Numerical Methods for Optimal Transportation 13frg167

Jean-David Benamou (INRIA)
Adam Oberman (McGill University)
Guillaume Carlier (Paris Dauphine University)

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1 Overview of the Field

Optimal Transportation is a mathematical research topic which began two centuries ago with the French mathematician Monge's work on "des remblais et déblais" in 1781. This engineering problem consists in minimizing the transport cost between two given mass densities. In the 40's, Kantorovitch [10] solved the dual problem and interpreted it as an economic equilibrium. The *Monge-Kantorovitch* problem became a specialized research topic in optimization and Kantorovitch obtained the 1975 Nobel prize in economics for his contributions to resource allocations problems. Following the seminal discoveries of Brenier in the 90's, Optimal Transportation has received renewed attention from mathematical analysts and the Fields Medal awarded in 2010 to C. Villani, who gave important contributions to Optimal Transportation[15], arrived at a culminating moment for this theory. *Optimal Transportation* is today a mature area of mathematical analysis connected with fields as diverse as:

- regularity theory for nonlinear elliptic equations [1],
- gradient flow formulation of nonlinear diffusion equations [11] [9],
- image warping [4] ,
- frontongenesis models in meteorology [5],
- mesh adaptation in weather forecasting models [3],
- cosmology [7],
- finance [8]
- mathematical economics, notably the principal-agent problem [2] [14] [6].

2 Presentation Highlights

We had presentations every morning on several topics. These were followed by discussions and work in small groups on open problems.

2.0.1 Numerical solution of the second boundary value problem for the Monge-Ampère equation

Speaker: Brittany Froese We present a numerical method for the Optimal Mass Transportation problem. The solution is obtained by solving the second boundary value problem for the Monge-Ampère equation, a fully nonlinear elliptic partial differential equation (PDE). Instead of standard boundary conditions the problem has global state constraints. These are reformulated as a tractable local PDE. We prove convergence of the numerical method using the theory of viscosity solutions.

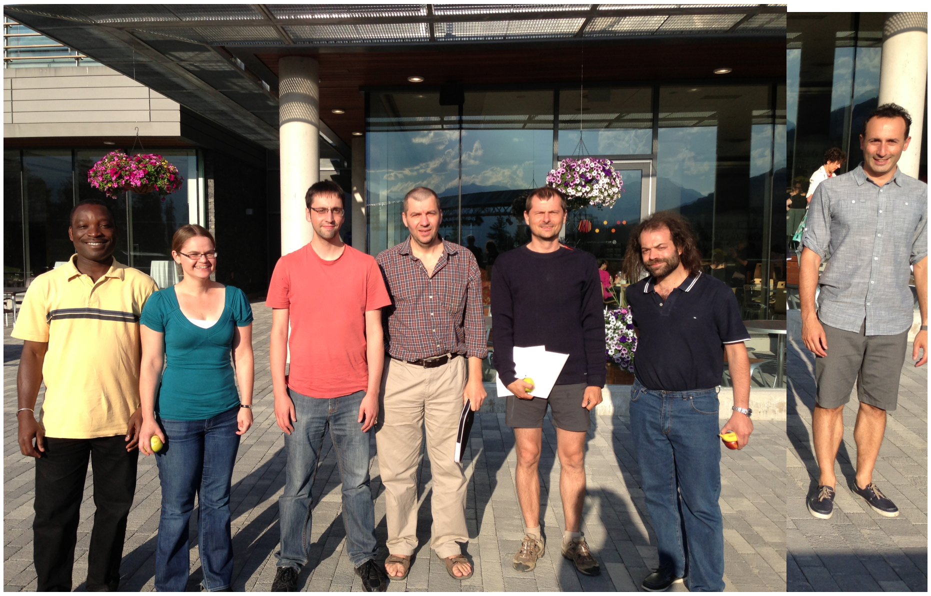

Figure 1: Photo of Participants: Numerical Methods for Degenerate Elliptic Equations and Applications 06w5095, From Left to Right : Agueh, Martial, U Victoria - Froese, Brittany, U. Texas, Austin - Pass, Brendan, U. Alberta - Benamou, Jean-David, INRIA - Oudet, Edouard - U Grenoble - Carlier, Guillaume, U. Paris Dauphine - Oberman, Adam, McGill. Absent : Ekeland Ivar U. Paris Dauphine.

2.0.2 Matching for teams, Principal agent problem

Speaker: Ivar Ekeland

The principal-agent problem the semi-analytic resolution method proposed in [13] is presented. The connection with the convexity constraint problem discussed below is stressed. Some parts of the paper are difficult to understand. Some theoretical indications that the proposed solution is not correct are given.

2.0.3 Numerical methods for Convexity constraints

Speaker: Édouard Oudet We provide a general framework to approximate constraints of convexity type. We give precise estimates of the distance between the approximation space and the admissible set. Our approach is not restricted to piecwise linear approximations and can easily extended to approximations of higher order. Recalling non smooth proximal algorithm, we describe how fast projections steps can be carried out based on our one dimensional discretization. This simple but crucial observation makes our approach relevant in the context of large scale computing where other approaches do not seem to be practicable. We imply our none smooth approach to denoising in three dimension. We demonstrate the versatility of the method by applying our algorithm to convex bodies. Up to our knowledge, these results are the first which are related to the rigorous numerical approximation of support functions. Namely, we obtained the first numerical descriptions of the projections, in the L^2, L^1 and L^∞ sense, of the support function of a regular simplex into the set of support functions associated to constant width body. These projections may be related to a famous geometrical conjecture addressed by Meissner in 1909.

2.0.4 Numerical methods for Convexity constraints

Speaker: Adam Oberman

External and internal representation of the cone of convex functions based on the wide stencil discretization is presented. The same technique is used to enforce convexity in the approximation of the Monge-Ampere operator. Problems like the principal agent can then be solved using standard QP or Cone programing algorithms.

2.0.5 Multimarginal Optimal transport - Matching for teams

Speaker: Brendan Pass

A general overview of the multi-marginal optimal transport problem with general cost functions was presented. Several applications were outlined. On the theoretical side, an update on the state of the art on existence and uniqueness of optimal maps was provided. In addition, an attempt was made to illustrate how and why the uniqueness and structure of solutions depends strongly on the cost function, a phenomenon largely absent in the classical, two marginal problem. In particular, cost functions arising in the Matching for Teams context (an economic problem explained in talks by I. Ekeland and G. Carlier) were contrasted with those arising in Density Functional Theory (a problem in chemical physics). For matching costs, under mild conditions, there exist unique optimal maps, as in the two marginal problem, whereas for DFT costs, the problem can exhibit non unique optimal plans with higher dimensional support.

2.0.6 Multimarginal Optimal transport - Matching for teams

Speaker: Guillaume Carlier We started presenting an equilibrium problem for the matching problem with more than two populations. We showed that equilibria can be obtained by convex duality arguments: one variational problem gives the equilibrium quality measure and its dual gives the equilibrium prices. We then related such problems with multimarginal Monge-Kantorovich problems in the spirit of Brendan Pass' talk and Wasserstein barycenters as in the work of Agueh and Carlier. We ended with some open questions on regularity of the barycenter and computational issues.

2.0.7 One-dimensional numerical algorithms for gradient flows in the p-Wasserstein spaces

Speaker: Martial Agueh We numerically approximate, on the real line, solutions to a large class of parabolic partial differential equations which are "gradient flows" of some energy functionals with respect to the L^p-Wasserstein metrics for all $p > 1$. Our method relies on variational principles involving the optimal transport problem with general strictly convex cost functions.

2.0.8 Numerical methods for fractional Laplacian

Speaker: Adam Oberman The fractional Laplacian is the seminal example of a nonlocal diffusion operator. It is the generator for a diffusion process with no second moment. There have been several highly cited numerical approaches to this operator in the scientific computing literature but no indication that the methods are convergent. In fact, in some cases these methods appear to be inconsistent, or fail to represent the long tails or singularity of the underlying measure. In the mathematical literature, the operator appear in nonlinear PDEs, but there is also no clear notion of how to represent it. We present a convergent numerical scheme based on quadrature and finite differneces, which also has a spectral interpretation.

3 Scientific Progress Made, Outcome of the Meeting, Open problem

3.1 Stability of approximations to Optimal Transportation problems

It is important to have convergence theorems for numerical approximations of Optimal Transportation problems. This result is currently missing from the literatture. G. Carlier discussed this open problem. He mentioned a related result, related to the stability of approximations in Gamma convergence, and cited a result from his previous work. We discussed extending this result: Carlier will write up a result to be used to prove convergence.

3.2 Numerical methods for the Multi-marginal and barycenter problem

The problem computing barycenters: it is an important problem with applications to physics: Density Functional Theory, probability: averaging of densities, and image processing. For regularity, Carlier and Pass discussed the open question of whether the support of the barycenter is convex when the support of the input densities is convex. A new numerical method for the Multi-Marginal **Barycenter** problem was proposed and

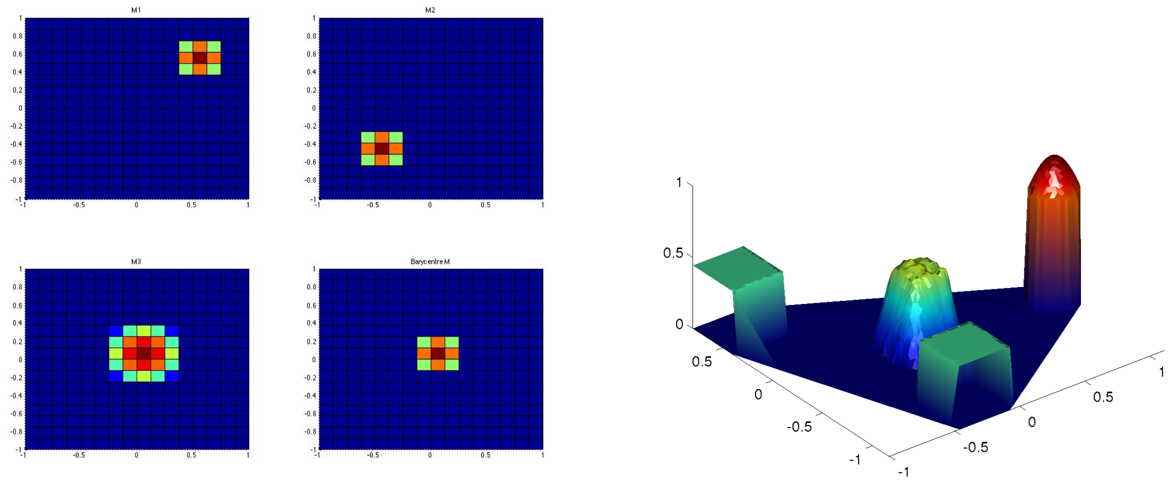

Figure 2: First figure: the first three images are some probability densities. The fourth is the numerically computed barycenter. Computed by Oberman using Linear programming. Second figure: barycenter of three densities computed a higher resolution by Oudet using nonsmooth optimization.

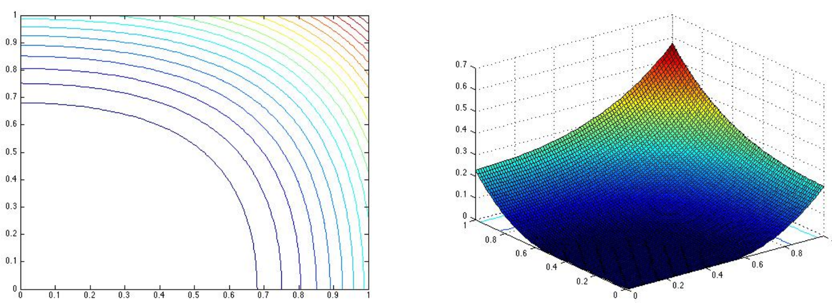

Figure 3: Solution of the Rochet Chone problem

implemented by A. Oberman. The method uses linear programming, in an implementation that was more efficient than expected: the cost is a multiple of the cost of the linear programming problem for Optimal Transportation. Sample numerical results are presented here. The method was improved by Carlier and Oudet: Carlier found the dual problem, and Oudet wrote a more efficent solver using nonsooth optimization for the dual. This allowed for larger sized problems to be solved, on a competitive basic with approximate and ad hoc methods used in image processing for textures [12] Carlier, Oberman, and Oudet are already partway to completing a paper on this subject. This is an exciting result which has the potential to have an impact in applications of the subject. The work would not have happenned if not for the meeting. See Figure 2.

3.3 Numerical results for Rochet-Chone problem

Numerical solutions of the Rochet Chone were presented, following the discussion by Ivar Ekeland over whether the exact solutions were correct. The numerical solutions obtained using the algorithm presented in the Convexity constraint (section 2.3) suggested strongly that the solution from the original paper [13] was incorrect. In particular, the second zone in the solution appears not to exist, or to be much smaller than was orginally suggested. Sample solutions (on $[0, 1]^2$) are presented in Figure 3.

4 Future plans

We already plan to organize several follow up meetings and structure the collaborations

- A one-day meeting about the numerical resolution of the PA problem will be organized at Universite Paris Dauphine.
- An application for a larger 2015 BIRS workshop on "Numerical Optimal" transportation will be proposed.
- We are seeking funds from INRIA Associate team program to support one ore more focused work group on the same topics in 2014.

We are most grateful to BIRS for kindly hosting this focused research group. It was a wonderful and fruitful experience.

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