 + \cdots + \bar{b}_{3m}x_e^m, \\
&\vdots \\
\bar{y}_n &= \lim_e \bar{b}_{n2}x_e^2 + \bar{b}_{n3}x_e^3 + \cdots + \bar{b}_{nm}x_e^m,
\end{aligned}$$

We continue this short of “Gaussian elimination” and we arrive at a system that contains one column and at least two non-trivial rows of the form

$$w_1 = \lim_e d_1 x_e^m$$
$$w_2 = \lim_e d_2 x_e^m,$$

where the data (w_1, w_2) is arbitrary. Therefore d_1, d_2 are non-zero, hence invertible. By letting $w_1 = 1$ we obtain that $\lim_e x_e^m = d_1^{-1}$. Therefore, if we let $w_2 = 0$, then we get that $0 = d_2 d_1^{-1}$, which is a contradiction.

THEOREM. Let (A, α) and (B, β) be multivariable dynamical systems consisting of $*$-epimorphisms. The tensor algebras $\mathcal{T}_+(A, \alpha)$ and $\mathcal{T}_+(B, \beta)$ are isometrically isomorphic if and only if the correspondences $((A, \alpha)$ and (B, β) are unitarily equivalent.

In light of the above result we ask

PROBLEM. Let (A, α) and (B, β) be multivariable dynamical systems consisting of $*$-monomorphis If the tensor algebras $\mathcal{T}_+(A, \alpha)$ and $\mathcal{T}_+(B, \beta)$ are isometrically isomorphic does it follow that the correspondences $((A, \alpha)$ and (B, β) are unitarily equivalent.

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