KMS states on the C*-algebras associated to finite graphs

Astrid an Huef

University of Otago

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This talk contains some results obtained in

A. an Huef, M. Laca, I. Raeburn and A. Sims, KMS states on C^*-algebras of finite graphs, *J. Math. Anal. Appl., 2013.*

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In physical models, observables of the system are represented by self-adjoint elements of A, and states of the system by positive functionals of norm 1 on A: $\phi(a)$ is the expected value of the observable a in the state ϕ (which is real because $a = a^*$ and $\phi \geq 0$).

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The action α represents the time evolution of the system: the observable a at time 0 moves to $\alpha_t(a)$ at time t, or the state ϕ at time 0 moves to $\phi \circ \alpha_t$.

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In statistical physics, an important role is played by *equilibrium states*, which are in particular invariant under the time evolution. In C^*-algebraic models equilibrium states are called *KMS states*, after Kubo, Martin and Schwinger.

Let $\alpha : \mathbb{R} \to \text{Aut}\, A$ be an action. Then $a \in A$ is an *analytic element* if the function $t \mapsto \alpha_t(a)$ from $\mathbb{R}$ to A has an extension to an entire function on $\mathbb{C}$.

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- The set of analytic elements is always a dense subalgebra of A. For $a \in A$ set

$$a_n := \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \alpha_t(a) e^{-nt^2}\, dt;$$

  then each a_n is analytic and $a_n \to a$.

A state ϕ on A is a *KMS state at inverse temperature β* if

$$\phi(ab) = \phi(b\alpha_{i\beta}(a)) \text{ for all analytic } a, b.$$

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- KMS states are α-invariant.
- It suffices to check the KMS_β condition on a set of analytic elements which span a dense subspace of A.
- The KMS_β states always form a simplex and the extremal KMS_β states are factor states.

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Example: Take the systems $(\mathcal{TO}_n, \alpha)$ and $(\mathcal{O}_n, \alpha)$ where the α are induced from the gauge actions.

- $(\mathcal{TO}_n, \alpha)$ has a unique KMS_β state for each $\beta \geq \ln n$ and no KMS_β state if $\beta < \ln n$.
- The only KMS state of $(\mathcal{TO}_n, \alpha)$ that factors through $\mathcal{O}_n$ is the $\ln n$ state.

Moral from Exel-Laca (2003), Laca-Neshveyev (2004): the Toeplitz algebra has a much richer supply of KMS states.

Suppose that $E = (E^0, E^1, r, s)$ is a directed graph. Today it is always finite. A *Toeplitz-Cuntz-Krieger E-family (Q, T)* consists of mutually orthogonal projections $\{Q_v : v \in E^0\}$ and partial isometries $\{T_e : e \in E^1\}$ such that $T_e^* T_e = P_{s(e)}$ and

$$Q_v \geq \sum_{r(e)=v} T_e T_e^* \quad \text{if } v \text{ is not a source.}$$

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It follows that the projections $\{T_e T_e^* : e \in E^1\}$ are mutually orthogonal. Then $T_e^* T_f = \delta_{e,f} Q_{s(e)}$ and

$$C^*(Q, T) = \overline{\operatorname{span}}\{T_\mu T_\nu^* : \mu, \nu \in E^*\}.$$

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The *Toeplitz algebra* $\mathcal{T}C^*(E)$ of E is the C^*-algebra generated by a universal Toeplitz-Cuntz-Krieger family (q, t). There is a *gauge action* $\gamma : \mathbb{T} \to \text{Aut}(\mathcal{T}C^*(E))$ satisfying $\gamma_z(t_e) = zt_e$ and $\gamma_z(q_v) = q_v$, which we can lift to an action α of $\mathbb{R}$ by $\alpha_t = \gamma_{e^{it}}$.

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Crucial for us:

$$t \mapsto \alpha_t(t_\mu t_\nu^*) = e^{it(|\mu|-|\nu|)} t_\mu t_\nu^*$$

extends to an analytic function (just replace t by z).

Let I be the ideal of $\mathcal{T}C^*(E)$ generated by

$$\{q_v - \sum_{r(e)=v} t_e t_e^* : v \text{ is not a source}\}.$$

View the *graph algebra* $C^*(E)$ as the quotient $\mathcal{T}C^*(E)/I$. There is a gauge action $\gamma : \mathbb{T} \to \mathrm{Aut}(C^*(E))$ which lifts to an action α and the quotient map is equivariant for γ (and hence α).

Let ϕ be a KMS$_\beta$ state on $(\mathcal{T}C^*(E), \alpha)$. Then ϕ is invariant for both α and γ. For $|\mu| \neq |\nu|$,

$$\phi(t_\mu t_\nu^*) = \int_{\mathbb{T}} \phi(\gamma_z(t_\mu t_\nu^*))\, dz = \Big(\int_{\mathbb{T}} z^{|\mu|-|\nu|}\, dz \Big) \phi(t_\mu t_\nu^*) = 0.$$

For $|\mu| = |\nu|$, the KMS condition and $t_\nu^* t_\mu = \delta_{\nu,\mu} q_{s(\mu)}$ gives

$$\phi(t_\mu t_\nu^*) = \phi(t_\nu^* \alpha_{i\beta}(t_\mu)) = e^{-\beta|\mu|} \phi(t_\nu^* t_\mu) = \delta_{\mu,\nu} e^{-\beta|\mu|} \phi(q_{s(\mu)}).$$

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We have proved one half of:

Lemma. A state ϕ of $\mathcal{T}C^*(E)$ is a KMS$_\beta$ state of $(\mathcal{T}C^*(E), \alpha)$ iff

$$\phi(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta|\mu|} \phi(q_{s(\mu)}) \quad \text{for all } \mu, \nu \in E^*.$$

Let ϕ be a KMS$_\beta$ state of $(\mathcal{T}C^*(E), \alpha)$. For $v \in E^0$ define $m = (m_v)$ by $m_v = \phi(q_v)$. Then m is a unit vector:

$$1 = \phi(1) = \sum_{v \in E^0} \phi(q_v) = \sum_{v \in E^0} m_v.$$

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Suppose $v \in E^0$ is not a source. Then

$$\begin{aligned} m_v = \phi(q_v) \geq \sum_{r(f)=v} \phi(t_f t_f^*) &= \sum_{r(f)=v} e^{-\beta} \phi(q_{s(f)}) \\ = \sum_{r(f)=v} e^{-\beta} m_{s(f)} &= e^{-\beta} \sum_{w \in E^0} A(v, w) m_w = e^{-\beta} (Am)_v. \end{aligned}$$

Hence $(Am^\phi)_v \leq e^\beta \phi(p_v) = e^\beta m_v$.

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Hence $(Am^\phi)_v \leq e^\beta \phi(p_v) = e^\beta m_v$.
If v is a source then $A(v, w) = 0 \ \forall w$ and $(Am)_v = 0 \leq e^\beta m_v$.

Let ϕ be a KMS$_\beta$ state of $(\mathcal{T}C^*(E), \alpha)$.

Lemma. For $v \in E^0$ define $m = (m_v)$ by $m_v = \phi(q_v)$. Then m is a unit vector satisfying the *subinvariance relation* $Am \leq e^\beta m$.

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Lemma. ϕ factors through $C^*(E)$ iff $(Am)_v = e^\beta m_v$ whenever v is not a source.

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Spse v is not a source. Then

$$\begin{aligned} e^\beta \phi\Big(q_v - \sum_{r(f)=v} t_f t_f^*\Big) &= e^\beta \Big(\phi(q_v) - \sum_{r(f)=v} e^{-\beta}\phi(q_{s(f)})\Big) \\ &= e^\beta m_v - (Am)_v \end{aligned}$$

By a technical lemma, ϕ factors through iff $\phi(q_v - \sum_{r(f)=v} t_f t_f^*) = 0$ for all such v.

Temporarily assume that E is strongly connected. Then A is an irreducible matrix. Perron-Frobenius Theory for $m \geq 0$:

- $Am = e^{\beta} m \implies m > 0$ is the PF eigenvector and $e^{\beta} = \rho(A)$, the spectral radius of A;
- $Am \leq e^{\beta} m$ and $\beta = \ln \rho(A) \implies m$ is the PF eigenvector;
- $Am \leq e^{\beta} m$ and $Am \neq e^{\beta} m \implies \beta > \ln \rho(A)$.

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Now we have proved half of:

Theorem (Enomoto-Fujii-Watatani 1984). Let E be a strongly connected finite graph with vertex matrix A. Then $(C^*(E), \alpha)$ has a unique KMS state. This state has inverse temperature $\beta = \ln \rho(A)$, where $\rho(A)$ is the spectral radius of A.

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We have shown there is at most one KMS$_\beta$ state of $(C^*(E), \alpha)$, when $\beta = \ln(\rho A)$. We still need to show existence.

Idea: Show there are lots of KMS$_\beta$ states of $(\mathcal{T}C^*(E), \alpha)$ when $\beta > \ln \rho(A)$, then take limits.

(No longer assuming E is strongly connected.) The KMS condition on a state ϕ places restraints on $m := (\phi(p_v))$. Note $Am \leq e^\beta m \iff (I - e^{-\beta}A)m \geq 0$.

Assume $\beta > \ln \rho(A)$. Then $\sum_{n=0}^{\infty} e^{-\beta n} A^n$ converges to $(I - e^{-\beta}A)^{-1}$.

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Take $\epsilon := (I - e^{-\beta}A)m$. Which $\epsilon \in [0,\infty]^{E^0}$ arise?

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For $v \in E^0$, set

$$y_v := \sum_{\mu \in E^* v} e^{-\beta|\mu|} = \sum_{n=0}^\infty \sum_{w \in E^0} e^{-\beta n} A^n(w,v)$$

and take $y = (y_v)$. Then:

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Lemma. Let $\beta > \ln \rho(A)$. Then $m := (I - e^{-\beta}A)^{-1}\epsilon$ is a unit vector in $\ell^1(E^0)$ satisfying $Am \leq e^\beta m$ if and only if $\epsilon \cdot y = 1$.

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Example. Consider the usual orthonormal basis $\{h_\mu : \mu \in E^*\}$ for $\ell^2(E^*)$ (by convention $E^0 \subset E^*$). There are projections Q_v and partial isometries T_e on $\ell^2(E^*)$ such that

$$Q_v h_\mu = \begin{cases} 0 & \text{unless } r(\mu) = v \\ h_\mu & \text{if } r(\mu) = v, \text{ and} \end{cases}$$

$$T_e h_\mu = \begin{cases} 0 & \text{unless } r(\mu) = s(e) \\ h_{e\mu} & \text{if } r(\mu) = s(e). \end{cases}$$

Then (Q, T) is a Toeplitz-CK family, and we have a representation $\pi_{Q,T}$ of $\mathcal{T}C^*(E)$ on $\ell^2(E^*)$ (in fact injective).

Theorem (an Huef-Laca-Raeburn-Sims, 2013). Suppose E is a finite graph with vertex matrix A, and $\beta > \ln \rho(A)$. Take $y = (y_v) \in [1, \infty)^{E^0}$ as above, and suppose $\epsilon \cdot y = 1$. Then there is a KMS$_\beta$ state ϕ_ϵ of $\mathcal{T}C^*(E)$ such that

$$\phi_\epsilon(a) = \sum_{\mu \in E^*} e^{-\beta|\mu|}\epsilon_{s(\mu)}(\pi_{Q,T}(a)h_\mu \mid h_\mu).$$

The map $\epsilon \mapsto \phi_\epsilon$ is an affine isomorphism of $\Delta_\beta = \{\epsilon \in [0,1]^{E^0} : \epsilon \cdot y = 1\}$ onto the simplex of KMS$_\beta$ states.

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Notice there is no hypothesis on E, hence no irreducibility assumption on A. So what happens at $\beta = \ln \rho(A)$? When A is irreducible, the series defining y diverges, so the simplex Δ_β contracts to $\{0\}$ as $\beta \to \ln \rho(A)$.

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Proof. Choose β_n decreasing to $\ln \rho(A)$, and KMS_{β_n} states ϕ_n of $\mathcal{T}C^*(E)$. By passing to a subsequence, $\phi_n \to \phi$, and ϕ is a $\mathrm{KMS}_{\ln \rho(A)}$ state of $\mathcal{T}C^*(E)$.

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$$\begin{aligned}
\rho(A)\phi(q_v) = \rho(A)m_v = (Am)_v &= \sum_{w \in E^0} A(v, w)\phi(q_w) \\
&= \sum_{r(e)=v} \phi(q_{s(e)}) = \sum_{r(e)=v} \rho(A)\phi(t_e t_e^*) \\
&= \rho(A)\phi\Big(\sum_{r(e)=v} t_e t_e^* \Big).
\end{aligned}$$

Corollary (Enomoto-Fujii-Watatani). If E is strongly connected, then $(C^*(E), \alpha)$ has a $\mathrm{KMS}_{\ln \rho(A)}$ state.

Proof. Choose β_n decreasing to $\ln \rho(A)$, and KMS_{β_n} states ϕ_n of $\mathcal{T}C^*(E)$. By passing to a subsequence, $\phi_n \to \phi$, and ϕ is a $\mathrm{KMS}_{\ln \rho(A)}$ state of $\mathcal{T}C^*(E)$. Then $m := (\phi(p_v))$ satisfies $Am \leq \rho(A)m$. PF implies $Am = \rho(A)m$. Thus

$$\rho(A)\phi(q_v) = \rho(A)m_v = (Am)_v = \sum_{w \in E^0} A(v,w)\phi(q_w)$$

$$= \sum_{r(e)=v} \phi(q_{s(e)}) = \sum_{r(e)=v} \rho(A)\phi(t_e t_e^*)$$

$$= \rho(A)\phi\Big(\sum_{r(e)=v} t_e t_e^*\Big).$$

So for all $v \in E^0$ which are not sources,

$$\phi\Big(q_v - \sum_{r(e)=v} t_e t_e^*\Big) = 0.$$

Now a technical lemma implies that ϕ factors through $C^*(E) = \mathcal{T}C^*(E)/I$.

This completes the proof of:

Theorem (Enomoto-Fujii-Watatani 1984). Let E be a strongly connected finite graph with vertex matrix A. Then $(C^*(E), \alpha)$ has a unique KMS state. This state has inverse temperature $\beta = \ln \rho(A)$, where $\rho(A)$ is the spectral radius of A.

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