Recent Progress in Applied and Computational Harmonic Analysis

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1 Discrete Wavelet Transform and Efficient Image Denoising Algorithms

Wavelet theory has been extensively developed in the function space L_2 and discrete wavelet transform has successful applications in many areas. However, to understand better the performance of different discrete wavelet transforms, it is important to investigate their underlying discrete wavelet systems in l_2. Though some preliminary results have been found recently, despite the fact that stability is a key issue in mathematical foundation of wavelet theory, results on stability of discrete wavelet systems in l_2 have been barely developed so far. In the meantime, in recent years, to better handle edge singularities, it is realized that redundant wavelet transform is often preferred by providing better directionality and flexibility. Many different directional systems such as curvelets, framelets, and shearlets have been proposed in the literature. The hard/soft thresholding is theoretically optimal for discrete wavelet transform using orthogonal wavelet filter banks. Because the associated redundant transform is no longer orthonormal, it is not known so far what is the theoretically optimal thresholding strategies for redundant transforms, in particular, for the problem of image denoising. Research teams at University of Calgary and University of Alberta are currently collaborating together to investigate the discrete wavelet system associated with complex tight framelet filter banks and are exploring the thresholding strategies for such redundant transforms using Gaussian scale mixture, which has the capability to cope with correlated noise with superior performance. The idea of using patches in the state-of-the-art denoising algorithm BM3D also takes the advantages of redundancy. We are currently also exploring the thresholding strategies for the patch-based approach in image denoising using higher order singular value decomposition.

2 Subdivision Methods in Scientific Computing

Subdivision method is a method which matured in tandem with wavelet analysis – a celebrated development in modern computational harmonic analysis.

A *linear subdivision scheme* is a procedure for constructing curves in $\mathbb{R}^n$ as the limit of a sequence of subdivisions of a polygon followed by an averaging procedure. The first such scheme was given by Georges de Rham in the early 1950's. The resulting curves, called *subdivision curves*, have a so-called multi-resolution structure and are closely related to wavelets. Subdivision curves and subdivision surfaces (their bi-variate generalization) have become a standard tool in many fields, including statistics, approximation theory, and computer aided geometric design, and there is a well-developed regularity and approximation theory for them.

In 2001, motivated by the explosion of manifold-valued data, Donoho presented a construction of a *nonlinear subdivision scheme* which served as the basis of a nonlinear wavelet transform for the muli-scale representation of such data. Each such nonlinear scheme is based on a linear subdivision scheme, and a fundamental question is to determine necessary and sufficient conditions for the regularity properties of the linear scheme to be inherited by the corresponding nonlinear scheme—this is the so-called *smoothness equivalence problem*.

Subdivision method is well-known for its capability in modeling smooth surfaces of arbitrary topology. While it has been a very successful tool in computer graphics, it has received little attention in other disciplines of science and engineering. The goal of this work is promote the application of subdivision surfaces to the realm of scientific computing, in particular in the simulation of the celebrated Helfrich model for biological membranes (such as a red blood cell.) Before this work, the Helfrich model has always been solved numerically based on approximation by simplicial (i.e. piecewise linear) surfaces. Not only does such an approximation suffer from a low approximation order, but also that the key quantities in the Helfrich model, namely the mean curvature, cannot be faithfully computed for such simplicial surfaces.

Various sufficient conditions have been given in earlier works by several authors including J. Wallner and N. Dyn (2005); G. Xie and T.Yu (2007); P. Grohs (2009); A. Weinmann (2010); T. Duchamp, G. Xie, and T.P.-Y. Yu (2013). Giving necessary and sufficient conditions have been an open problem. During the past two years, Tom Duchamp, Thomas P.Y Yu and Gang Xie have succeeded in solving the smoothness equivalence problem.

This project brings modern computational harmonic analysis techniques to life by applying it to an important scientific computing application in biophysics. It also stimulates a deeper understanding of the Helfrich model, thanks to the superb accuracy of the newly developed method.

The authors of this project greatly benefited from the interactions that happened in the Banff center. In particular, the question of "Gamma-convergence" of the proposed method (and in fact of all earlier proposed methods) was crystallized through extensive interactions of Thomas Yu (Drexel University) with Tom Duchamp (University of Washington), another participant of the BIRS workshop. The workshop also gave them the opportunity to discuss some of the open problems in the area. The next steps include the following:

(i) Study the underlying geometry of nonlinear schemes

(ii) Classify nonlinear subdivision schemes with a large symmetry group

3 Quantization of Random Frame Expansions

Motivated by compressed sensing, the main focus of the talk of Ozgur Yilmaz (UBC) was on random frames. It was explained why classical quantization methods are bound to be substantially sub-optimal and shown that by using sigma-delta quantizers along with reconstruction via "Sobolev duals", we can improve the quantization error substantially when we quantize sub-Gaussian frame expansions. Specifically, we prove that using an r-th order sigma-delta scheme, we get an accuracy of order $(-r)$-th power of the aspect ratio of the frame. Furthermore, if we optimize the order of the scheme depending on the aspect ratio, this yields root-exponential accuracy.

4 Nonlinear Frame Theory and Sparse Signals

Frame theory plays important roles in Mathematics, such as Harmonic Analysis and Functional Analysis and has significant impact in Engineering applications, such as Signal and Image Processing. The analysis operator associated with various frames, such as Hilbert frames, p-frames, Banach frames, g-frames and fusion frames, is one of most important tool in frame theory, which has bi-Lipschitz property. Motivated by nonlinear sampling theory, phase retrieval and nonlinear scattering recently developed, we discuss nonlinear extension of frame theory, by considering maps between Banach spaces which has bi-Lipschitz property.

Some necessary and sufficient conditions are given and most of those conditions are optimal. Also we discuss the stable reconstruction algorithm with exponential decay, which made our nonlinear linear frame applicable in engineering community. Sparsity is one of common features in lots of data representations. There are few publications on stable recovery of sparse signals from their nonlinear observations. Qiyu Sun and his collaborators proposed a general framework in Banach space setting and establish the exponential convergence of iterative thresholding algorithm which was discussed at the workshop.

5 Wavelets with Crystal Symmetry Shifts

Wavelets in two dimensions are now established as useful to image processing. In the classical theory, the wavelets are moved by integer shifts and powers of a dilation matrix compatible with those integer shifts. In [1], a theory was introduced where the shift group is generalized to the symmetry group of a crystal (in two dimensions, there are 17 distinct crystal groups). The introduction of this concept has opened up several significant lines of investigation:

1. Construction of a variety of different wavelet systems for each crystal group. Haar-like systems have been successfully constructed, but the construction of smooth wavelets is open.

2. Adaption of the non-abelian Fourier analysis on a crystal group to provide the essential tools for the development of the full range of basic theorems in the crystal shift group situation.

3. Characterization of the shift invariant subspaces of $L^2(R^n)$ for crystal shift groups.

The work presented at BIRS began with motivation and an introduction to the most general types of shifts that generate wavelets with a practical Fourier transform. All Haar type wavelets from crystal shifts were constructed, leading to the problem of understanding their corresponding filters. A complete classification of such Haar type filters was then given. This prompted a discussion about the problems associated with generalizing these ideas and the difficulties that arise when trying to understand the general theory of non-abelian shifted filters. Currently such a theory is still in the works, however it has yet to be developed.

The Harmonic Analysis meeting at BIRS was a great opportunity to share the recent work that has been done with wavelets from crystal shifts with other experts in the construction and application of filters and filter banks. The importance of the representation theory of crystallographic groups was stressed as well as the potential applications of crystal shifted filters to quasicrystals, crystal growth, spectroscopy, seismic imaging and medical imaging. This led to a number of fruitful discussions with the other participants of the meeting.

References

[1] J. MacArthur and K. F. Taylor, Wavelets with crystal symmetry shifts, *J. Fourier Analysis and Applications* **17** (2011), 1109–1118.