Arithmetic Groups
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Abstract. We present detailed summaries of the talks that were given during a week-long workshop on *Arithmetic Groups* at the Banff International Research Station in April 2013, organized by Kai-Uwe Bux (University of Bielefeld), Dave Witte Morris (University of Lethbridge), Gopal Prasad (University of Michigan), and Andrei Rapinchuk (University of Virginia). The vast majority of these reports are based on abstracts that were kindly provided by the speakers. Video recordings of lectures marked with the symbol ▷ are available online at

  http://www.birs.ca/events/2013/5-day-workshops/13w5019/videos/

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1. Introduction and Overview

The theory of arithmetic groups deals with groups of matrices whose entries are integers, or more generally, S-integers in a global field. This notion has a long history, going back to the work of Gauss on integral quadratic forms. The modern theory of arithmetic groups retains its close connection to number theory (for example, through the theory of automorphic forms) but also relies on a variety of methods from the theory of algebraic groups, particularly over local and global fields (this area is often referred to as the arithmetic theory of algebraic groups), Lie groups, algebraic geometry, and various aspects of group theory (primarily, homological methods and the theory of profinite groups). At the same time, results about arithmetic groups have numerous applications in differential and hyperbolic geometry (as the fundamental groups of many important manifolds often turn out to be arithmetic), combinatorics (expander graphs), and other areas. There are also intriguing connections and parallels (which are currently not so well-understood) between arithmetic groups and other important classes of groups, such as Kac-Moody groups, automorphism groups of free groups, and mapping class groups.

The objective of the workshop was to survey the most significant results in the theory of arithmetic groups obtained primarily in the last five years, in order to make the new concepts and methods accessible to a broader group of mathematicians whose interests are closely related to arithmetic groups. The workshop brought together 34 mathematicians, from the world's leading experts to recent PhD recipients and graduate students, working on a variety of problems involving arithmetic groups. This resulted in very active exchanges between and after the lectures. The scientific program of the workshop consisted of 3 mini-courses (two 45-min lectures each), 17 survey and research talks (30 or 45 minutes), one of which (by Bertrand Remy) was not planned in advance, a Q&A session, and an open problem session.

The subjects of the mini-courses were: *Pseudo-reductive groups and their arithmetic applications* (Brian Conrad), *Towards an arithmetic Kac-Moody theory* (Ralf Köhl), and *Homological finiteness properties of arithmetic groups in positive characteristic* (Kevin Wortman). The mini-course on the pseudo-reductive groups focused on arithmetic applications of the theory of pseudo-reductive groups, developed by Conrad, Gabber and Prasad, which include the proof of fundamental finiteness theorems (finiteness of the class number, finiteness of the Tate-Schafarevich set, etc.) for *all* algebraic groups over the fields of positive characteristic, not just reductive ones. The course on Kac-Moody groups contained a series of results extending the known properties of (higher rank) arithmetic groups, such as property (T) and (super)rigidity, to Kac-Moody groups over rings. The course on homological finiteness properties contained an account of a major breakthrough in the area — the proof of the Rank Conjecture.

Most major areas within the theory of arithmetic groups were represented by at least one talk at the workshop. Topics that were not discussed, but should be included in the program of future meetings on the subject, include:

- connections between the cohomology of arithmetic groups and the theory of automorphic forms,
- the virtual positivity of the first Betti number of certain rank one lattices and the growth of higher Betti numbers,
- the analysis on homogeneous spaces modulo arithmetic groups.

We conclude this introduction with a brief overview of the main themes that were discussed in the lectures. More details of the three mini-courses are in Section 2, and the other lectures are described individually in Section 3.

(a) Structural and homological properties. In addition to the mini-course on homological finiteness properties for S-arithmetic subgroups of semi-simple algebraic groups in positive characteristic, which contained an exposition of the work of Bux-Köhl-Witzel-Wortman on the rank conjecture, there was a talk by Stefan Witzel on finiteness properties for proper actions of arithmetic groups.

The talk of Ted Chinburg demonstrated how to use the Lefschetz Theorem from algebraic geometry to show, in certain situations, that a "large" arithmetic group can be generated by smaller arithmetic subgroups.

In his talk, Vincent Emery showed how to bound the torsion homology of non-uniform arithmetic lattices in characteristic zero.

(b) Profinite techniques for arithmetic groups. The method of analyzing arithmetic groups via the study of their finite quotients has a long history. One aspect of this approach, known as the congruence subgroup problem, is focused on understanding the difference between the profinite completion and the congruence completion. This difference is measured by the congruence kernel. In his talk, Andrei Rapinchuk gave a survey of the concepts and results pertaining to the congruence subgroup problem.

In their talks, which together virtually constituted another mini-course, Benjamin Klopsch and Christopher Voll presented their new results on the representation growth of S-arithmetic groups satisfying the congruence subgroup property (i.e., for which the congruence kernel is finite). These results are formulated in terms of the representation zeta function of the S-arithmetic group, and show, in particular, that the abscissa of convergence depends only on the root system.

The talk of Pavel Zalesskii was devoted to the general question of when the profinite topology on an arithmetic group should be considered to be "strong." The cohomological aspect of this question boils down to the notion of "goodness" introduced by Serre. The central conjecture here asserts that if an S-arithmetic subgroup in characteristic zero fails the congruence subgroup property, then it should be good, and the talk contained an account of the results supporting this conjecture.

(c) Connections with Kac-Moody, automorphism groups of free groups, and mapping class groups. As we already mentioned, Ralf Köhl gave a mini-course on Kac-Moody groups, in which he described how Kac-Moody groups are obtained from Chevalley groups by amalgamation, defined the Kac-Peterson topology on them, and established their important properties, including property (T) and a variant of superrigidity. The lectures generated so much interest in Kac-Moody groups among the conference participants that Bertrand Rémy was asked to give a survey talk on the subject. He discussed various approaches to Kac-Moody groups, and stressed the utility of buildings in their analysis.

In his talk, Alan Reid showed how methods from Topological Quantum Field Theory can be used to prove that every finite group is a quotient of a suitable finite index subgroup of the mapping class group (for any genus).

Lizhen Ji reported on the construction of complete geodesic metrics on the outer space X_n that are invariant under $\mathrm{Out}(F_n)$, where F_n is the free group of rank n.

(d) Applications to geometry, topology, and beyond. Mikhail Belolipetsky gave a survey of the long line of research on hyperbolic reflection groups. (These are discrete isometry groups that are generated by reflections of hyperbolic n-space). One of the central results here is that there are only finitely many conjugacy classes of maximal arithmetic hyperbolic reflection groups. This opens the possibility of classifying such groups.

Matthew Stover reported on his results that provide a lower bound on the number of ends (cusps) of an arithmetically defined hyperbolic manifold/orbifold or a locally symmetric space. In particular, one-end arithmetically defined hyperbolic n-orbifolds do not exist for any $n > 31$. The question about the existence of one-ended nonarithmetic finite volume hyperbolic manifolds remains wide open.

T. N. Venkataramana spoke about the monodromy groups associated with hypergeometric functions. One of the central questions is when the monodromy group is a finite index subgroup of the corresponding integral symplectic group. A criterion was described for this to be the case.

(e) Rigidity. Generally speaking, a group-theoretic rigidity theorem asserts that, in certain situations, any abstract homomorphism of a special subgroup (e.g, an arithmetic subgroup or a lattice) of the group of rational points of an algebraic group (resp., a Lie group, or a Kac-Moody group) can be extended to an algebraic (resp., analytic or continuous) homomorphism of the ambient group. The pioneering and most unexpected result of this type is Margulis's Superridity Theorem for irreducible lattices in higher rank semi-simple Lie groups. As was pointed out by Bass, Milnor, and Serre, rigidity statements can also be proved for S-arithmetic groups if the corresponding congruence kernel is finite.

In his mini-course, Ralf Köhl showed how to use the known rigidity results for higher rank arithmetic groups to prove a rigidity theorem for Kac-Moody groups over $\mathbb{Z}$.

In his talk, Igor Rapinchuk discussed his rigidity results for the finite-dimensional representations of elementary subgroups of Chevalley group of rank > 1 over arbitrary commutative rings. These results settle a conjecture of Borel and Tits about abstract homomorphisms for split algebraic groups of rank > 1 over fields of characteristic $\neq 2, 3$, and also have applications to character varieties of some finitely generated groups.

(f) Weakly commensurable groups and connections to algebraic groups. The notion of weak commensurability for Zariski-dense subgroups of the group of rational points of semi-simple algebraic groups over a field of characteristic zero was introduced by G. Prasad and A. Rapinchuk. They were able to provide an almost complete answer to the question of when two S-arithmetic subgroups of absolutely almost simple algebraic groups are weakly commensurable. Using Schanuel's conjecture from transcendental number theory, they connected this work to the analysis of length-commensurable and isospectral locally symmetric spaces, and in fact obtained new important results about isospectral spaces. In his talk, Rajan reported on his work which is based on a new notion of representation equivalence of lattices. While the condition of representation equivalence is generally stronger than isospectrality, it enables one to obtain results about representation equivalent locally symmetric spaces without using Schanuel's conjecture (which is still unproven).

The work of Prasad-Rapinchuk also attracted attention to a wide range of questions in the theory of algebraic groups asking about a possible relationship between two absolutely almost simple algebraic groups G_1 and G_2 over the same field K, given the fact that they have the

same isomorphism/isogeny classes of maximal K-tori. In his talk, Vladimir Chernousov reported on the recent results on a related problem of characterizing finite-dimensional central division algebras over the same field K that have the same isomorphism classes of maximal subfields. He also formulated a conjecture that would generalize these results to arbitrary absolutely almost simple groups, and indicated that the results on division algebras enable one to prove this conjecture for inner forms of type A_n.

(g) Applications to combinatorics. In his talk, Alireza Salehi Golsefidy discussed applications of arithmetic groups and their Zariski-dense subgroups to the construction of highly connected but sparse graphs known as *expanders*. It was pointed out by G. A. Margulis that families of expanders can be constructed from a discrete Kazhdan group Γ by fixing a finite system S of generators of Γ and considering the Cayley graphs $\mathrm{Cay}(\Gamma/N, S)$ of the finite quotients of Γ with respect to this generating system. Later, using deep number-theoretic results, Lubotzky, Phillips, and Sarnak showed that one also obtains a family of expanders from the non-Kazhdan group $SL_2(\mathbb{Z})$ by fixing a system S of generators of the latter and considering the Cayley graphs $\mathrm{Cay}\big(SL_2(\mathbb{Z}/d\mathbb{Z}), S\big)$ for the congruence quotients.

Lubotzky raised the question of whether one gets a family of expanders if one takes, for example, an arbitrary finitely generated Zariski-dense subgroup $\Gamma \subset SL_2(\mathbb{Z})$, and considers the Cayley graphs of the congruence quotients $SL_2(\mathbb{Z}/d\mathbb{Z})$ with respect to a fixed finite generating set of the subgroup. (It is known that Γ will map surjectively onto $SL_2(\mathbb{Z}/d\mathbb{Z})$ for all d prime to some d_0 that depends on Γ.)

Alireza Salehi Golsefidy surveyed the important progress on this question in the context of general arithmetic groups, including the groundbreaking work of Bourgain and Gamburd, and his recent results with Varjú.

2. Mini-Courses

Brian Conrad (Stanford University):
Pseudo-reductive groups and their arithmetic applications ▷

The talk first described some motivation for, and examples of, the theory of pseudo-reductive groups [39], culminating in the main structure theorem (ignoring subtleties that arise in characteristics 2 and 3). Then, it discussed how to apply the main structure theorem in the context of proving some finiteness theorems over global function fields [38]. Such results include affirmative solutions to open questions which render unconditional several results in [86] on the behavior of sizes of degree-1 Tate–Shafarevich sets and Tamagawa numbers for linear algebraic groups over global function fields.

A very highly-recommended survey of both the general theory and arithmetic applications is given in the Bourbaki report [99] by Bertrand Rémy. This includes a user-friendly overview of the contents of [39], indicating where main results can be found and how the logical development of the main proofs proceeds.

Definition of pseudo-reductivity: triviality of the so-called *k-unipotent radical $\mathscr{R}_{u,k}(G)$* (the largest smooth connected normal unipotent k-subgroup of a smooth connected affine k-group), with k a general field. The formation of $\mathscr{R}_{u,k}(G)$ commutes with separable extension on k, including $K \to K_v$ for a global function field K and place v. We say G is *pseudo-reductive* if it is smooth connected affine and $\mathscr{R}_{u,k}(G) = 1$. This coincides with "connected reductive" when k is perfect. Several examples were given over any imperfect field, both commutative as well as perfect ($G = \mathscr{D}(G)$). The most basic example is the Weil restriction $\mathrm{R}_{k'/k}(G')$ for a finite extension k'/k and a connected reductive k'-group G'; this is never reductive if $G' \neq 1$ and k' is not separable over k.

Surprises: the purely inseparable Weil restriction may fail to preserve dimension (e.g., $\mathrm{R}_{k'/k}(\mu_p)$ has positive dimension for a nontrivial purely inseparable finite extension k'/k in characteristic $p > 0$). It may also fail to preserve surjectivity or perfectness: $\mathrm{R}_{k'/k}(\mathrm{PGL}_p)$ has a nontrivial commutative quotient modulo the image of $\mathrm{R}_{k'/k}(\mathrm{SL}_p)$ for a nontrivial purely inseparable finite extension k'/k in characteristic $p > 0$. Other surprises: pseudo-reductivity can be lost under central quotient, and also some quotients by smooth connected normal k-subgroups.

Good news: Cartan k-subgroups are always commutative, $\mathrm{R}_{k'/k}(G')$ is perfect when G' is simply connected, and there is a theory of root systems (which can be non-reduced in characteristic 2, even if $k = k_s$). A related concept is pseudo-semisimplicity; this has some surprises (there are two possible definitions, one of which is "wrong"). Overall, pseudo-reductivity is not a particularly robust concept, so its main purpose is the role it plays in trying to prove theorems about rather general linear algebraic groups when one might not have much control (e.g. for Zariski closures, working with stabilizer schemes for a group action, etc.).

A general principle: we cannot hope to understand the commutative pseudo-reductive groups, but we will aim to describe the general structure modulo that ignorance. Life could be worse; at least we are remaining ignorant only about something commutative.

The standard construction: this is a procedure which is a kind of pushout that replaces a Weil-restricted maximal torus $\mathrm{R}_{k'/k}(T')$ from a simply connected semisimple k'-group G' with another commutative pseudo-reductive group C according to a very specific kind of procedure. The final output of this process is a central quotient presentation

$$G = (\mathrm{R}_{k'/k}(G') \rtimes C)/\mathrm{R}_{k'/k}(T').$$

This can be generalized to allow several extensions k'_i/k, and admits a precise uniqueness aspect as well, determines all of the data $(G', k'/k, T', C)$ in terms of a choice of maximal k-torus of T.

A general principle for applying the structure theory of pseudo-reductive groups: if a theorem is known in the smooth connected solvable affine case over k and in the connected semisimple case over all finite extensions of k then "probably" one can use the structure theorem via standard presentations (plus extra care in characteristics 2 and 3) to prove the result for all smooth connected linear algebraic groups (and something without smoothness or connectedness, depending on the specific assertion).

Some of the finiteness questions one would like to settle (all of which have long been known in the affirmative in the connected reductive case): finiteness for "class numbers" of smooth connected linear algebraic groups over global function fields, finiteness for degree-1 Tate–Shafarevich sets for all affine algebraic group schemes over global function fields, finiteness for Tamagawa numbers of smooth connected affine groups, and finiteness for obstruction sets to a local-global principle for orbits over global function fields (i.e., if $x, x' \in X(k)$ are in the same $G(k_v)$-orbit for all $v \notin S$ then the images of x and x' in $G(k)\backslash X(k)$ might not coincide but are at least constrained within a finite set, depending on S). In this final question it is important that we do not impose smoothness hypotheses (or any hypotheses at all) on the stabilizer schemes for the geometrically transitive action. The finiteness question for orbits reduces to finiteness for Tate–Shafarevich sets for the *stabilizer* scheme, so it is important for the latter finiteness result that we do not demand smoothness hypotheses (but they can be imposed by a trick).

Example applications: finiteness for Tate–Shafarevich sets can be reduced to the pseudo-reductive case, and the *form* of the "standard presentation" is very well-suited to then pulling up the known result for the semisimple and commutative cases, essentially using vanishing theorems for simply connected groups. Second: finiteness for Tamagawa numbers is settled in general by a different kind of argument with the "standard presentation" for pseudo-reductive groups to pull it up from the known semisimple and commutative cases. Third: the original finiteness question for the local-global principle with orbits. That indeed works out affirmatively, due to the established case for Tate–Shafarevich sets. Fourth: various formulas for the behavior of Tamagawa numbers under exact sequences that were proved in Oesterlé's paper [86] conditionally on certain unknown finiteness results are now all valid unconditionally.

Ralf Köhl (University of Gießen):
Kac-Moody groups

By theorems of Tits and Curtis, a Chevalley group over a field with at least four elements is the product of its rank two subgroups amalgamated over the rank one subgroups. The following result goes to show that in the context of Chevalley groups over local fields, the topology is also forced by the rank one subgroups:

Theorem 1 (Glöckner-Hartnick-Köhl [49])**.** *Let $\mathbb{F}$ be a local field, and let G be a Chevalley group over $\mathbb{F}$. Then the Lie group topology on G is the finest group topology making the embeddings of the fundamental rank one subgroups (endowed with their Lie group topologies) continuous.*

Now let Δ be a 2-spherical Dynkin diagram without loops and let $\mathbb{F}$ be a field with at least four elements. For each node $\alpha \in \Delta$ let G_α be a copy of $\mathrm{SL}_2(\mathbb{F})$; and for each pair $\alpha, \beta \in \Delta$ let $G_{\alpha,\beta}$ be a simply connected Chevalley group over $\mathbb{F}$ of the type given in Δ. There are obvious inclusions $G_\alpha \rightarrowtail G_{\alpha,\beta}$. The Kac-Moody group $G_\Delta(\mathbb{F})$ can then be described as the product of the $G_{\alpha,\beta}$ amalgamated over the G_α and is uniquely determined by Δ since Δ does not contain loops. Every 2-spherical split Kac-Moody group arises this way.

Definition 2. *The* Kac-Peterson *topology on $G_\Delta(\mathbb{F})$ is the finest group topology that makes the canonical embeddings $G_\alpha \rightarrowtail G_\Delta(\mathbb{F})$ continuous.*

Theorem 3 (Hartnick-Köhl-Mars [56])**.** *The group $G_\Delta(\mathbb{F})$ with the Kac-Peterson topology is Hausdorff and a k_ω-space, i.e., it is the direct limit of an ascending sequence of compact Hausdorff subspaces. If Δ is not spherical, then the Kac-Peterson topology is neither locally compact nor metrizable.*

In particular, the existence of a Haar measure is not guaranteed for non-spherical Kac-Moody groups with the Kac-Peterson topology.

Theorem 4 (Hartnick-Köhl [55])**.** *Let $\mathbb{F}$ be a local field and let G_Δ be an irreducible (i.e., Δ is connected), 2-spherical split Kac-Moody group. Then $G_\Delta(\mathbb{F})$ with the Kac-Peterson topology has Kazhdan's property (T).*

The subgroup $G_\Delta(\mathbb{Z})$ is discrete and finitely generated. One would like to think of $G_\Delta(\mathbb{Z})$ in analogy to arithmetic lattices. It is, however, an open question whether it inherits property (T) from $G_\Delta(\mathbb{R})$.

The following follows easily from a theorem of Caprace and Monod on Chevalley groups acting on CAT(0) polyhedral complexes applied to the Davis realization of the twin building for $G_\Delta(\mathbb{R})$:

Proposition 5. *Let L be an irreducible Chevalley group of rank at least two, let $G_\Delta(\mathbb{R})$ be a Kac-Moody group, and let $\varphi\colon L(\mathbb{Z}) \to G_\Delta(\mathbb{R})$ be a group homomorphism. Then the image $\varphi(L(\mathbb{Z}))$ is a bounded subgroup, i.e., it lies in the intersection of two parabolic subgroups of opposite sign. In particular, $\varphi(L(\mathbb{Z}))$ is contained in an algebraic subgroup of $G_\Delta(\mathbb{R})$.*

This can be extended to yield an analogue of Margulis superrigidity:

Theorem 6 (Farahmand-Horn-Köhl)**.** *Let $G_\Delta(\mathbb{R})$ and $G_{\Delta'}(\mathbb{R})$ be irreducible 2-spherical Kac-Moody groups and let $\varphi\colon G_\Delta(\mathbb{Z}) \to G_{\Delta'}(\mathbb{R})$ be a group homomorphism with Zariski dense image. Then there exists $n \in \mathbb{N}$ such that the restriction of φ to $G_\Delta(n\mathbb{Z})$ extends uniquely to a continuous homomorphism $G_\Delta(\mathbb{R}) \to G_{\Delta'}(\mathbb{R})$ with respect to the Kac-Peterson topologies.*

The proof proceeds by first dealing with the case that G_Δ is a Chevalley group, where the Kac-Peterson topology is the Lie group topology. For general G_Δ, the statement is reduced to the rank two subgroups, which are Chevalley groups. Using the the presentations of G_Δ and $G_{\Delta'}$ as products of their respective rank two subgroups amalgamated along their rank one subgroups, one constructs the extension of φ. Since the Kac-Peterson topology is universal with respect to the Lie topologies on the rank one subgroups, it follows that φ is continuous.

Kevin Wortman (University of Utah):
Finiteness properties of arithmetic groups over function fields ▷

Recall that a group Γ has finiteness length $\leq m$ if it has a classifying space whose m-skeleton is finite. In this case, the cohomology of Γ is clearly finitely generated in dimensions $\leq m$.

Let K be a global function field of characteristic $p > 0$, let $\mathbb{F}_p$ be the finite field with p elements, and let $\mathbf{G}$ be a connected noncommutative absolutely almost simple K-isotropic K-group. Let $d := \sum_{p \in S} \mathrm{rk}_{K_p}(\mathbf{G})$ denote the sum of the local ranks of $\mathbf{G}$. With this notation fixed, the two main results are:

Theorem 1 ("Rank Theorem", Bux-Köhl-Witzel [26]). *The finiteness length $\phi(\Gamma)$ of the S-arithmetic subgroup $\Gamma = \mathbf{G}(\mathcal{O}_S)$ is $d - 1$.*

Theorem 2 (Wortman). *For some subgroup Γ' of finite index in Γ, the cohomology $\mathrm{H}^d(\Gamma', \mathbb{F}_p)$ is not finitely generated.*

(At the time of the conference, a mild restriction on the K-type of $\mathbf{G}$ was needed in Theorem 2, but Wortman was soon able to remove this restriction.)

Results on finiteness properties of arithmetic groups have a long history. The Euclidean algorithm shows that $\mathrm{SL}_n(\mathbb{Z})$ is finitely generated. Finite presentability of these groups is a classical application of Siegel domains. Raghunathan [90] proved that arithmetic groups in characteristic 0 enjoy all finiteness properties. In fact, he showed that they have a torsion-free subgroup of finite index that is the fundamental group of a compact aspherical manifold with boundary. Borel-Serre [20] have shown that S-arithmetic subgroups of reductive groups in characteristic 0 also enjoy all finiteness properties.

The picture in positive characteristic is different. Nagao [81] showed that $\mathrm{SL}_2(\mathbb{F}_q[t])$ is not even finitely generated. Behr [11] proved that Γ as in the Rank Theorem is finitely generated if and only if $d > 1$. Stuhler [107] showed that $\mathrm{SL}_2(\mathcal{O}_S)$ has finiteness length $|S| - 1 = d - 1$. Abels [1] and Abramenko [3] independently showed that $\mathrm{SL}_n(\mathbb{F}_q[t])$ has finiteness length $n - 2 = d - 1$, provided q is large enough. Sometime during the 1980s, the pattern became transparent. Behr turned it into a serious conjecture when he proved in [12] that the S-arithmetic subgroup Γ of the Rank Theorem is finitely presented if and only if $d > 2$.

A significant step toward the Rank Theorem was the proof of its "negative half" by Wortman and Bux in [27], where they showed that $\phi(\Gamma) < d$. In 2008 (published in [28]), Wortman and Bux also settled the Rank Theorem in full for groups of global rank 1. The major improvement was a geometric filtration of the Bruhat-Tits building for Γ defined by Busemann functions. The relative links of this filtration are larger than those occurring in combinatorially defined filtrations used previously: the new relative links are hemi-sphere complexes in spherical buildings, whose connectivity properties have been established by

Schulz [102]. The proof of the Rank Theorem for arbitrary groups follows this line of thought. Here, Behr-Harder reduction theory is the source of the Busemann functions and the associate filtration.

Theorem 2 above is a considerable strengthening of the negative half of the Rank Theorem. For $\mathrm{SL}_2(\mathcal{O}_S)$, Stuhler succeeded in proving that the homology in the critical dimension (here $d = |S|$) is infinitely generated, by using of a spectral sequence argument. In the other works cited above, the finiteness length was deduced by combinatorial or geometrical means that do not detect homology in the critical dimension.

The main difficulty is that the action of Γ on its associated Bruhat-Tits building X is not free; in fact, the orders of the point stabilizers are not bounded. Wortman uses the height function from the Rank Theorem to pass to a cocompact subspace $X(0)$, which is $(d-2)$-connected. Gluing in free Γ-orbits of cells of dimensions d and $d+1$, he obtains a d-connected space Y on which Γ acts with stabilizers of uniformly bounded order, and he obtains a Γ-equivariant map $Y \to X$. He can now pass to a finite index subgroup $\Gamma' \leq \Gamma$ that acts freely on Y. He then constructs an infinite family of cocycles on Y and a "dual" family of cycles on X paired via the comparison map $Y \to X$. The supports of the cycles in X increase in height and "low" cocycles evaluate trivially on higher cycles. On the other hand, each cocycle evaluates non-trivially on the corresponding cycle. This shows that the cocycles are non-trivial and linearly independent.

3. Research Lectures and Survey Talks

Mikhail Belolipetsky (IMPA, Brazil):
Arithmetic hyperbolic reflection groups ▷

A *hyperbolic reflection group* Γ is a discrete subgroup of the group of isometries of the hyperbolic n-space $\mathrm{Isom}(\mathbf{H}^n)$ generated by reflections in the faces of a hyperbolic polyhedron $P \subset \mathbf{H}^n$. If P has finite volume, then $\mathbf{H}^n/\Gamma = \mathcal{O}$ is a finite volume hyperbolic orbifold, which is obtained by "mirroring" the faces of P. A reflection group Γ is *maximal* if there does not exist a reflection group $\Gamma' \subset \mathrm{Isom}(\mathbf{H}^n)$ that properly contains Γ. A reflection group is called *arithmetic* if it is an arithmetic subgroup of $\mathrm{Isom}(\mathbf{H}^n) = \mathrm{PO}(n,1)$.

This talk was about finiteness results for maximal arithmetic hyperbolic reflection groups. After a brief review of previous foundational work by Vinberg and Nikulin, it focused attention on the results obtained in this area in the last 10 years. The important breakthrough was achieved by Maclachlan–Long–Reid [69] and Agol [5], who proved that there exist only finitely many conjugacy classes of arithmetic maximal hyperbolic reflection groups in dimensions $n = 2$ and $n = 3$, respectively. Later, this was proved for all dimensions:

Theorem 1 (Agol-Storm-Belolipetsky-Whyte [6], Nikulin [83]). *There are only finitely many conjugacy classes of arithmetic maximal hyperbolic reflection groups in any fixed dimension n.*

The finiteness theorem allows us to talk about a classification of arithmetic hyperbolic reflection groups. Two types of problems are considered here: proving quantitative bounds for the invariants of the groups, and constructing examples that fit into the bounds. Results in these directions were obtained in the papers [13–16, 75, 78, 84]. The end of the talk discussed the main open problems in the area.

Vladimir Chernousov (University of Alberta):
A finiteness theorem for the genus ▷

Given a finite-dimensional central division algebra D over a field K, the *genus* $\mathbf{gen}(D)$ is defined to be the set of isomorphism classes of central division K-algebras having the same (isomorphism classes of) maximal subfields as D. One would like to have qualitative and quantitative results for $\mathbf{gen}(D)$ over arbitrary fields.

This general question is related to other interesting problems in division algebras, quadratic forms, Galois cohomology and even differential geometry (a question of this nature was raised in the paper [88] on length-commensurable and isospectral locally symmetric spaces). Since every division algebra is the union of its maximal subfields, questions about the genus can informally be thought of questions about ways to use the same subfields and construct a different algebra (so, in some sense, these are analogs of the Banach-Tarski paradox for division algebras).

More precisely, we have the following questions.

Question 1. *When is $|\mathbf{gen}(D)| = 1$?*

We note that $|\mathbf{gen}(D)| = 1$ if and only if D is uniquely determined by its maximal subfields. Since D and its opposite algebra D^{op} have the same maximal subfields, an affirmative answer to Question 1 is possible only if $\exp(D) = 2$.

Question 2. *When is $|\mathbf{gen}(D)| < \infty$?*

Various people including Garibaldi, Rost, Saltman, Shacher, Wadsworth, have described a method for constructing non-isomorphic quaternion algebras over very large fields (having infinite transcendence degree over the prime field) with the same quadratic subfields, which actually shows that the genus of a quaternion algebra over such a field can be infinite. This suggests that Question 2 should be considered primarily over finitely generated fields.

Work of Vladimir Chernousov with Andrei and Igor Rapinchuk developed a general approach to proving the finiteness of the genus of a division algebra, and estimating its size, based on an analysis of the unramified Brauer group with respect to an appropriate set of discrete valuations of K. This approach yields, in particular, the following two theorems.

Theorem 3 ("Stability Theorem", Chernousov-Rapinchuk-Rapinchuk [34, 35]). *Let K be a field of characteristic not 2. If $|\mathbf{gen}(D)| = 1$ for any central division K-algebra D of exponent 2, then the same is true for any division algebra of exponent 2 over the field of rational functions $K(x)$.*

Corollary 4. *Let k be either a finite field of characteristic not 2 or a number field, and let $K = k(x_1, \ldots, x_r)$ be a finitely generated purely transcendental extension of k. Then, for any central division K-algebra D of exponent 2, we have $|\mathbf{gen}(D)| = 1$.*

Theorem 5 (Chernousov-Rapinchuk-Rapinchuk [34, 35]). *Let K be a finitely generated field. If D is a central division K-algebra of exponent prime to char K, then $\mathbf{gen}(D)$ is finite.*

Furthermore, the authors proposed a generalization of the notion of genus to arbitrary absolutely almost simple algebraic K-groups, based on the isomorphism classes of maximal K-tori. (Possible variations of this notion can be based on the consideration of isogeny classes and/or some special classes of maximal K-tori, e.g., generic tori.)

In view of Theorem 5, the following seems natural.

Conjecture 6 (Chernousov-Rapinchuk-Rapinchuk [34, 35]). *Let G be an absolutely almost simple simply connected algebraic group over a finitely generated field K of characteristic zero (or of "good" characteristic relative to G). Then there exists a finite collection $G_1, \ldots, G_r$ of K-forms of G such that if H is a K-form of G having the same isomorphism classes of maximal K-tori as G, then H is K-isomorphic to one of the G_i's.*

The proof of Theorem 5 yields a proof of this conjecture for inner forms of type A_ℓ.

Ted Chinburg (University of Pennsylvania):
Generating arithmetic groups by small subgroups using Lefschetz Theorems ▷

The following result provides interesting examples of large arithmetic groups that are generated by specific small arithmetic subgroups.

Theorem 1 (Chinburg-Stover [37]). *Let G be $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SU}(2,1)$ with corresponding hermitian symmetric space X, and let Γ be a cocompact arithmetic lattice in G. Assume that the complex algebraic surface $S = \Gamma \backslash X$ contains a holomorphically immersed totally geodesic projective algebraic curve. Then there exist finitely many such curves $C_1, \ldots, C_r \subset S$ and positive integers $\alpha_1, \ldots, \alpha_r$ such that:*

(1) *The divisor $D = \sum \alpha_j C_j$ is a connected effective divisor on S.*
(2) *The image of $\pi_1(|D|)$ in Γ under the natural homomorphism is a finite index subgroup.*

Statement 1 is shown using the commensurator of Γ and the fact that there are infinitely many commensurability classes of Fuchsian curves on S once one exists. The proof of statement 2 uses work of Nori [85] and of Napier and Ramachandran [82] on Lefschetz Theorems for sufficiently positive divisors on complex varieties.

One consequence of the theorem is the following structure theorem for the Albanese varieties of arithmetic complex hyperbolic 2-manifolds.

Theorem 2 (Chinburg-Stover [37]). *Let Γ be as in Theorem 1 and suppose $G = \mathrm{SU}(2,1)$. There exists $r \geq 1$ and Fuchsian curves $C_1, \ldots, C_r$ on $S = \Gamma \backslash \mathbf{H}_{\mathbb{C}}^2$ such that if the Albanese variety $\mathrm{Alb}(S)$ is nontrivial, then every simple factor of $\mathrm{Alb}(S)$ is isogenous to a factor of the Jacobian of the normalization $C_j^{\#}$ of (at least) one of the curves C_j.*

If Γ is a congruence arithmetic lattice of simple type, Gelbart and Rogawski [47] proved that the first cohomology group of $\Gamma \backslash \mathbf{H}_{\mathbb{C}}^2$, which determines $\mathrm{Alb}(\Gamma \backslash \mathbf{H}_{\mathbb{C}}^2)$, arises from the theta correspondence. Murty and Ramakrishnan then used Faltings' work on the Mordell conjecture in [80] to show that the simple factors of $\mathrm{Alb}(\Gamma \backslash \mathbf{H}_{\mathbb{C}}^2)$ are, in fact, CM abelian varieties. This gave a positive answer to a question of Langlands.

Theorem 2 provides information of a different nature about $\mathrm{Alb}(\Gamma \backslash \mathbf{H}_{\mathbb{C}}^2)$ for both congruence and noncongruence Γ, namely that $\mathrm{Alb}(\Gamma \backslash \mathbf{H}_{\mathbb{C}}^2)$ is built from the Jacobians of Fuchsian curves on $\Gamma \backslash X$. By work of Kazhdan in [63], one can always find elements in the commensurability class with nontrivial Albanese variety, and there are always noncongruence groups in the commensurability classes under consideration. It would be interesting to know whether or not the factors which appear for noncongruence Γ must also have complex multiplication.

Vincent Emery (Stanford University):
Bounds for torsion homology of arithmetic groups ▷

Let $\mathbf{G}$ be a connected semisimple real algebraic group, such that $\mathbf{G}(\mathbb{R})$ has trivial center and no compact factor. A result of Gelander [46] can be used to prove the following theorem that bounds the torsion homology of nonuniform arithmetic lattices $\Gamma \subset \mathbf{G}(\mathbb{R})$, without requiring Γ to be torsion-free (but with a restriction on $\mathbf{G}$). Results of this type for Betti numbers hold in much greater generality (see, for instance, [101]).

Theorem 1 (Emery [43]). *Let $\mathbf{G}$ be as above, and assume, for all irreducible lattices $\Gamma \subset \mathbf{G}(\mathbb{R})$, that we have $H_q(\Gamma, \mathbb{Q}) = 0$ for $q = 1, \ldots, j$. Then there exists a constant $C_{\mathbf{G}} > 0$, such that, for each irreducible nonuniform arithmetic lattice $\Gamma \subset \mathbf{G}(\mathbb{R})$, the following bound on torsion homology holds:*

$$\log |H_j(\Gamma, \mathbb{Z})| \leq C_{\mathbf{G}} \, \mathrm{vol}\big(\Gamma \backslash \mathbf{G}(\mathbb{R})\big).$$

The following theorem about K-theory of number fields is obtained by combining Theorem 1 with recent results of Calegari and Venkatesh [29]. It improves — for totally imaginary fields — previous bounds due to Soulé [105].

Theorem 2 (Emery [43]). *Let $d > 5$ be an integer. There exists a constant $C = C(d) > 0$, such that, for each totally imaginary field F of degree d, we have:*

$$\log |K_2(\mathcal{O}_F) \otimes R| \leq C |D_F|^2 (\log |D_F|)^{d-1},$$

where $R = \mathbb{Z}[\frac{1}{6w_F}]$, D_F is the discriminant, and w_F the number of roots of unity in F.

Alireza Salehi Golsefidy (University of California, San Diego):

Expansion properties of linear groups

Highly connected sparse graphs are extremely useful in the theory of communication, theoretical computer science, and pure mathematics (see the beautiful surveys [58], [71] and [66]). A family $\{X_i\}_{i=1}^{\infty}$ of finite k-regular graphs is called a family of *expanders* if, for some positive number c, we have

$$c \min\{|A|, |V(X_i) \setminus A|\} < |\partial A| \text{ for any } i \text{ and } A \subseteq V(X_i).$$

Margulis gave the first explicit construction of expanders. He made the fundamental observation that the Cayley graphs of finite quotients of a discrete group with Kazhdan's property (T) form a family of expanders. Based on his ideas, Selberg's 3/16-theorem implies that $\{\mathrm{Cay}(\pi_m(\mathrm{SL}_2(\mathbb{Z})), \pi_m(\Omega))\}_m$ is a family of expanders, where

$$\Omega := \left\{ \begin{bmatrix} 1 & \pm 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \pm 1 & 1 \end{bmatrix} \right\},$$

m runs through all the positive integers, and π_m is the reduction map modulo m. Many others (Burger, Sarnak and Clozel, to name a few) have studied the analytic behavior of congruence quotients of arithmetic lattices using automorphic forms and representation theory. Their work resulted in the following.

Theorem 1. *Let $\mathbb{G} \subseteq \mathbb{GL}_n$ be a semisimple $\mathbb{Q}$-group and $\Gamma := \mathbb{G} \cap \mathrm{GL}_n(\mathbb{Z}_S)$, where S is a finite set of primes. Assume Γ is an infinite group that is generated by a finite (symmetric) set $\Omega = \Omega^{-1}$. Then the Cayley graphs $\mathrm{Cay}(\pi_m(\Gamma), \pi_m(\Omega))$ form a family of expanders as m runs through positive integers.*

Lubotzky was the first to ask if Theorem 1 holds for a *thin group*. Specifically he asked if $\{\mathrm{Cay}(\pi_p(\Gamma), \pi_p(\Omega))\}_p$ is a family of expanders, where $\Gamma = \langle \Omega \rangle$,

$$\Omega = \left\{ \begin{bmatrix} 1 & \pm 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \pm 3 & 1 \end{bmatrix} \right\},$$

and p runs through all the primes. This question essentially asks if the *analytic behavior* of the congruence quotients of a linear group (under suitable conditions) is dictated by its Zariski topology.

In a groundbreaking work based on a result of Helfgott [57], Bourgain and Gamburd [22] answered Lubotzky's question affirmatively. These works were the starting point of a chain of fundamental results on the expansion properties of linear groups and their applications in other branches of mathematics (see the surveys [50, 66, 71]). The following result was an essential part of the proof of the fundamental theorem of affine sieve [51].

Theorem 2 (Golsefidy-Varjú [52])**.** *Let $\Omega \subseteq \mathrm{GL}_n(\mathbb{Z}_S)$ be a finite symmetric set and $\Gamma = \langle \Omega \rangle$. Then $\{\mathrm{Cay}(\pi_q(\Gamma), \pi_q(\Omega))\}_q$ is a family of expanders as q runs through square-free integers if and only if the Zariski connected component $\mathbb{G}^{\circ}$ of the Zariski closure $\mathbb{G}$ of Γ is perfect, i.e. $\mathbb{G}^{\circ} = [\mathbb{G}^{\circ}, \mathbb{G}^{\circ}]$.*

Lizhen Ji (University of Michigan):
Outer automorphisms of free groups and tropical geometry ▷

Let F_n be the free group on n generators (with $n \geq 2$), and let $\mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n)$ be its outer automorphism group. The group $\mathrm{Out}(F_n)$ is one of the most basic groups in combinatorial group theory, and it has been extensively studied.

One key tool for understanding the properties of $\mathrm{Out}(F_n)$ is its action on a certain space X_n, called *outer space*. By definition (see [40]), X_n is the space of equivalence classes of marked metric graphs whose fundamental group is F_n. (A *metric* on a graph Γ is an assignment of lengths to the edges of the graph, such that the sum is 1. A *marking* of Γ is a homotopy equivalence from Γ to the wedge of n circles. Roughly speaking, this is an identification of $\pi_1(\Gamma)$ with F_n, but no basepoint has been fixed, so the identification is only well-defined up to an inner automorphism. Equivalence classes are taken with respect to a natural notion of isomorphism of marked metric graphs.) Since $\mathrm{Out}(F_n)$ acts on X_n (by changing the marking), information about X_n can yield information about $\mathrm{Out}(F_n)$.

According to the celebrated Erlangen program of Klein, an essential part of the geometry of a space is concerned with invariants of the isometries (or symmetries) of the space. Similarly, an essential part of the geometry of a group is to find and understand spaces on which the group acts and preserves suitable additional structures of the space. It is often natural to require the space to be a metric space and the action to be by isometries.

In classical geometry, a metric space is a complete Riemannian manifold such as the Euclidean space, the sphere, or the hyperbolic space. But it is also important to consider metric spaces that are not manifolds. Examples include Tits buildings and Bruhat-Tits buildings for real and p-adic semisimple Lie groups. These are simplicial complexes with a natural complete Tits metric so that the groups act isometrically and simplicially on them. Of course, the rich combinatorial structure also makes their geometry interesting.

The outer space X_n can be realized in a natural way as a subset of a finite-dimensional simplicial complex (with infinitely many simplices) and hence admits a natural simplicial metric d_0. This metric is invariant under $\mathrm{Out}(F_n)$, but it is not complete, because X_n is missing the faces of some simplices. Since the completeness of metrics is a basic condition that is important for many applications, the following problem is very natural (see [24, Question 2] for more discussion).

Problem 1. *Construct complete geodesic metrics on X_n that are invariant under $\mathrm{Out}(F_n)$.*

As explained above, solutions to this problem provide geometries of $\mathrm{Out}(F_n)$ in a certain sense. Another motivation for this problem comes from the analogy with other important groups in geometric group theory: arithmetic subgroups Γ of semisimple Lie groups G and mapping class groups $\mathrm{Mod}_{g,n}$ of surfaces of genus g with n punctures. *A lot of work on* $\mathrm{Out}(F_n)$ *is motivated by results obtained for these families of groups.*

An arithmetic group Γ acts on the symmetric space G/K, and the mapping class group $\mathrm{Mod}_{g,n}$ acts on the Teichmüller space $\mathcal{T}_{g,n}$ of Riemann surfaces of genus g with n punctures. The symmetric space $X = G/K$ admits a complete G-invariant Riemannian metric and Γ acts isometrically and properly on X. The Teichmüller space $\mathcal{T}_{g,n}$ admits several complete Riemannian and Finsler metrics, such as the Bergman and Teichmüller metrics, and $\mathrm{Mod}_{g,n}$ acts isometrically and properly on them.

Though X_n is not a manifold, it is a locally finite simplicial complex and hence is a canonically stratified space with a smooth structure. Therefore, the following problem also seems to be natural in view of the above analogy.

Problem 2. *Construct piecewise smooth Riemannian metrics on X_n that are invariant under* $\mathrm{Out}(F_n)$ *and whose induced length metrics are complete geodesic metrics. Furthermore, the quotient* $\mathrm{Out}(F_n) \backslash X_n$ *has finite volume.*

With respect to the smooth structure on X_n as a canonically stratified space, it is natural to require the Riemannian metric on X_n to be smooth in the sense of stratified spaces. Problems 1 and 2 are solved by the following theorem.

Theorem 3 (Ji [60]). *There exist several explicitly constructed complete geodesic metrics and complete piecewise-smooth Riemannian metrics on X_n that are invariant under* $\mathrm{Out}(F_n)$.

The proof of this theorem utilizes tropical geometry, which is algebraic geometry over the tropical semifield. (This is a rapidly developing subject — see [59, 79] and the references therein.) The theory is applicable because metric graphs can be identified with tropical curves. There is a tropical Jacobian map from the moduli space of tropical curves to the moduli space of principally polarized tropical abelian varieties, and the desired metrics are obtained by combining this map with the simplicial metric d_0 on X_n. This application of tropical geometry to geometric group theory might be of independent interest.

Now that complete invariant geodesic metrics have been constructed on X_n, one natural problem is to understand how these metrics can be used to study X_n and $\mathrm{Out}(F_n)$. Their construction might be the first step towards a metric theory of outer space [24, Question 1.2].

**Benjamin Klopsch (Heinrich Heine University Düsseldorf) and
Christopher Voll (Bielefeld University):**
Representation growth of arithmetic groups ▷

The talks of B. Klopsch and C. Voll were coordinated to essentially constitute another mini-course, so they are summarized here as a single unit.

For a group Γ, let $r_n(\Gamma)$ denote the number of irreducible complex representations (up to isomorphism) in dimension n. The group Γ is called *representation rigid* if $r_n(\Gamma)$ is finite for every n.

A rigid group Γ has *polynomial representation growth* (PRG) if $r_n(\Gamma)$ grows at most polynomially. Equivalently, one can ask that the partial sums $R_n(\Gamma) := \sum_{i=1}^n r_i(\Gamma)$ grow at most polynomially. In this case, it is profitable to encode the numbers $r_n(\Gamma)$ into the *(representation) zeta function* of Γ:

$$\zeta_\Gamma(s) := \sum_{n=1}^\infty \frac{r_n(\Gamma)}{n^s}.$$

The PRG-condition ensures that ζ_Γ converges absolutely on some complex right half-plane. Conversely, the abscissa of convergence determines the rate of growth of the sequence $\{R_n(\Gamma)\}$. Computing this abscissa is therefore a central problem in the subject of representation growth of groups. In particular, it is important to understand how the abscissa varies (and how other analytic invariants of ζ_Γ vary), as Γ ranges over interesting classes of groups.

Consider the case that Γ is an arithmetic group in characteristic zero. For simplicity, assume that $\Gamma = \mathbf{G}(\mathcal{O}_S)$, where $\mathbf{G}$ is a connected, simply connected, semisimple algebraic group defined over a number field k with ring of S-integers $\mathcal{O}_S$.

Theorem 1 (Lubotzky-Martin [72])**.** *The group Γ has PRG if and only if it has the Congruence Subgroup Property (CSP).*

From now on we assume, again for simplicity, that the congruence kernel of Γ is trivial. In this case, the zeta function has an Euler product decomposition that is a consequence of Margulis superrigidity:

Proposition 2 (Larsen-Lubotzky [67])**.** *If Γ has trivial congruence kernel then the zeta function ζ_Γ has an Euler product decomposition:*

$$(1) \qquad\qquad \zeta_\Gamma(s) = \zeta_{\mathbf{G}(\mathbb{C})}(s)^{[k:\mathbb{Q}]} \prod_{v \notin S} \zeta_{\mathbf{G}(\mathcal{O}_v)}$$

The archimedean factor $\zeta_{\mathbf{G}(\mathbb{C})}$ enumerates *rational* representations of the algebraic group $\mathbf{G}(\mathbb{C})$. This factor, known as the Witten zeta function, is comparatively well understood in terms of highest weight theory and the Weyl character formula. The non-archimedean factors $\zeta_{\mathbf{G}(\mathcal{O}_v)}$ enumerate *continuous* representations of the p-adic analytic groups $\mathbf{G}(\mathcal{O}_v)$, and there exists a well-developed Lie theory for their principal congruence subgroups $\mathbf{G}^m(\mathcal{O}_v)$. In particular, the Kirillov orbit method sets up a 1-1-correspondence between continuous, irreducible complex representations of these pro-p groups and finite co-adjoint orbits in the dual of their Lie algebras. Deligne-Lusztig theory governs the representation theory of the finite groups of Lie type $\mathbf{G}(\mathcal{O}_v)/\mathbf{G}^1(\mathcal{O}_v)$.

Avni, Klopsch, Onn, and Voll have recently created a framework for the study of the local representation zeta functions $\zeta_{\mathbf{G}(\mathcal{O}_v)}$, with a view toward analyzing their Euler products. Formulas for the zeta functions of pro-p-groups of the form $\mathbf{G}^m(\mathcal{O}_v)$ can be obtained by developing methods from p-adic integration. These formulas yield, in particular, proofs of *local functional equations* upon inversion of the residue field characteristic [7].

For analysis of the analytic properties of Euler products such as (1), control over the zeta functions of congruence subgroups is not sufficient. In [8], powerful machinery from model theory (viz., integrals of quantifier-free definable functions) is developed to "approximate", uniformly over large sets of places, the Clifford theory connecting the representation theory of the groups $\mathbf{G}(\mathcal{O}_v)$ with the representation theory of their congruence subgroups. This approach requires new insights into the behavior of representation growth of groups such as $\mathbf{G}(\mathcal{O})$ under base change, combined with a detailed analysis of representation zeta functions of finite groups of Lie type. This yields, in particular, a proof of the following theorem, which implies that the degree of representation growth is invariant under ring extensions.

Theorem 3 (Avni-Klopsch-Onn-Voll [8])**.** *For every irreducible root system Φ, there is a constant α_Φ that equals the abscissa of convergence of the representation zeta function for any group $\Gamma = \mathbf{G}(\mathcal{O}_S)$ satisfying the CSP, such that $\mathbf{G}$ has absolute root system Φ.*

This is related to the following conjecture, which is a refinement of Serre's conjecture on the Congruence Subgroup Property of lattices in higher-rank groups:

Conjecture 4 (Larsen-Lubotzky [67])**.** *Let G be a higher-rank semisimple locally compact group and let Γ_1 and Γ_2 be two irreducible lattices in G. Then the corresponding representation zeta functions have the same abscissa of convergence.*

For groups that have the CSP (which is required by the theorem, but not by the conjecture), the theorem's hypothesis is weaker than the conjecture's hypothesis. Namely, the conjecture requires Γ_1 and Γ_2 to be contained in a common ambient group G, but the theorem only requires the ambient groups to have the same absolute root system. Further details can be found in the survey [64].

C. S. Rajan (Tata Institute of Fundamental Research):
Representation and characteristically equivalent arithmetic lattices

The inverse spectral problem is to recover the properties of a compact Riemannian manifold M from the knowledge of the spectrum of the Laplace operator (or of a more general Laplacian type operator) acting on the space of smooth functions on M. It is known, for example, that the spectra on functions determines the dimension, volume and the scalar curvature of M.

Examples of non-isometric compact Riemannian manifolds which are isospectral on functions have been given by Milnor in the context of flat tori, and by Vigneras for compact hyperbolic surfaces [112]. Sunada gave a general method in analogy with a construction in arithmetic [108].

In many of these constructions, the manifolds are quotients by finite groups of a fixed Riemannian manifold. The question arises whether isospectral manifolds are indeed commensurable, i.e., have a common finite cover. In the context of Riemannian locally symmetric spaces this question has been studied by various authors [36, 73, 88, 96] assuming that the spaces are isospectral for the Laplace-Beltrami operator acting on functions. In [88], Gopal Prasad and A. S. Rapinchuk address this question in full generality, and get conditional commensurability type results for isospectral, compact locally symmetric spaces. For this when the locally symmetric spaces are of rank at least two, they have to assume the validity of Schanuel's conjecture on transcendental numbers. Another hypothesis they are required to make is that the base field is totally real and the group is anisotropic at all but one real place.

Chandrasheel Bhagwat, Supriya Pisolkar, and C. S. Rajan [18] considered this question, assuming the stronger hypothesis that the lattices defining the locally symmetric spaces are representation equivalent, rather than isospectral on functions (see [42]). They were able to obtain unconditionally similar conclusions as in [88] for representation equivalent lattices, for example without invoking Schanuel's conjecture, and also extend the application to representation equivalent S-arithmetic lattices. In the process, they introduced a new relation of characteristic equivalence of lattices, stronger than weak commensurability.

The proofs are distilled from the arguments given in [88]. The stronger hypothesis simplifies the arguments used in [88].

Andrei Rapinchuk (University of Virginia):
On the congruence subgroup problem ▷

The talk was a brief survey of, and a progress report on, the congruence subgroup for algebraic groups over global fields. Let G be an absolutely almost simple algebraic group defined over a global field K, and let S be a (not necessarily finite) set of places containing all

the archimedean ones when K is a number field. Then one considers the completions $\widehat{G}^S$ and $\overline{G}^S$ of the group $G(K)$ of rational point with respect to the S-arithmetic and the S-congruence topologies (see [89, 91] for precise definitions), and defines the *S-congruence kernel* $C^S(G)$ to be the kernel of the natural continuous surjective homomorphism $\widehat{G}^S \xrightarrow{\pi} \overline{G}^S$. The congruence subgroup problem in this situation is the question about the computation of $C = C^S(G)$. The main conjecture, due to Serre, states that C *should be* finite *if* $\mathrm{rk}_S\, G := \sum_{v \in S} \mathrm{rk}_{K_v}\, G$ *is* $\geqslant 2$ *and* $\mathrm{rk}_{K_v}\, G > 0$ *for all nonarchimedean* $v \in S$, *and* infinite *if* $\mathrm{rk}_S\, G = 1$ (here $\mathrm{rk}_{K_v}\, G$ denotes the rank of G over the completion K_v). The talk focused primarily on the higher rank case of Serre's conjecture (the structure of C in the rank one situation has been determined in many cases by O. V. Mel'nikov, A. Lubotzky and P. A. Zalesskii with various co-authors - see the references in [89]). First, it was explained that modulo the Margulis-Platonov conjecture on the structure of normal subgroups of $G(K)$, which has already been proved in the majority of cases, the finiteness of C is equivalent to its *centrality*, i.e., to the fact that it lies in the center of $\widehat{G}^S$, in which case C (or more precisely, its Pontrjagin dual) is isomorphic to the *metaplectic kernel* $M(S, G)$ (see [89], [91] for the details). Second, the effort to compute $M(S, G)$ initiated by C. Moore and continued by Matsumoto, Deodhar, Prasad-Raghunathan and others was completed in [87]. In essence, the final result says that $M(S, G)$ is always finite and is isomorphic to a subgroup of the group μ_K of roots of unity in K (in particular, it is always a finite cyclic group), and in fact is trivial under some rather general additional assumptions. So, the focus in the higher rank case of the congruence subgroup problem is currently on finding a general approach to the proof of centrality. The centrality has been established in a number of cases using a variety of techniques (cf. [89, 91]), but there are still anisotropic groups (e.g., K-groups of the form $\mathrm{SL}_{1,D}$ where D is a finite-dimensional central division algebra over K) that defy all efforts. The talk discussed some approaches to proving the centrality that do not require any case-by-case considerations. One of the approaches relies on the analysis of the centralizers of elements from $G(K)$ in $\widehat{G}^S$. To formulate a result in this direction, recall that, without loss of generality, we may assume that $\mathrm{rk}_S\, G > 0$, and then the S-congruence completion $\overline{G}^S$ can be identified with the group of S-adeles $G(\mathbb{A}(S))$ by the strong approximation theorem (see [92] for a recent survey on strong approximation).

Proposition 1. *Assume that there exists $n \geqslant 1$ such that for any regular semi-simple element $t \in G(K)$ and its centralizer $T = Z_G(t)$ we have*

$$\pi(Z_{\widehat{G}^S}(t)) \supset T(\mathbb{A}(S))^n.$$

Then C is central.

If we let $\widehat{}$ and $\overline{}$ denote the closure in $\widehat{G}^S$ and $\overline{G}^S$, respectively, then clearly, $Z_{\widehat{G}^S}(t)$ contains $\widehat{T(K)}$, and since $\pi(\widehat{T(K)}) = \overline{T(K)}$, we obtain the following.

Corollary 2. *If there exists $n \geqslant 1$ such that*

$$\overline{T(K)} \supset T(\mathbb{A}(S))^n$$

for any K-torus T in G then C is central.

Thus, the centrality would be a consequence of the property of almost strong approximation in all K-tori of G for a given S. The bad news is that this property *never* holds for a nontrivial K-torus if S is finite. More precisely, one uses the fact that T admits coverings

$T \to T$ of any degree to show that the quotient $T(\mathbb{A}(S))/\overline{T(K)}$ has infinite exponent (cf. [92]). But there is a little bit of good news at the other end of the spectrum, viz. when S is co-finite, i.e. $S = V^K \setminus S_0$ where S_0 is a finite set of nonarchimedean places. In this case,

$$T(\mathbb{A}(S)) = T_{S_0} := \prod_{v \in S_0} T(K_v),$$

and one shows that $T_{S_0}/\overline{T(K)}$ has finite exponent which can be bounded by a function depending only on $\dim T$.

Corollary 3 (Semi-local case). *If S is co-finite then C is central (in fact, trivial).*

In fact, the property of almost strong approximation holds in tori if S almost contains a generalized arithmetic progression (subject to some natural assumptions), and one can formulate the corresponding result for the centrality of C.

Corollary 3 leads to another condition for the centrality of C. To formulate it, we observe that using the identification $\overline{G}^S = G(\mathbb{A}(S))$, one can think of $G(K_v)$ for any $v \notin S$ as a subgroup of $\overline{G}^S$.

Theorem 4. *Assume that for each $v \notin S$ there exists a subgroup $\mathcal{G}_v \subset \widehat{G}^S$ so that*

(1) $\pi(\mathcal{G}_v) = G(K_v)$;
(2) $\mathcal{G}_{v_1}$ *and* $\mathcal{G}_{v_2}$ *commute elementwise for any $v_1 \neq v_2$;*
(3) *the subgroup generated by the $\mathcal{G}_v$'s is dense in $\widehat{G}^S$.*

Then C is central.

This result can be used to give a relatively short proof of Serre's conjecture for K-isotropic groups and also for K-anisotropic groups of exceptional types $\mathsf{E}_7, \mathsf{E}_8$ and F_4 (recall that these types split over a quadratic extension of K).

Finally, the ideas involved in the proof of Theorem 4 can be used to provide some information about C in the rank one case. First, recall the following general dichotomy (assuming the truth of the Margulis-Platonov conjecture): *C is either central and finite, or is* not *finitely generated* (e.g., the congruence kernel for the group $\mathrm{SL}_2(\mathbb{Z})$ is known to be the free profinite group of countable rank). Nevertheless, Lubotzky proved that if $\Gamma = G(\mathcal{O}(S))$ is the corresponding S-arithmetic subgroup then C is *finitely generated* as a normal subgroup of $\widehat{\Gamma}$ (this is a consequence of finite presentation of the group of integral ideles $\overline{\Gamma}$). G. Prasad and A. Rapinchuk showed that in some situations C is virtually generated by a *single* element as a normal subgroup of $\widehat{G}^S$. (Note that, between this result and that of Lubotzky, neither one implies the other.)

Theorem 5. *Assume that K is a number field and G is K-isotropic. Then there exists $c \in C$ such that if $D \subset C$ is the closed normal subgroup of $\widehat{G}^S$ generated by c then C/D is a finite cyclic group.*

(For $G = \mathrm{SL}_2$ this element c can be written down explicitly.)

Igor Rapinchuk (Yale University):
On the conjecture of Borel and Tits for abstract homomorphisms of algebraic groups ▷

The general philosophy in the study of abstract homomorphisms between groups of rational points of algebraic groups is as follows. Suppose $\mathbf{G}$ and $\mathbf{G}'$ are algebraic groups that are defined over infinite fields K and K', respectively. Let

$$\varphi \colon \mathbf{G}(K) \to \mathbf{G}'(K')$$

be an abstract homomorphism between their groups of rational points. Then, under appropriate assumptions, one expects to be able to write φ essentially as a composition $\varphi = \beta \circ \alpha$, where $\alpha \colon \mathbf{G}(K) \to {}_K\mathbf{G}(K')$ is induced by a *field homomorphism* $\tilde{\alpha} \colon K \to K'$ (and ${}_K\mathbf{G}$ is the group obtained from $\mathbf{G}$ by base change via $\tilde{\alpha}$), and $\beta \colon {}_K\mathbf{G}(K') \to \mathbf{G}'(K')$ arises from a K'-defined *morphism of algebraic groups* ${}_K\mathbf{G} \to \mathbf{G}'$. Whenever φ admits such a decomposition, one generally says that it has a *standard description*.

The following conjecture of Borel and Tits is a major open question. Recall that, for an algebraic group $\mathbf{G}$ defined over a field k, one denotes by $\mathbf{G}^+$ the subgroup of $\mathbf{G}(k)$ generated by the k-points of split (smooth) connected unipotent k-subgroups.

Conjecture 1 (Borel-Tits [21, 8.19]). *Let $\mathbf{G}$ and $\mathbf{G}'$ be algebraic groups defined over infinite fields k and k', respectively. If $\rho \colon \mathbf{G}(k) \to \mathbf{G}'(k')$ is any abstract homomorphism, such that $\rho(\mathbf{G}^+)$ is Zariski-dense in $\mathbf{G}'(k')$, then there exists a commutative finite-dimensional k'-algebra B and a ring homomorphism $f \colon k \to B$, such that*

$$\rho|_{\mathbf{G}^+} = \sigma \circ r_{B/k'} \circ F,$$

where

- $F \colon \mathbf{G}(k) \to {}_B\mathbf{G}(B)$ *is induced by f (and ${}_B\mathbf{G}$ is obtained by change of scalars),*
- $r_{B/k'} \colon {}_B\mathbf{G}(B) \to R_{B/k'}({}_B\mathbf{G})(k')$ *is the canonical isomorphism (here $R_{B/k'}$ denotes the functor of restriction of scalars), and*
- σ *is a rational k'-morphism of $R_{B/k'}({}_B\mathbf{G})$ to $\mathbf{G}'$.*

In their fundamental paper [21], Borel and Tits proved the conjecture for $\mathbf{G}$ an absolutely almost simple k-isotropic group and $\mathbf{G}'$ a reductive group. Shortly after the conjecture was formulated, Tits [109] sketched a proof of 1 in the case that $k = k' = \mathbb{R}$. Prior to the recent work of I. Rapinchuk that is described below, the only other available result was due to L. Lifschitz and A. S. Rapinchuk [68], where the conjecture was essentially proved in the case where k and k' are fields of characteristic 0, $\mathbf{G}$ is a universal Chevalley group, and $\mathbf{G}'$ is an algebraic group with commutative unipotent radical.

While the above results only deal with abstract homomorphisms of groups of points over *fields*, it should be pointed out that there has also been considerable interest and activity in analyzing abstract homomorphisms of higher rank arithmetic groups and lattices (e.g., the work of Bass, Milnor, and Serre [10] on the congruence subgroup problem and Margulis's Superrigidity Theorem [76, Chap. VII]). However, relatively little was previously known about abstract homomorphisms of groups of points over *general commutative rings*, which has been the primary focus of I. Rapinchuk's work in this area.

To state the new results, we first need to fix some notation. Let Φ be a reduced irreducible root system of rank ≥ 2 and $\mathbf{G}$ be the corresponding universal Chevalley-Demazure group scheme over $\mathbb{Z}$. For any commutative ring R, we denote by $\mathbf{G}(R)^+$ the subgroup of $\mathbf{G}(R)$

generated by the R-points of the canonical one-parameter root subgroups (usually called the *elementary subgroup*).

The first theorem below is a rigidity result for abstract representations

$$\rho \colon \mathbf{G}(R)^+ \to GL_n(K),$$

where K is an algebraically closed field. In its statement, for a finite-dimensional commutative K-algebra B, the group of rational points $\mathbf{G}(B)$ is viewed as an algebraic group over K by using the functor of restriction of scalars. Furthermore, given a commutative ring R, we will say that (Φ, R) is a *nice pair* if $2 \in R^\times$ whenever Φ contains a subsystem of type B_2, and $\{2, 3\} \subseteq R^\times$ if Φ is of type G_2.

Theorem 2 (I. Rapinchuk [93, Main Theorem])**.** *Let Φ be a reduced irreducible root system of rank ≥ 2, R a commutative ring such that (Φ, R) is a nice pair, and K an algebraically closed field. Assume that R is noetherian if $\operatorname{char} K > 0$. Furthermore let $\mathbf{G}$ be the universal Chevalley-Demazure group scheme of type Φ and let $\rho \colon \mathbf{G}(R)^+ \to GL_n(K)$ be a finite-dimensional linear representation over K of the elementary subgroup $\mathbf{G}(R)^+ \subset \mathbf{G}(R)$. Set $H = \overline{\rho(\mathbf{G}(R)^+)}$ (Zariski closure), and let H° denote the connected component of the identity of H. Then in each of the following situations*

   (1) *H° is reductive;*
   (2) *$\operatorname{char} K = 0$ and R is semilocal;*
   (3) *$\operatorname{char} K = 0$ and the unipotent radical U of H° is commutative,*

there exists a commutative finite-dimensional K-algebra B, a ring homomorphism $f \colon R \to B$ with Zariski-dense image, and a morphism $\sigma \colon \mathbf{G}(B) \to H$ of algebraic K-groups such that for a suitable subgroup $\Delta \subset \mathbf{G}(R)^+$ of finite index, we have

$$\rho|_\Delta = (\sigma \circ F)|_\Delta,$$

where $F \colon \mathbf{G}(R)^+ \to \mathbf{G}(B)^+$ is the group homomorphism induced by f.

Thus, if $R = k$ is a field of characteristic $\neq 2$ or 3, then R is automatically semilocal and (Φ, R) is a nice pair. Hence, Theorem 1 provides a proof of Conjecture 1 in the case that $\mathbf{G}$ is split and K is an algebraically closed field of characteristic zero.

Let us now describe some applications of Theorem 2 to the study of character varieties of elementary subgroups of Chevalley groups. Let K be an algebraically closed field of characteristic 0 and R be a finitely generated commutative ring. As above, suppose that Φ is a reduced irreducible root system of rank ≥ 2 and let $\mathbf{G}$ be the corresponding universal Chevalley-Demazure group scheme. Then the elementary subgroup $\mathbf{G}(R)^+$ has Kazhdan's property (T) (see [44]), hence is in particular a finitely generated group, and therefore, for any integer $n \geq 1$, one can consider the character variety $X_n(\Gamma)$.

Theorem 3 (I. Rapinchuk [94])**.** *Let Φ be a reduced irreducible root system of rank ≥ 2, R a finitely generated commutative ring such that (Φ, R) is a nice pair, and $\mathbf{G}$ the universal Chevalley-Demazure group scheme of type Φ. Denote by Γ the elementary subgroup $\mathbf{G}(R)^+$ of $\mathbf{G}(R)$ and consider the nth character variety $X_n(\Gamma)$ of Γ over an algebraically closed field K of characteristic 0. Then there exists a constant $c = c(R)$ (depending only on R) such that $\kappa_n(\Gamma) := \dim X_n(\Gamma)$ satisfies*

$$\kappa_n(\Gamma) \leq c \cdot n$$

for all $n \geq 1$.

The proof of Theorem 3 makes extensive use of Theorem 2's description of the representations with non-reductive image.

Another application of Theorem 2 has to do with the problem of realizing complex affine varieties as character varieties of suitable finitely generated groups. This question was previously considered by M. Kapovich and J. Millson [62], who showed that any affine variety S defined over $\mathbb{Q}$ is birationally isomorphic to an appropriate character variety of some Artin group Γ. By using Theorem 2 with $K = \mathbb{C}$, it is possible to prove the following result.

Theorem 4 (I. Rapinchuk [95]). *Let S be an affine algebraic variety defined over $\mathbb{Q}$. There exist a finitely generated group Γ having Kazhdan's property (T) and an integer $n \geq 1$ such that there is a biregular isomorphism of complex algebraic varieties*

$$S(\mathbb{C}) \to X_n(\Gamma) \setminus \{[\rho_0]\},$$

where ρ_0 is the trivial representation.

Alan Reid (University of Texas at Austin):
All finite groups are involved in the mapping class group ▷

Let Σ_g be a closed orientable surface of genus $g \geq 1$, and Γ_g its Mapping Class Group.

A group H is *involved* in a group G if there is a finite index subgroup $K < G$ so that K subjects onto H. The question as to whether every finite group is involved in a fixed Γ_g was raised by U. Hamenstädt in her talk at the 2009 Georgia Topology Conference. This is easily seen to hold for the case $g = 1$ (since $\Gamma_1 = \mathrm{SL}(2, \mathbf{Z})$ is virtually free) and for $g = 2$ (since Γ_2 is large, see [65]). In fact, it holds for all g:

Theorem 1 (Masbaum-Reid [77]). *For all $g \geq 1$, every finite group is involved in Γ_g.*

Although Γ_g is well-known to be residually finite [53], and therefore has a rich supply of finite quotients, very little seems known about what finite groups can arise as quotients of Γ_g (or of subgroups of finite index), other than those finite quotients obtained from

$$\Gamma_g \to \mathrm{Sp}(2g, \mathbb{Z}) \to \mathrm{Sp}(2g, \mathbb{Z}/N\mathbb{Z}).$$

It should be emphasized that one cannot expect to prove Theorem 1 by simply using the subgroup structure of the groups $\mathrm{Sp}(2g, \mathbb{Z}/N\mathbb{Z})$. The reason for this is that, since $\mathrm{Sp}(2g, \mathbb{Z})$ has the Congruence Subgroup Property [10], it is well-known that not all finite groups are involved in $\mathrm{Sp}(2g, \mathbb{Z})$ (see [70, Chapter 4.0], for example).

The main new idea in the proof of Theorem 1 is to exploit the unitary representations arising in Topological Quantum Field Theory (TQFT), first constructed by Reshetikhin and Turaev [100]. (Actually, the proof uses the so-called SO(3)-TQFT, following the skein-theoretical approach of [19] and the Integral TQFT refinement [48].)

Since, as was mentioned above, the case $g = 1$ and the case $g = 2$ are easy, it suffices to deal with the case where $g \geq 3$. Therefore, Theorem 1 easily follows from the next result, which gives many new finite simple groups of Lie type as quotients of Γ_g.

Theorem 2 (Masbaum-Reid [77]). *For each $g \geq 3$, there exist infinitely many N, such that, for each such N, there exist infinitely many primes q, such that Γ_g surjects onto the finite group $\mathrm{PSL}(N, \mathbb{F}_q)$, where $\mathbb{F}_q$ denotes the finite field of order q.*

In addition, [77] also shows that Theorem 2 holds for the Torelli group (with $g \geq 2$).

Bertrand Rémy (Institut Camille Jordan):
Informal talk on Kac-Moody groups ▷

There are two kinds of Kac-Moody groups: the complete groups and the minimal groups. Both are discussed in J. Tits' Bourbaki seminar [111], and they both have the same algebraic origin, namely Kac-Moody Lie algebras [61]. (Kac-Moody algebras are infinite-dimensional analogues of finite-dimensional reductive Lie algebras. In the classical finite-dimensional setting, the Serre presentation produces generators and relations for the Lie algebra from a Cartan matrix. In the infinite-dimensional setting, there is an analogous presentation that is produced from a generalized Cartan matrix.) The two kinds of groups also share a method of construction that imitates, in an infinite-dimensional context, the definition of Chevalley-Demazure group schemes for reductive groups [41]. Namely, the Kac-Moody group of either kind is a functor that is defined by a presentation. It is essentially a Steinberg presentation, generalizing an abstract presentation of the rational points of a split reductive group [110]. The main ingredients in the presentation are various completions of integral forms of (pieces of) universal enveloping algebras of Kac-Moody algebras [97].

Instead of working with the presentation of a Kac-Moody group, it is much more efficient to use its nice combinatorics (the existence of two twinned Tits systems). The geometric counterpart to these rich combinatorial properties is the existence of buildings on which the Kac-Moody group acts highly transitively. By definition, buildings are cell complexes that are the union of subcomplexes all isomorphic to a given Coxeter tiling; some additional incidence properties are required [4]. They admit very useful metrics that are complete and non-positively curved. Moreover, non-spherical buildings are contractible, which suggests a fruitful analogy with symmetric spaces of non-compact type (an important tool in the study of Lie groups and their discrete subgroups).

One of the valuable features of Kac-Moody theory is that it leads to intriguing new examples of groups. For example, the minimal Kac-Moody groups over finite fields provide infinitely many quasi-isometry classes of finitely presented simple groups [32]. (The proof of simplicity has two main parts. First is the fact that non-central normal subgroups have finite index [9, 98], which is analogous to a well-known result in the theory of arithmetic groups. Then the proof exploits a crucial "weakly hyperbolic" property of the geometry of non-Euclidean infinite Coxeter groups [31].) The maximal pro-p subgroups in complete Kac-Moody groups are another class of groups that pose interesting challenges. Their first homology has just been computed [30], but their higher finiteness properties still need to be investigated.

Matthew Stover (Temple University):
Counting ends of rank one arithmetic orbifolds ▷

If N is a noncompact negatively curved locally symmetric space of finite volume, then it has a finite number of topological ends, or cusps. The following question, remarkably, remains wide open:

Question 1. *Is there a one-cusped complete finite-volume hyperbolic n-manifold for every n?*

One-ended finite-volume orbifold quotients of hyperbolic n-space $\mathbf{H}^n$ are known for $n < 10$, but no one-cusped n-manifold is known for $n > 4$. For $n = 2, 3$, it is relatively easy to find examples with arithmetic fundamental group. For example, one can interpret the fact that

the modular surface $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbf{H}^2$ has one cusp in terms of the fact that $\mathbb{Z}$ is a principal ideal domain, and it is relatively easy to find a one-cusped manifold cover. The number of cusps of arithmetic hyperbolic 3-manifolds is closely related to the class number of an imaginary quadratic field k, which is an invariant that measures 'how far' its ring of integers is from being a PID.

Given the ease with which one can build one-cusped arithmetic orbifolds in dimensions 2 and 3, one might hope that arithmetic techniques could provide one-cusped hyperbolic n-manifolds, or at least orbifolds, for all n. The following theorem shows that this is impossible.

Theorem 2 (Stover [106])**.** *One-cusped arithmetic hyperbolic n-orbifolds do not exist for any $n > 31$.*

The proof relates the number of ends to the so-called class number of a certain quadratic form, then studies related number theoretic invariants that yield a lower bound on the number of ends. In fact, it gives an exact formula for the number of cusps for $\Gamma\backslash\mathbf{H}^n$ when Γ is a natural generalization of the usual congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. The paper also constructs new one-cusped examples in dimensions 10 and 11.

Theorem 2 is actually a precise special case of the following much stronger finiteness theorem.

Theorem 3 (Stover [106])**.** *Fix $k > 0$. There are only finitely many commensurability classes of negatively curved arithmetic locally symmetric spaces that contain an element with k ends.*

The negatively curved locally symmetric spaces are hyperbolic n-space, complex hyperbolic n-space $\mathbf{H}^n_{\mathbb{C}}$, quaternionic n-space $\mathbf{H}^n_{\mathbb{H}}$, and the exceptional Cayley hyperbolic plane $\mathbf{H}^2_{\mathbb{O}}$. All finite-volume quaternionic hyperbolic n-orbifolds and Cayley hyperbolic 2-orbifolds are arithmetic, so the arithmetic assumption in Theorem 3 is superfluous and finiteness holds over all finite-volume quotients. In particular, for each $k > 0$, there is a constant c_k such that finite-volume quaternionic hyperbolic n-orbifolds with k ends do not exist for $n > c_k$.

T. N. Venkataramana (Tata Institute of Fundamental Research):
Monodromy of arithmetic groups ▷

Let $f, g \in \mathbb{Z}[X]$ be polynomials of degree n that are monic with constant term one, and have no common root. Also assume that every root of fg is a root of unity, and that $\{f, g\}$ is a "primitive pair" (see [104]).

Theorem 1 (Beukers-Heckman [17])**.** *The companion matrices A, B of f, g preserve a non-degenerate symplectic form Ω on $\mathbb{Q}^n$ and generate a Zariski dense subgroup Γ of the integral symplectic group $\mathrm{Sp}_n(\Omega, \mathbb{Z})$.*

It is known [17] that Γ is the monodromy group of a suitable hypergeometric equation of type $_nF_{n-1}$. The following theorem determines when the subgroup Γ has finite index in $\mathrm{Sp}_n(\Omega, \mathbb{Z})$:

Theorem 2 (Singh-Venkataramana [104])**.** *If the leading coefficient of the polynomial $f - g$ does not exceed two, then Γ is arithmetic.*

The method of proof also shows that for the 14 examples of Calabi-Yau threefolds listed in [33], the monodromy group is arithmetic.

Stefan Witzel (University of Münster):
Bredon finiteness properties of arithmetic groups ▷

Classifying spaces $\mathrm{E}G$ of groups G have been studied for a long time. Its usefulness in understanding G partly depends on how finite $\mathrm{E}G$ is (or how finite it can be chosen to be). One natural measure of finiteness is dimension, which leads to the notion of the *geometric dimension* $\mathrm{gd}\,G$ of the group G. In a different direction, one investigates up to which dimension the action of G on $\mathrm{E}G$ is cocompact, which is encoded in the *finiteness properties* F_n.

For groups with torsion, it is natural to allow the actions to have finite stabilizers. Thus, instead of classifying spaces $\mathrm{E}G$ for free actions, one studies classifying spaces $\underline{\mathrm{E}}G$ for proper actions. (All actions are by cell-permuting homeomorphisms on CW-complexes.) In the same way as for free actions, this gives rise to the notions of *proper geometric dimension* $\underline{\mathrm{gd}}$ and *proper finiteness properties* $\underline{\mathrm{F}}_n$ (see [23] and [74]).

For the study of classical (free) finiteness properties, there is a very useful criterion due to Brown [25]. If one can let G act on a contractible space, in such a way that the stabilizers have good finiteness properties themselves, then the criterion relates the finiteness properties of G to the essential connectivity of an orbit. (Essential connectivity is a technical property that measures how highly connected an orbit is in a coarse sense.)

Fluch and Witzel [45] translated a homological version of Brown's criterion to actions with arbitrary families of stabilizers, in particular, to proper actions. Algebraically, proper finiteness properties correspond to finiteness properties of normalizers of finite groups. Topologically, they are reflected in the connectivity of fixed point sets of finite subgroups. The essential connectivity in this case is measured uniformly over all finite groups.

A concrete family of examples illustrates how and why the classical finiteness properties of a group can differ from its proper finiteness properties. Namely, consider the stabilizers in $\mathrm{GL}_{n+1}\!\left(\mathbb{Z}[1/p]\right)$ of two horospheres in the associated Bruhat–Tits building. In a special case, these groups were known to be of type F_{n-1}, but not of type F_n, by work of Abels, Brown and others [2]. The classical finiteness properties generalize to the whole family. In contrast, the proper finiteness properties depend on the position of the horospheres. This is because the fixed-point sets of finite subgroups decompose as products of buildings, and the connectivity of horospheres in these depends on which direct factors are contained in one of the horospheres. In addition, the amount of torsion in the groups can also vary. A detailed analysis yields the following theorem, which shows that the two types of finiteness properties can vary completely independently of each other.

Theorem 1 (Witzel [114])**.** *For $0 < m \leq n$, there is a solvable algebraic group $\mathbf{G}$, such that, for every odd prime p, the group $\mathbf{G}\!\left(\mathbb{Z}[1/p]\right)$ is*

- *of type $\underline{\mathrm{F}}_{m-1}$, but not $\underline{\mathrm{F}}_m$, and, also,*
- *of type F_{n-1} but not F_n.*

Pavel Zalesskii (University of Brasilia):
Profinite topology on arithmetic groups ▷

Let G be a group. We can make G into a topological group by considering all normal finite index subgroups of G as a fundamental system of neighborhoods of the identity. This topology is called the *profinite topology* on G.

Question 1. *How strong is the profinite topology?*

Question 2. *To what extent does the profinite completion*

$$\widehat{G} = \varprojlim{}_{N \triangleleft_f G} \ G/N$$

determine G?

Definition 3. *If every finitely generated subgroup of G is closed in the profinite topology, then G is called* subgroup separable.

Definition 4. *If the conjugacy class of every element is closed, then G is called* conjugacy separable.

Cohomological aspect of Question 2. According to J.-P. Serre [103], a group G is called *good* if $G \longrightarrow \widehat{G}$ induces isomorphisms $H^n(\widehat{G}, M) \longrightarrow H^n(G, M)$ for every finite G-module M.

Subgroup separability, conjugacy separability, goodness are indications (or features) of *strong* profinite topology.

The term "strong profinite topology" has a precise meaning for S-arithmetic groups. Namely, the profinite topology on an S-arithmetic group Γ is *strong* if Γ does not have the Congruence Subgroup Property.

Remark 5. *If an S-arithmetic group Γ has CSP then it is* not *subgroup separable, conjugacy separable or good.*

Conjecture 6. *If an S-arithmetic group Γ does not have CSP then Γ is conjugacy separable and subgroup separable. If in addition the characteristic of the ground field is zero, then Γ is good.*

Supporting result. The conjecture is true for arithmetic lattices in $SL_2(\mathbb{C})$.

Recent progress in the study of 3-manifolds allows one to deduce from results of Wilton-Zalesskii [113] that the fundamental group of a compact 3-manifold is good and from results of Hamilton-Wilton-Zalesskii [54] that it is conjugacy separable.

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