Doubly-periodic Monopoles and their Moduli Spaces

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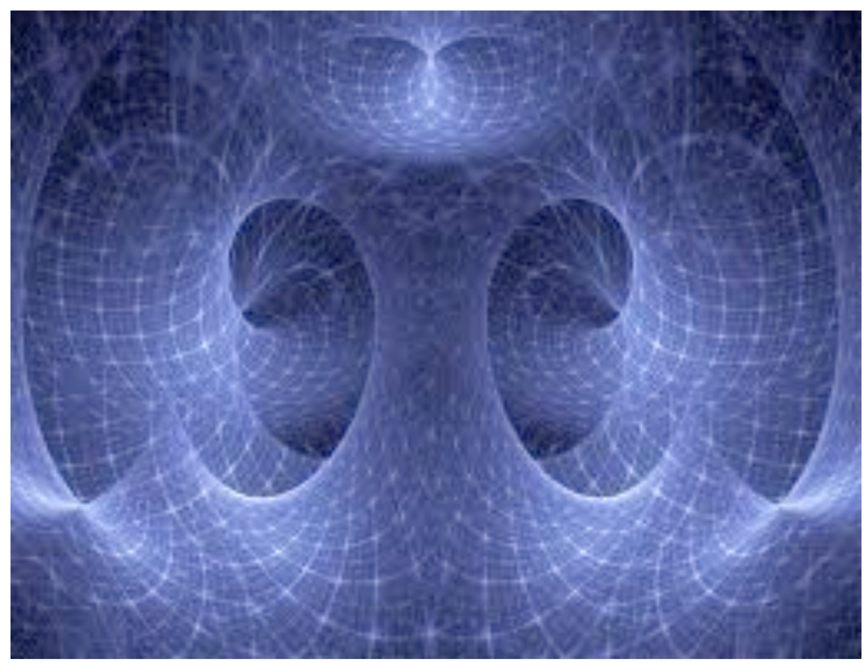
collaboration with Richard Ward



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Advances in hyperkähler and holomorphic symplectic geometry
BIRS, Banff
March 12, 2012

Motivation

- Self-dual Gravitational Instantons are complete hyperkähler manifolds of real dimension four with finite Chern number $\int_M R \wedge R < \infty$

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}R^{\mu\nu}{}_{\gamma\delta}$$

- There are only two compact cases: T^4 and K3.

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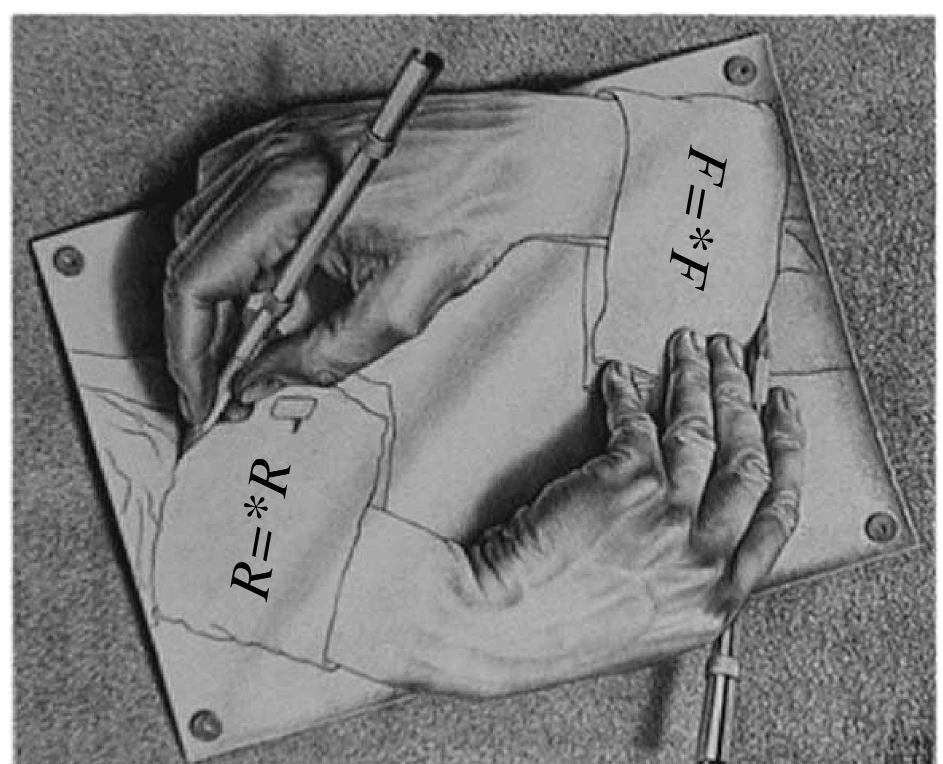
- Self-dual Yang-Mills equations are the moment map conditions for the gauge group action.

- Self-dual Yang-Mills moduli spaces are infinite hyperkähler quotients and thus carry hyperkähler metrics.

- This is a natural place to look for Gravitational Instantons.



Self-dual Yang-Mills Gauge Theory

Vector bundle $E \to M^4$ over a space-time four manifold with structure group G and a connection $\nabla = d + A$

For most of this talk G is unitary.
A is a one form valued in $\mathrm{End}\,(E)$
and E is a Hermitian vector bundle.

Curvature two-form

$$F = dA + A \wedge A$$

Self-duality Equation:

$$F = *F$$

Self-dual Yang-Mills and its reductions

Self-dual Yang-Mills (Instanton)

$M^4 = \mathbb{R}^4$ $\qquad F = F_{\mu\nu} dx^\mu dx^\nu$ $\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

$$\begin{cases} \partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] & = \partial_2 A_3 - \partial_3 A_2 + [A_2, A_3] \\ \partial_0 A_2 - \partial_2 A_0 + [A_0, A_2] & = \partial_3 A_1 - \partial_1 A_3 + [A_3, A_1] \\ \partial_0 A_3 - \partial_3 A_0 + [A_0, A_3] & = \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2] \end{cases}$$

Bogomolny Equation (Monopole)

Hitchin System

Nahm Equation

ADHM Equations

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Bogomolny Equation (Monopole)

$\mathbb{R}^3$

$\partial_0 = 0$

$$\Phi = -A_0$$

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Nahm Equation

$\mathbb{R}$

$\partial_1 = \partial_2 = \partial_3 = 0$

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ADHM Equations

Self-dual Gravitational Instantons as Moduli Spaces

Volume Growth
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$r^a, a < 2$

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r^3

r^4

Self-dual Gravitational Instantons as Moduli Spaces

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Volume Growth of a radius r ball	Moduli spaces of
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Self-dual Gravitational Instantons as Moduli Spaces

Volume Growth of a radius r ball	Moduli spaces of
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ADHM-Nahm Transform

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Instantons ⟷ Quivers (ADHM)

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ADHM-Nahm Transform

Monowalls (aka doubly-periodic monopoles)

Definition:

Monowall is a BPS monopole on $T^2_{(x,y)} \times \mathbb{R}_z$,

i.e. an n-dimensional hermitian vector bundle $E \to T^2 \times \mathbb{R}$
with a connection A and an endomorphism Φ satisfying
Bogomolny equation $$\boxed{*D_A\Phi = -F}$$

We also impose a condition:
$|\Phi|$ has no zeros for large enough z .

Bogomolny Equation: $*D_A\Phi = -F$

Abelian case (gauge group U(1)) $\Phi = \mathrm{i}\phi \quad A = \mathrm{i}a$

Bogomolny Equation is linear $*d\phi = -da$

thus the Higgs field ϕ is harmonic

Based on: http://www.phys.uu.nl/~thooft/

$$\phi = 2\pi(Qz + M), \quad a = 2\pi(Qy\,dx - p\,dx - q\,dy)$$

$$\boxed{\begin{aligned}
\phi &= \phi_0 + \pi z - \frac{1}{2r} + \frac{1}{2}\sum_{j,k\in\mathbb{Z}}\left[\frac{1}{e_{jk}} - \frac{1}{r_{jk}}\right], \\
a_+ &= \frac{1}{2}\sum_{j,k\in\mathbb{Z}}\frac{(y-k)dx + (j-x)dy}{r_{jk}(z+r_{jk})} + \frac{\pi}{2}(3y\,dx + x\,dy) \quad \text{for } z \geq 0, \\
a_- &= \frac{1}{2}\sum_{j,k\in\mathbb{Z}}\frac{(y-k)dx + (j-x)dy}{r_{jk}(z-r_{jk})} + \frac{\pi}{2}(y\,dx - x\,dy) \quad \text{for } z < 0.
\end{aligned}}$$

$$\mathbf{e}_{jk} = (j,k,0) \text{ and } e_{jk} = |\mathbf{e}_{jk}|, \quad r = |\vec{x}|, \quad r_{ij} = |\vec{x} - \vec{e}_{ij}|.$$

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- A typical abelian solution (Constant Energy Density solution)

$$\phi = 2\pi(Qz + M), \quad a = 2\pi(Qy\,dx - p\,dx - q\,dy)$$

- Dirac Monowall

$$\boxed{\begin{aligned} \phi &= \phi_0 + \pi z - \frac{1}{2r} + \frac{1}{2}\sum_{j,k\in\mathbb{Z}}\left[\frac{1}{e_{jk}} - \frac{1}{r_{jk}}\right], \\ a_+ &= \frac{1}{2}\sum_{j,k\in\mathbb{Z}}\frac{(y-k)dx + (j-x)dy}{r_{jk}(z+r_{jk})} + \frac{\pi}{2}(3y\,dx + x\,dy) \quad \text{for } z \geq 0, \\ a_- &= \frac{1}{2}\sum_{j,k\in\mathbb{Z}}\frac{(y-k)dx + (j-x)dy}{r_{jk}(z-r_{jk})} + \frac{\pi}{2}(y\,dx - x\,dy) \quad \text{for } z < 0. \end{aligned}}$$

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Our Asymptotic Conditions:

For a U(n) monopole wall, as $z \to \pm\infty$,

$$\mathrm{EigVal}\ \Phi = \{2\pi\mathrm{i}\,(Q_{\pm,l}z + M_{\pm,l}) + o(1/z)\,|\,l = 1,\ldots,n\}\,.$$

Simplest case is Maximal Symmetry Breaking: the pairs $(Q_{+,l}, M_{+,l})$ are all distinct, and so are $(Q_{-,l}, M_{-,l})$. It splits the bundle at infinity into line bundles.

Asymptotic holonomy eigenvalues are $e^{2\pi i p_{\pm,l}}$ around the x-direction and $e^{2\pi i q_{\pm,l}}$ around y.

$$E|_z = \oplus_{j=1}^{f_\pm} E_{\pm j}; \qquad \int_{T_z} c_1(E_{\pm j}) = \frac{\mathrm{i}}{2\pi}\int_{T_z} \operatorname{tr} F_{\pm j} = -\frac{\mathrm{i}}{2\pi}\int_{T_z} \operatorname{tr} * D\Phi_{\pm j} = \operatorname{rk}(E_{\pm j})\, Q_{\pm j}.$$

$$Q_{\pm j} = \frac{\alpha_{\pm j}}{\beta_{\pm j}},$$

$$\operatorname{rk}(E_{\pm j}) = r_{\pm j}\beta_{\pm j}.$$

We also allow prescribed positive and negative Dirac singularities at some points $\vec{\rho}_\alpha \in T^2 \times \mathbb{R}$ with the Higgs field behavior

$$\Phi = \mathrm{i}\begin{pmatrix} \frac{+1}{2|\mathbf{r}-\mathbf{r}_{+,\nu}|} & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} \end{pmatrix} + O(|\mathbf{r}-\mathbf{r}_{+,\nu}|),$$

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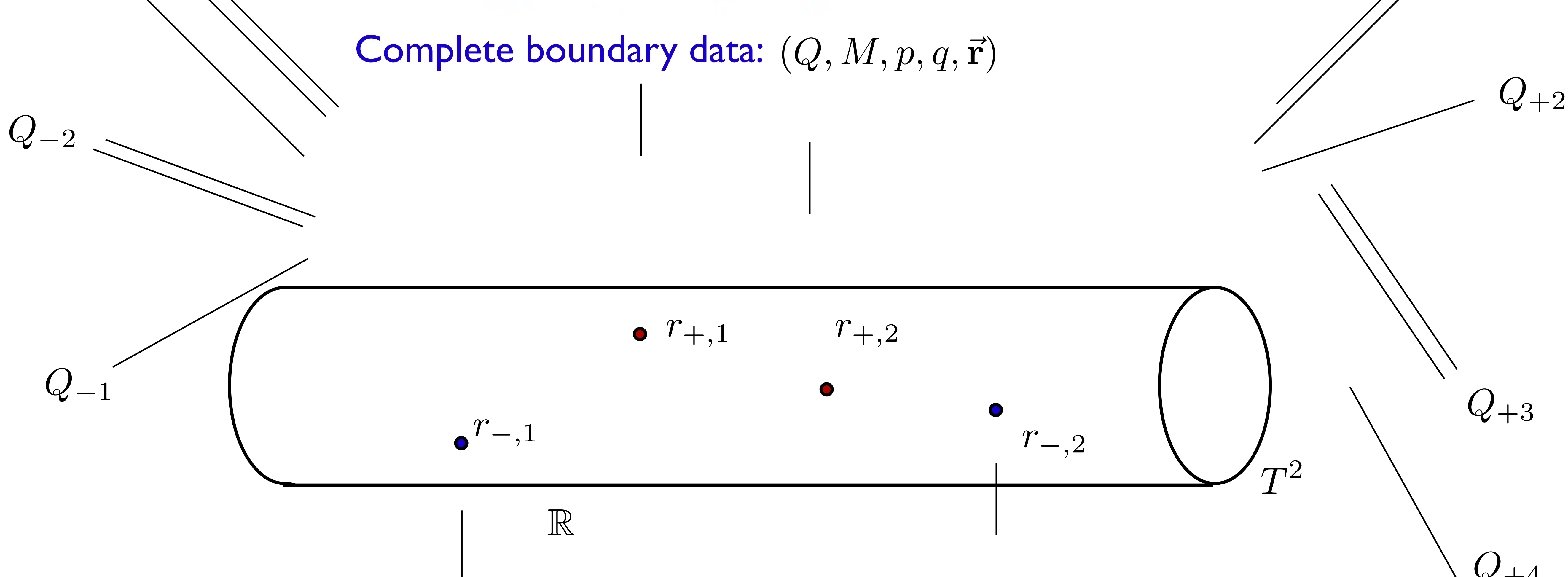
Complete boundary data: $(Q, M, p, q, \vec{\mathbf{r}})$



The charges $Q_{\pm,l}$ are rational, with the denominator equal to the multiplicity of $(Q_{\pm,l}, M_{\pm,l})$.

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Singularities:

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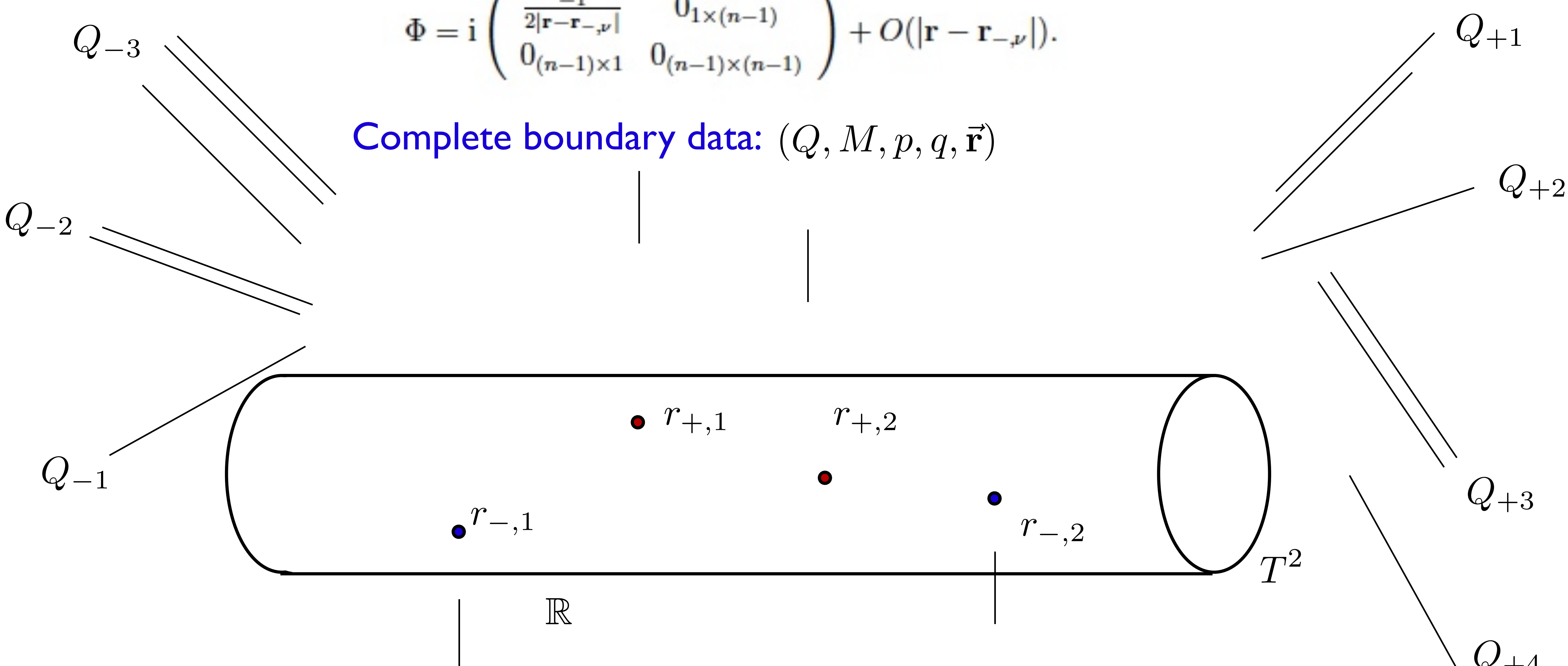
Complete boundary data: $(Q, M, p, q, \vec{\mathbf{r}})$



Moduli Problem

Once we fix
1) the asymptotic conditions $\{(Q_\pm, M_\pm, p_\pm, q_\pm)\}$ and
2) positions of positive and negative singularities $\vec{\rho}_+$ and $\vec{\rho}_-$,

We would like to explore the moduli spaces of solutions with these conditions.

In particular aim to identify
- the moduli of these solutions = coordinates on the moduli space and
- combinations of the asymptotic parameters that are parameters of the moduli space.

Demonstrate that the moduli space is ALH by computing its asymptotic.

So far there are two known examples of ALH spaces:
- D-type: $\widetilde{T^3 \times \mathbb{R}}/\mathbb{Z}_2$ and
- 1/2 K3.

with Marcos Jardim

Aside: D-type emerges as the moduli space of two centered SU(2) monopoles on T^3 with two positive and two negative singularities.

Questions:
1. What monopole wall has D-type ALH as their moduli space?
2. Is there a transform from such monopole wall to a monopole on T^3?
3. What is are the topologies of the other moduli spaces.

Spectral Description

Bogomolny equation $*D_A\Phi = -F$ can be written in the form

$$\begin{cases} \left[D_z - iD_y, D_x + i\Phi\right] = 0 & (1) \\ \left[D_z - iD_y, (D_z - iD_y)^\dagger\right] + \left[D_x + i\Phi, (D_x + i\Phi)^\dagger\right] = 0 & (2) \end{cases}$$

Eq. (1) implies that the holonomy $V_x(y,z)$ of $D_x + i\Phi$ is meromorphic in $s = e^{2\pi(z-iy)} \in \mathbb{C}^*$

$$F_x(s,t) = \det\left(V_x(y,z) - t\right)$$

is a degree n polynomial in t with coefficients being rational functions in s.

Spectral curve:

$$\Sigma_x = \{(s,t)|F_x(s,t) = 0\} \in \mathbb{C}^* \times \mathbb{C}^*$$

formed by the eigenvalues of the holonomy and equipped with
a holomorphic eigen line bundle $M_x \to \Sigma_x$

(Σ_x, M_x) form complete x-spectral data equivalent to the (A, Φ) solution.

Note: Analogously, we define the y-spectral data (Σ_y, M_y) .

There is a 1-to-1 map $(\Sigma_x, M_x) \to (\Sigma_y, M_y)$.

Let $G_x(s,t)=P(s)F_x(s,t)$ be a minimal monic polynomial in s and t, so that the spectral curve is given by a polynomial equation $G_x(s,t)=0$.

Newton Polygon

Monopole Wall Moduli = Moduli of Σ_x + Moduli of M_x

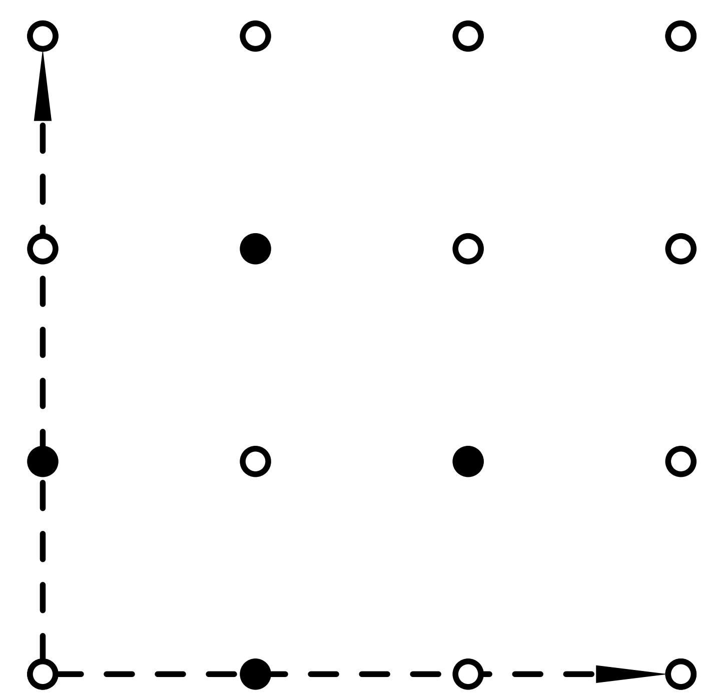
Newton polygon N_x of $G_x(s,t)$ is a minimal convex polygon containing all points (a,b) such that the monomial $s^a t^b$ is present in $G_x(s,t)$.

Example: U(2) monopole wall with charges $(Q_-, Q_+) = \begin{pmatrix} 1 & & 1 \\ & , & \\ -1 & & -1 \end{pmatrix}$

Reducible

$$\boxed{\begin{aligned} &\Phi = 2\pi i \sigma_3, \\ &A = \pi i (y dx - x dy) \sigma_3 \end{aligned}}$$

$G_x(s,t)=st^2-s^2t-t-s$



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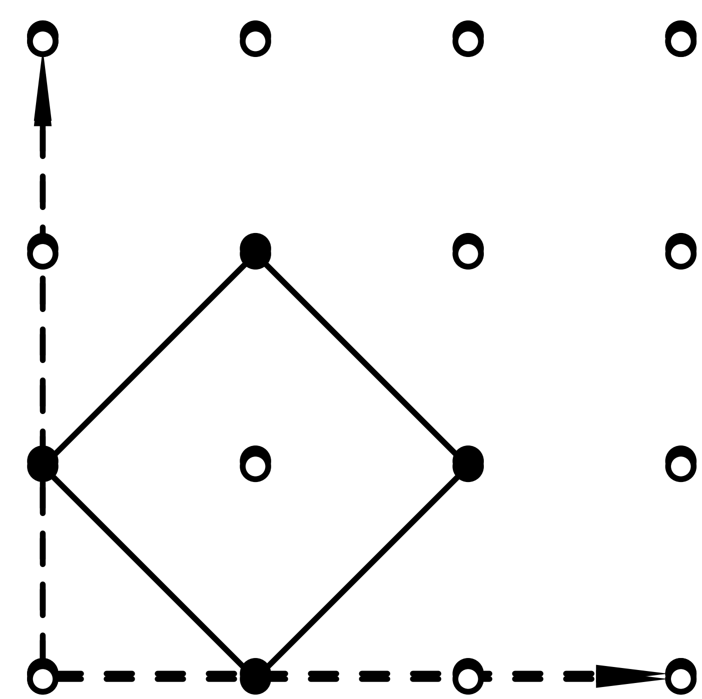
Reducible

$$\boxed{\begin{aligned} \Phi &= 2\pi i \sigma_3, \\ A &= \pi i (ydx - xdy)\sigma_3 \end{aligned}}$$

$G_x(s,t) = st^2 - s^2t - t - s$



Newton Polygon

Monopole Wall Moduli = Moduli of Σ_x + Moduli of M_x

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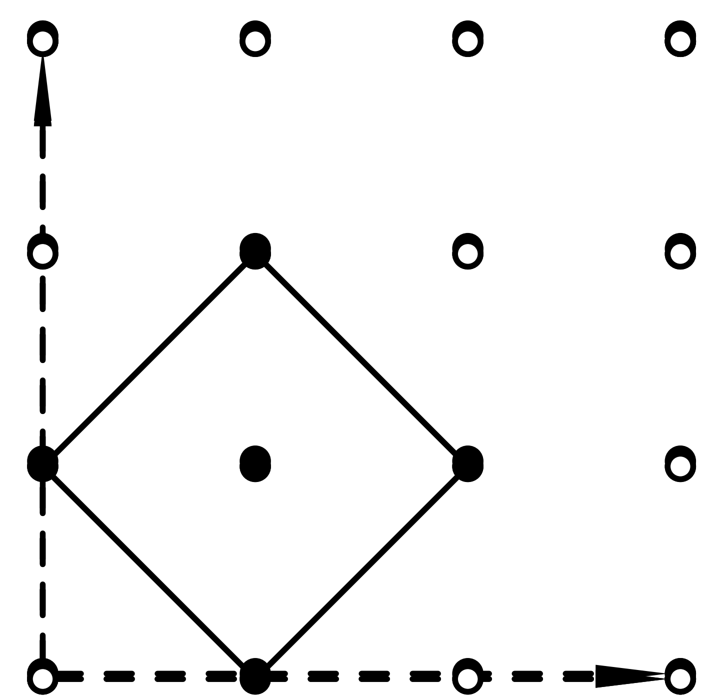

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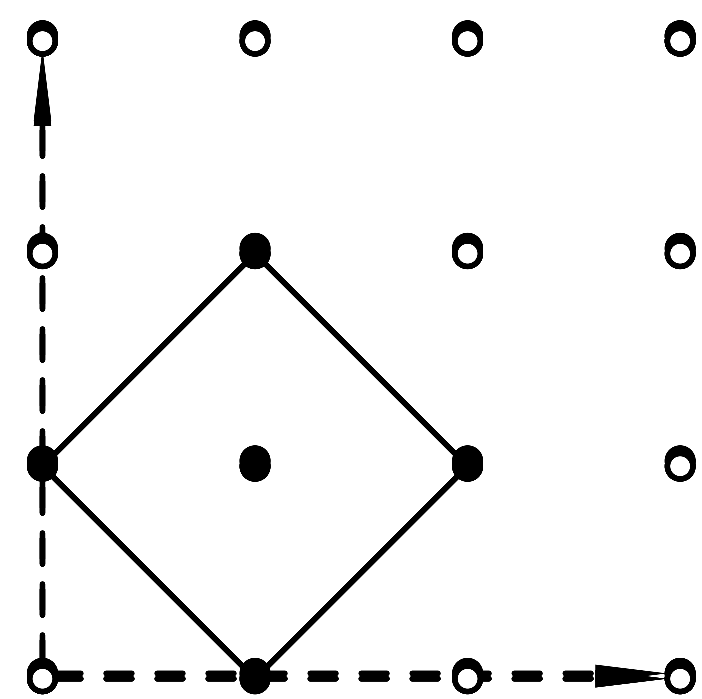
General



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General

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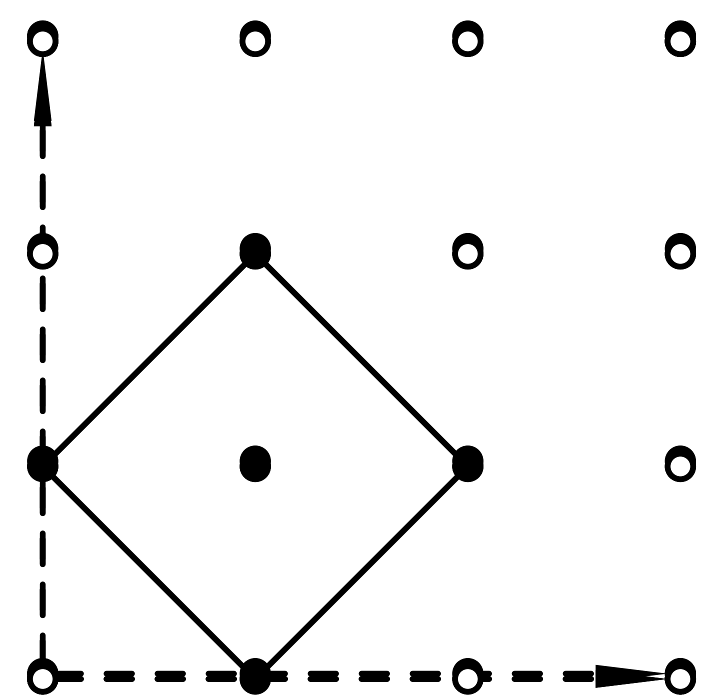
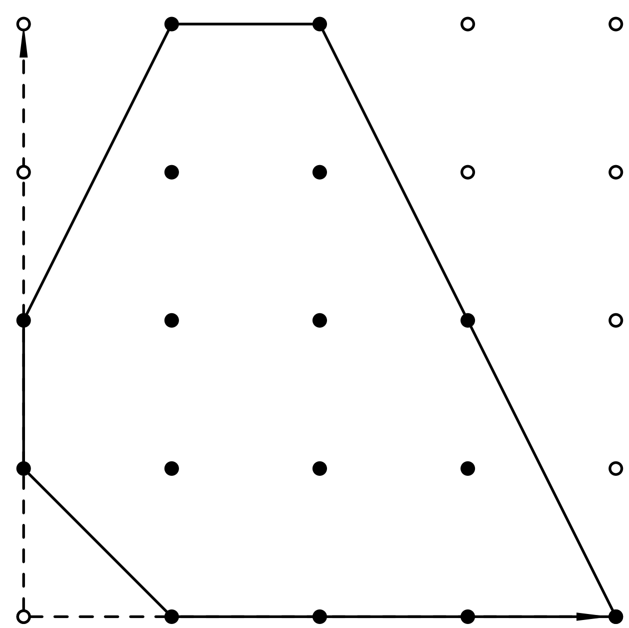
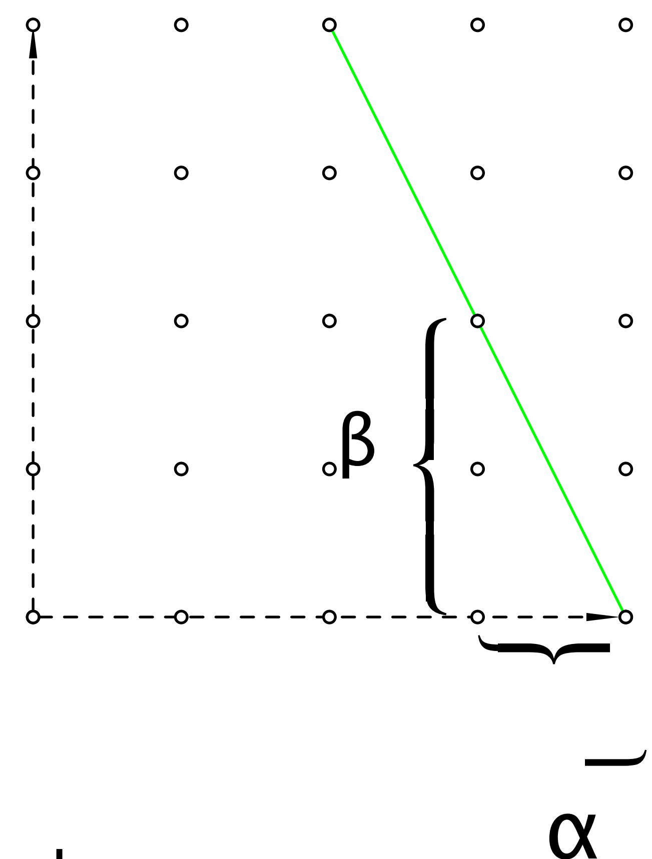


- Horizontal edges of Newton polygon correspond to the singularities: Northern edges to positive; and Southern edges, to negative.
- Vertical edges - constant eigenvalues of Φ as $z=\pm\infty$.
- Rank of the gauge group is the hight of the Newton polygon.





Consider one edge of N_x

a single edge of N_x directed along (α,β) produces
asymptotic satisfying $\alpha \log s+\beta \log t= 2\pi\beta (M+i\, p)$ => Charge $Q=\alpha/\beta$.

$m_l^- = \exp\left[-2\pi\beta_{\pm,l}(M_{\pm,l}+ip_{\pm,l})\right]$ are roots of the edge polynomial.

Dressed Newton polygon N => Monowall Boundary data

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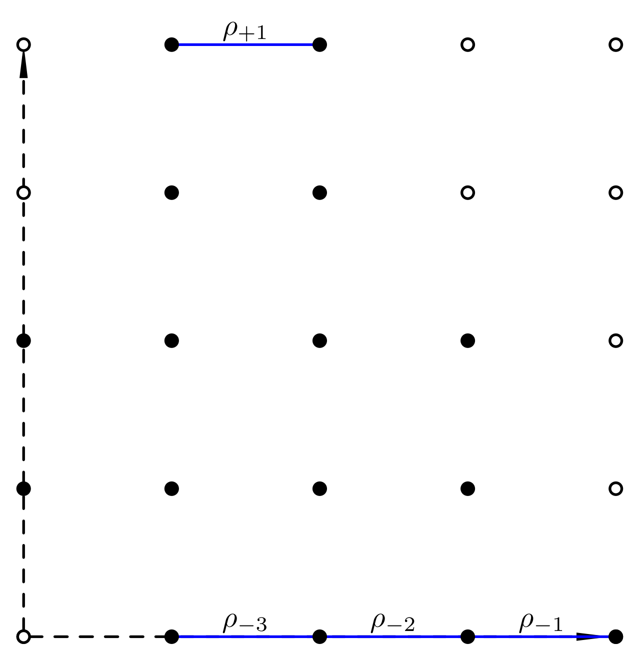

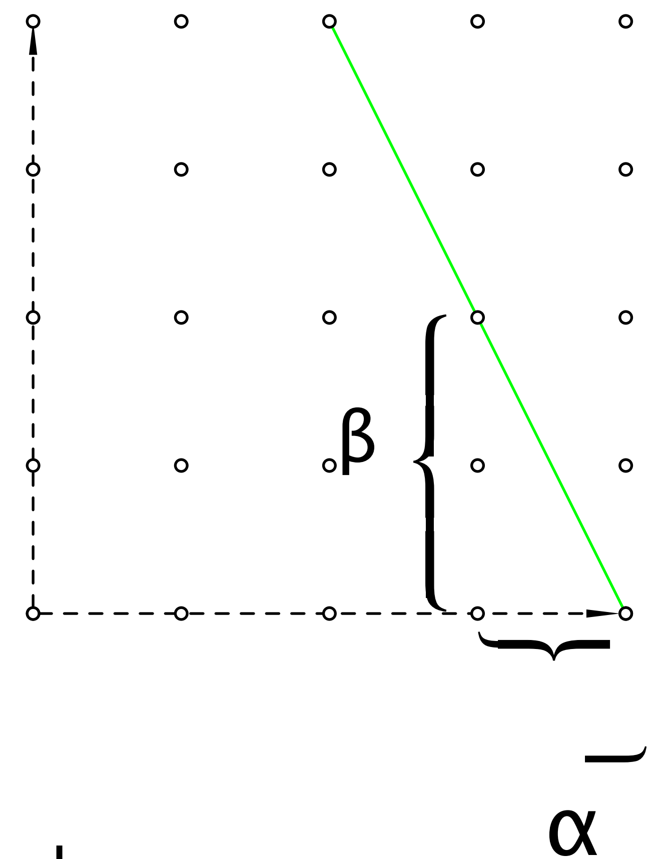

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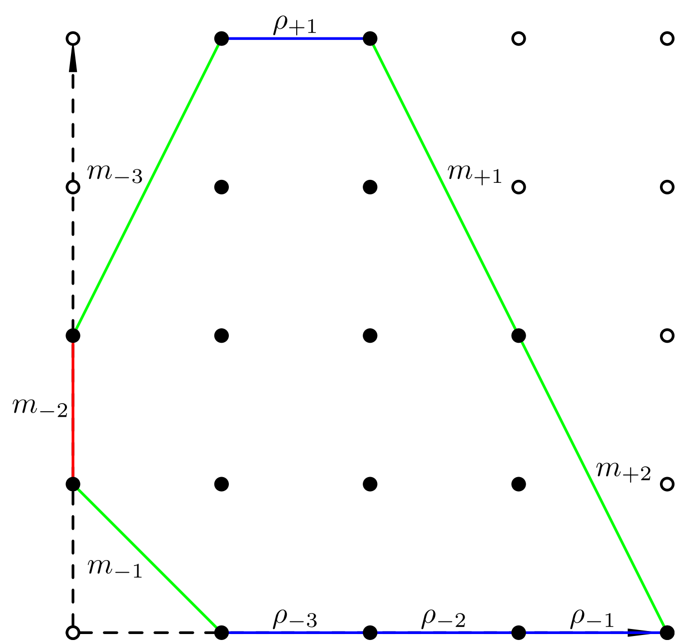

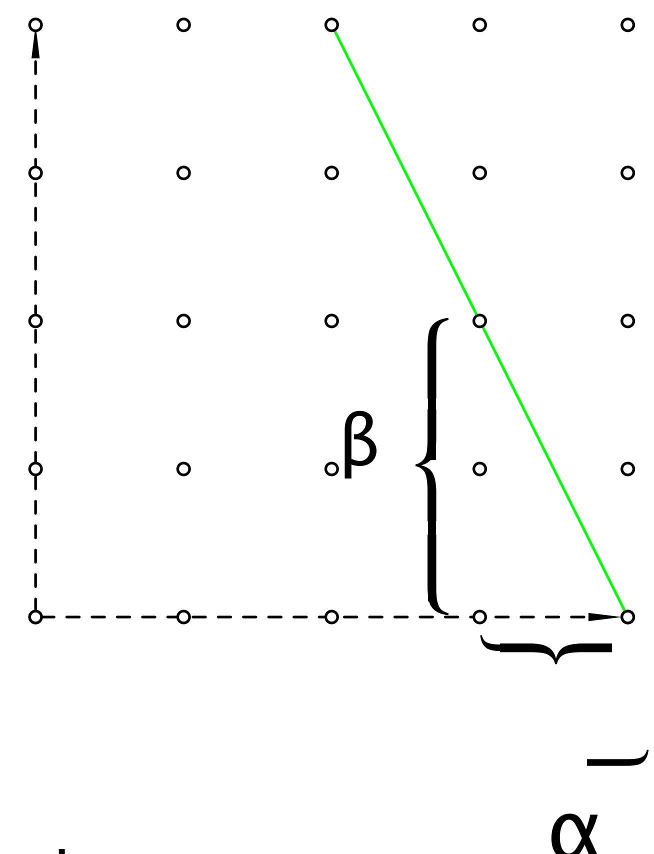

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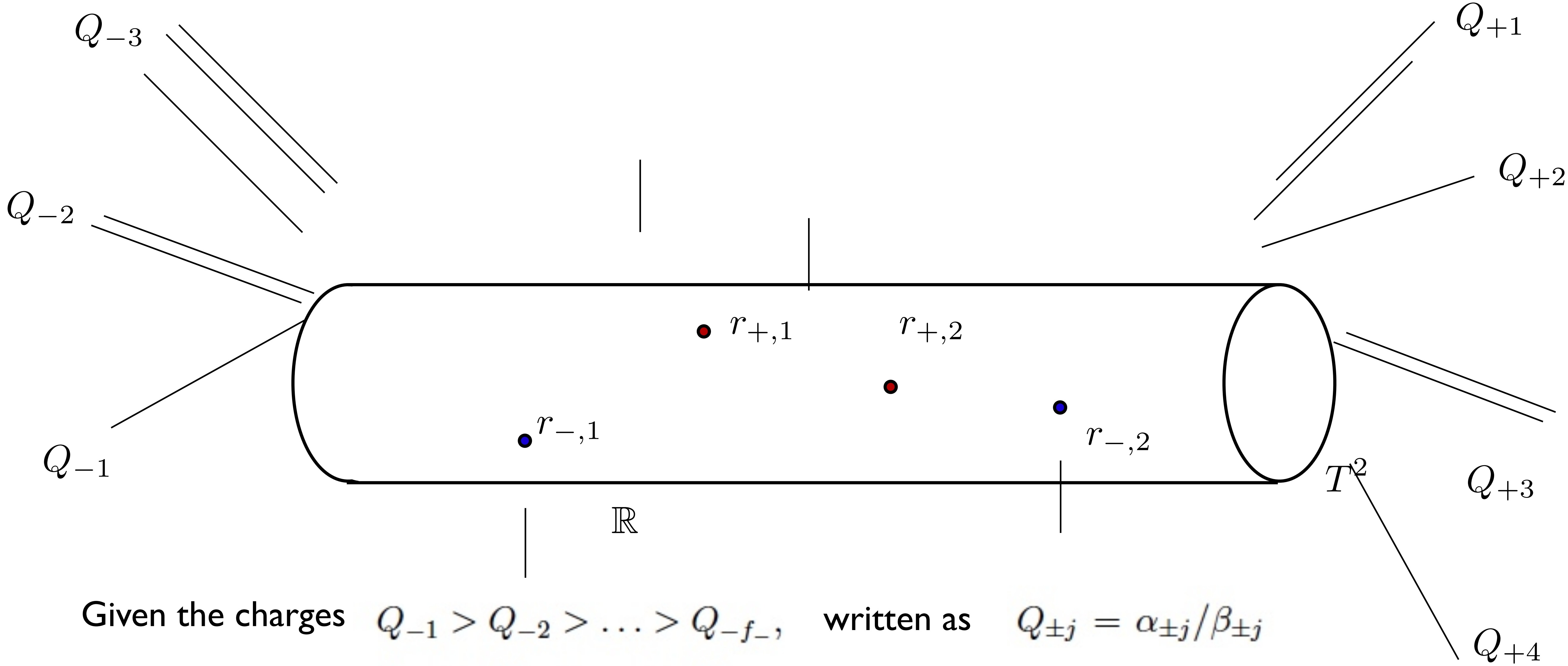


Given the charges $Q_{-1} > Q_{-2} > \ldots > Q_{-f_-}$, written as $Q_{\pm j} = \alpha_{\pm j}/\beta_{\pm j}$

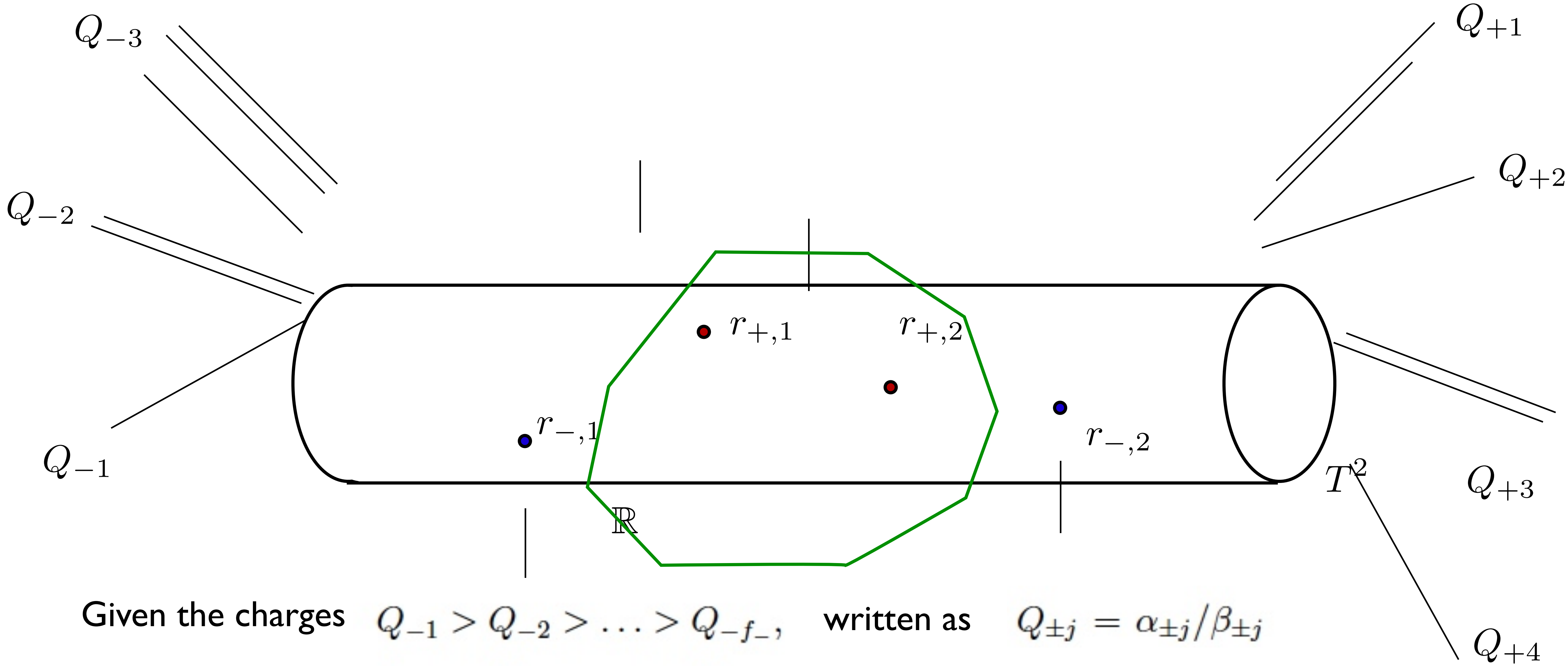
$Q_{+1} > Q_{+2} > \ldots > Q_{+f_+}$. of multiplicities $r_{\pm j}\beta_{\pm j}$

form orthogonal vectors

$$e_{-0} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad e_{-j} = \begin{pmatrix} -\alpha_{-j} \\ \beta_{-j} \end{pmatrix}, \qquad e_{+0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_{+j} = \begin{pmatrix} \alpha_{+j} \\ -\beta_{+j} \end{pmatrix}.$$

then the Newton polygon edges are

$$r_{-0}e_{-0}, r_{-1}e_{-1}, \ldots, r_{-f_-}e_{-f_-}, r_{+0}e_{+0}, r_{+1}e_{+1}, \ldots, r_{+f_+}e_{+f_+}$$



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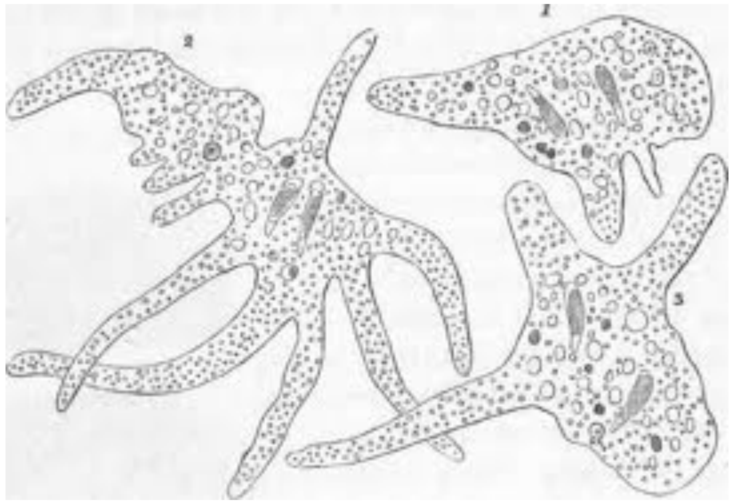
Amoeba

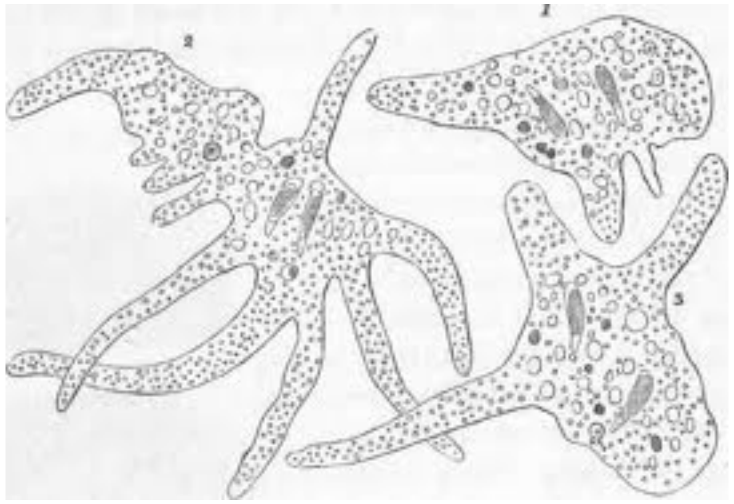


An amoeba A_x of $G_x(s,t)$ is a the image of Σ_x under the map

$$\begin{aligned}\mathbb{C}^* \times \mathbb{C}^* &\to \mathbb{R}^2\\ (s,t) &\mapsto (\log|s|, \log|t|).\end{aligned}$$

- $\mathrm{Area}(A_x) \leq \pi^2\, \mathrm{Area}(N_X)$.

Amoeba



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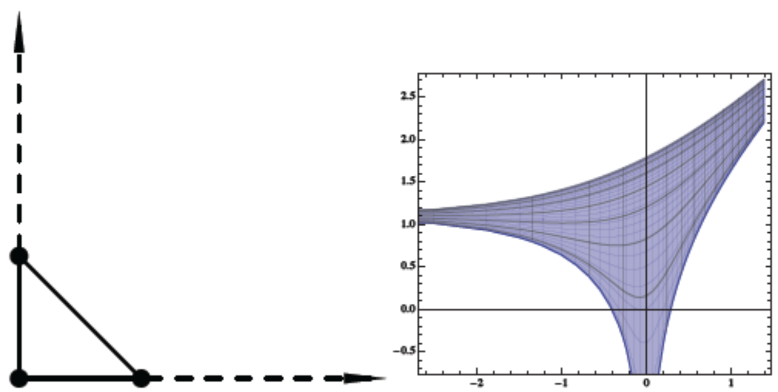
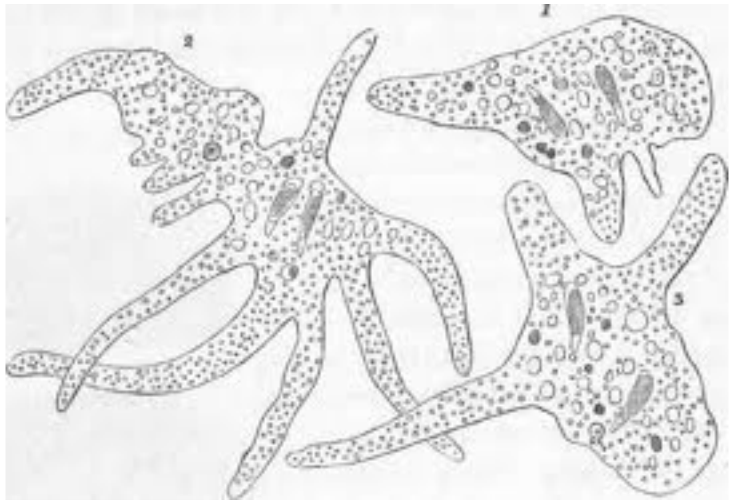


(0,1) U(1) monopole wall with one negative singularity.

- Area(A_x) ≤ π^2 Area(N_X).

Amoeba



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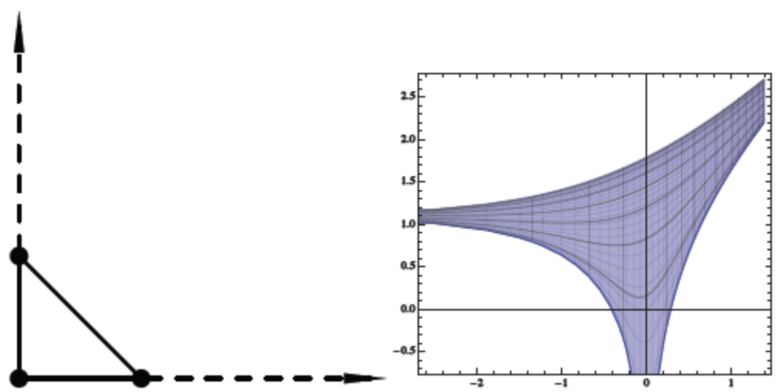

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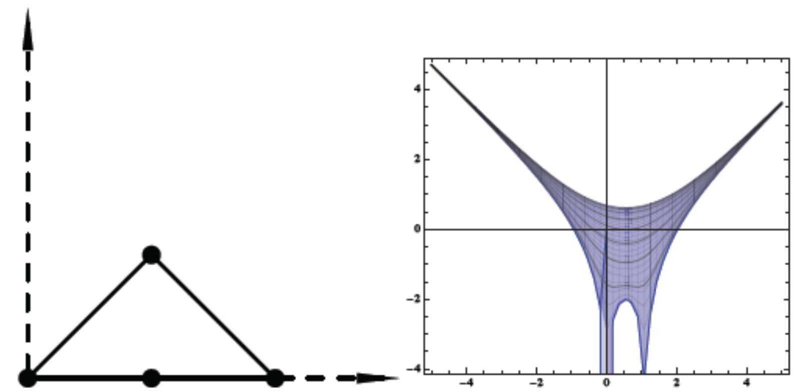
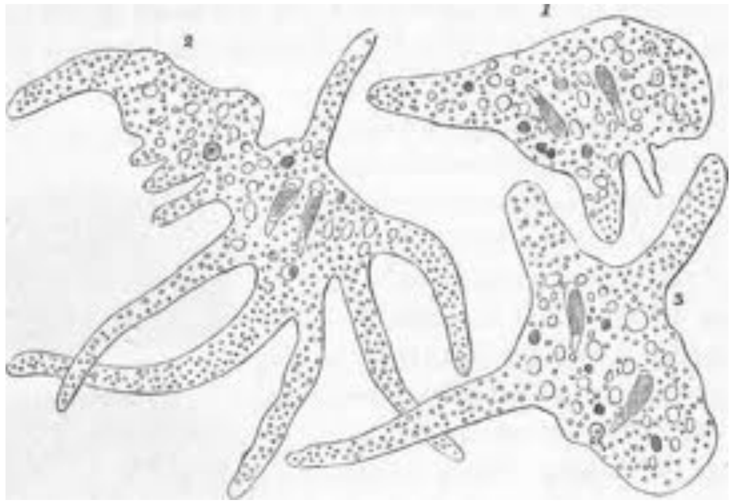


(1,1) U(1) monopole wall with two negative singularities.

- Area(A_x) $\leq \pi^2$ Area(N_X).

Amoeba



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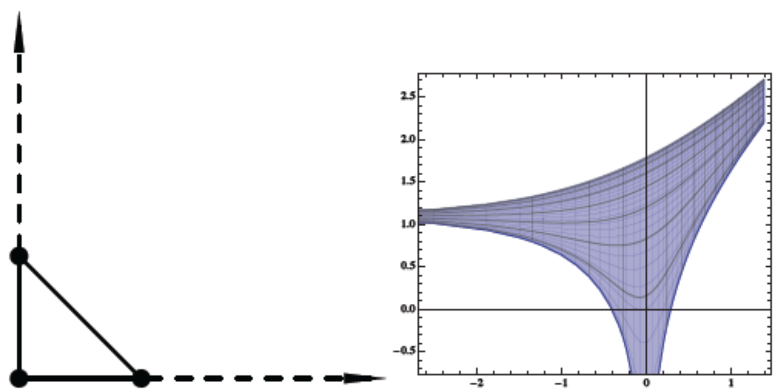

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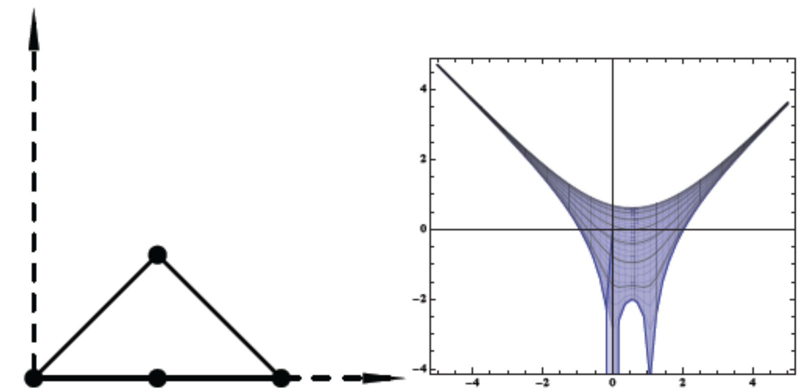

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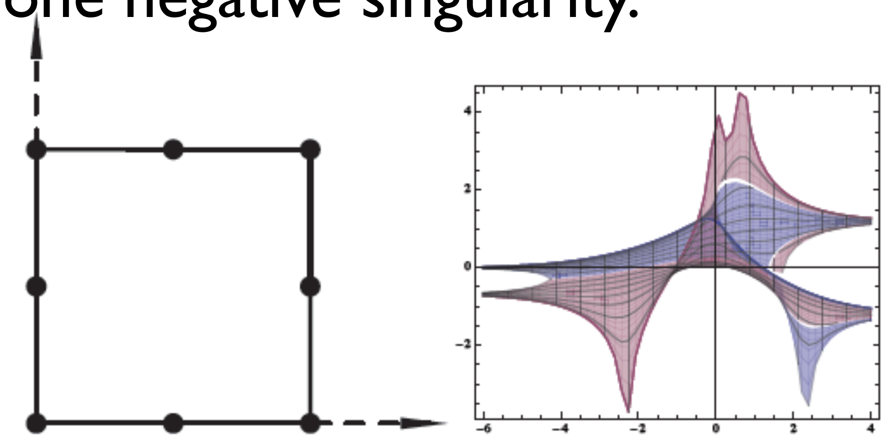
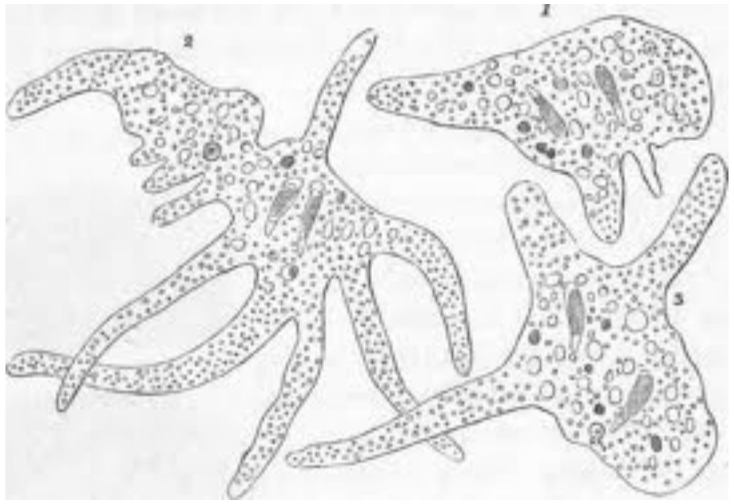


Balanced U(2) monopole with 2 positive and two negative singularities.

- $\mathrm{Area}(A_x) \leq \pi^2\, \mathrm{Area}(N_X)$.

Amoeba



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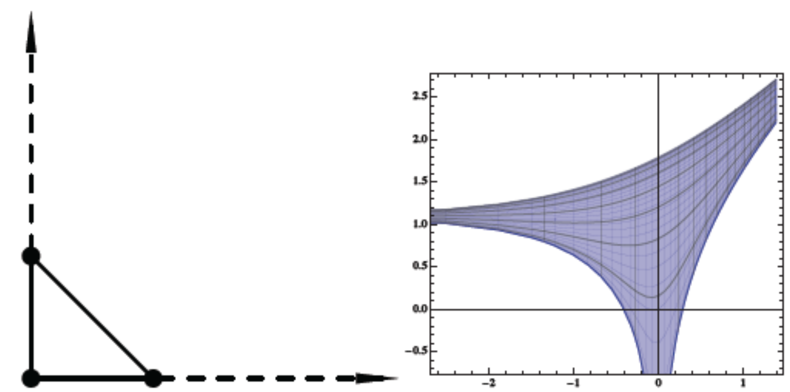

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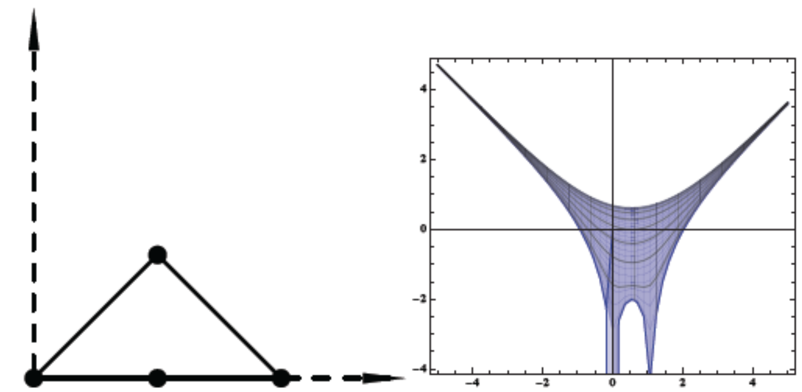

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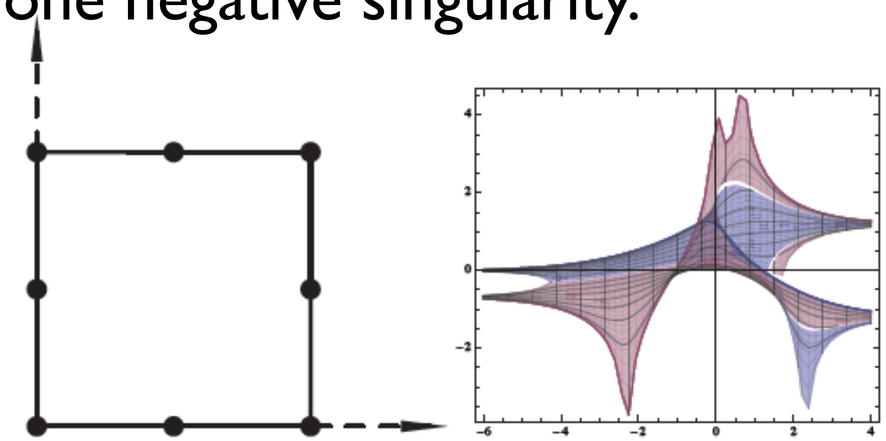

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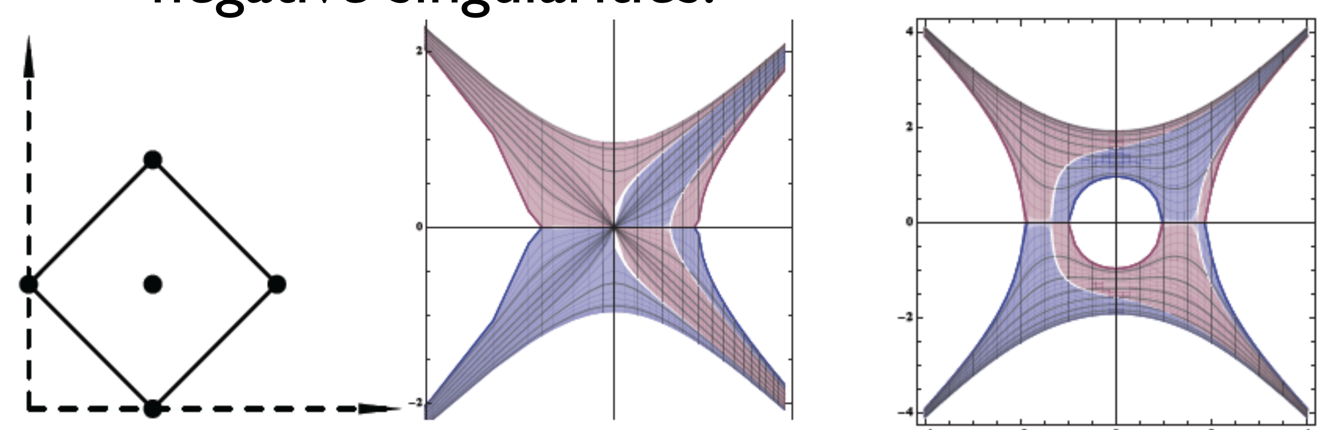

Reducible U(2) monopole with charges (Q_+; Q_-)=(1,-1; 1,-1)

- $\mathrm{Area}(A_x) \leq \pi^2\, \mathrm{Area}(N_X)$.

- Amoeba's tentacles are orthogonal to the edges of the Newton polygon.
- Slope of a tentacle is the charge Q and its position is M.
- Vertical tentacles are positioned at the z-positions ρ_α of the Dirac singularities.
- Tentacle multiplicity is the multiplicity of the corresponding (Q,M) pair, which is the hight of the corresponding edge.

Statement: Introducing or changing terms in the interior of N_x does not change the asymptotic, tentacles, of its amoeba.

Asymptotic and singularity data => Amoeba tentacles => Newton Polygons N_x and N_y.
(charges and number of singularities)

Since tentacle slopes are determined by the charges and the singularities
N_x=N_y !

Moduli Count:

Number of Moduli = 4 x (Number of internal points of N_x).

Number of internal points is given by the Pick's relation

$$\mathrm{Int}\, N = A(N) - \frac{p}{2} + 1.$$

Parameter Count

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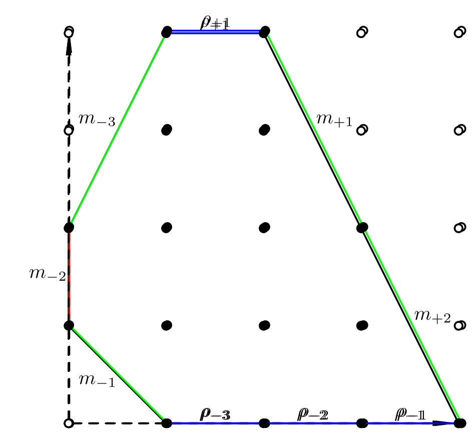
$$m_l^- = \exp\left[-2\pi\beta_{\pm,l}(M_{\pm,l} + ip_{\pm,l})\right]$$



Parameter Count:

Number of Parameters = 3 x (Number of perimeter points of N_x - 1).

Charges Q=α/β satisfy:

$$\sum_{j=1}^{f_-} r_{-j}\beta_{-j} = \sum_{j=1}^{f_+} r_{+j}\beta_{+j} = n, \qquad r_{-0} + \sum_{j=1}^{f_-} r_{-j}\alpha_{-j} = r_{+0} + \sum_{j=1}^{f_+} r_{+j}\alpha_{+j},$$

Singularities and constant terms satisfy:
(Vietta theorem)

$$\sum_{\nu=1}^{r_+} z_{+,\nu} - \sum_{\nu=1}^{r_-} z_{-,\nu} = \sum_{l=1}^{n} M_{+,l} - \sum_{l=1}^{n} M_{-,l},$$

$$\sum_{\pm}\sum_{\nu=1}^{r_\pm} \pm y_{\pm,\nu} + \sum_{\pm}\sum_{l=1}^{n} \pm p_{\pm,l} + \frac{1}{2}\sum_{\substack{l_1,l_2=1 \\ l_1 < l_2}}^{2n} (Q_{,l_1} - Q_{,l_2}) \in \mathbb{Z},$$

$$\sum_{\pm}\sum_{\nu=1}^{r_\pm} \pm x_{\pm,\nu} + \sum_{\pm}\sum_{l=1}^{n} \pm q_{\pm,l} + \frac{1}{2}\sum_{\substack{l_1,l_2=1 \\ l_1 < l_2}}^{2n} (Q_{,l_1} - Q_{,l_2}) \in \mathbb{Z}.$$

SL(2,Z) Isometric Action

There is a natural SL(2,Z) action on C*xC*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (s,t) \mapsto (s^d t^c, s^b t^a),$$

it induces a map on the spectral data (Σ,M), under which

a monomial $s^{-\alpha} t^{\beta} \mapsto (s')^{-\alpha'} (t')^{\beta'}$ with $\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$

and the resulting Newton polygon is $N' = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} N$

and the boundary data transforms as

$$g : (Q, M, p, q) \mapsto \left(\frac{aQ+b}{cQ+d}, \frac{M}{cQ+d}, \frac{p}{cQ+d}, \frac{q}{cQ+d} \right).$$

Nahm Transform

maps a monopole wall to another monopole wall keeping the spectral curve fixed.

$$G_x(s,t)=P(s,t) \quad \longleftrightarrow \quad \check{G}_x(s,t)=c\,P(t,s)$$

The Nahm transform is the S element of SL(2,Z).

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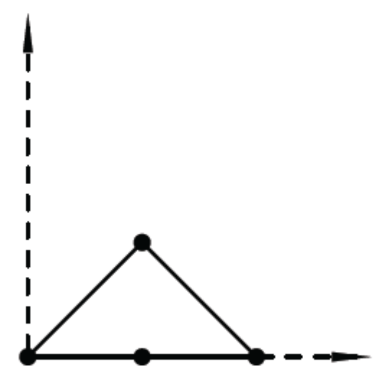

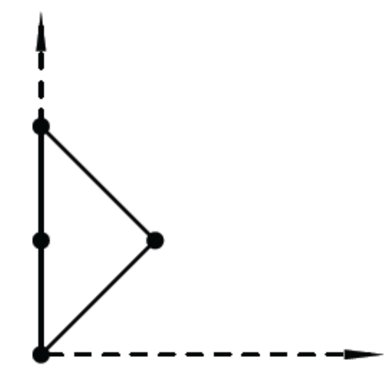

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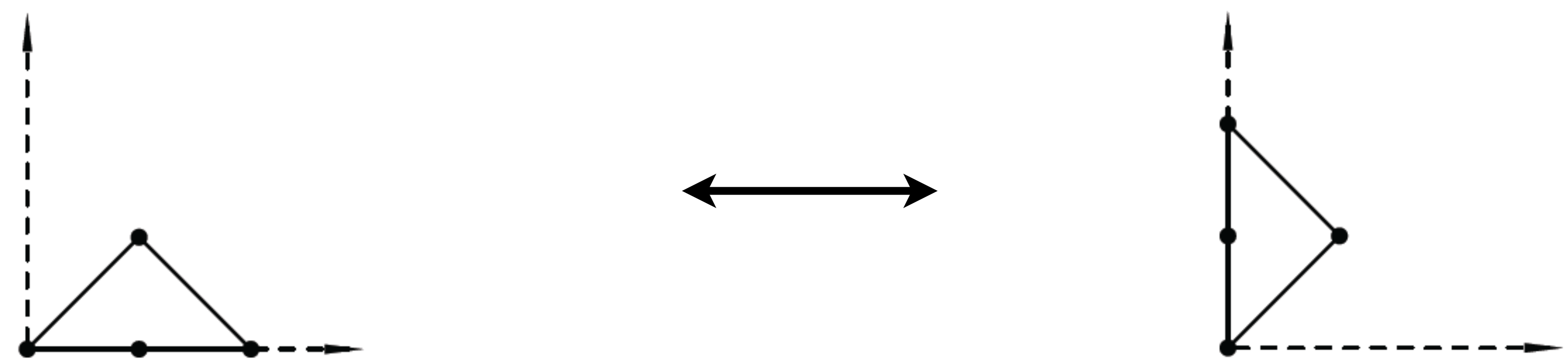

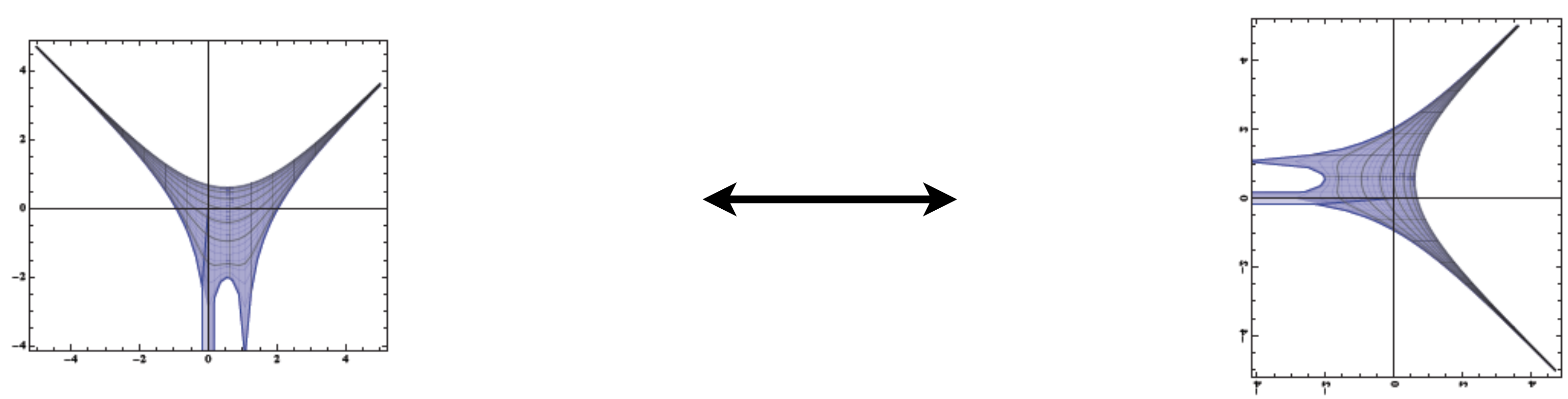

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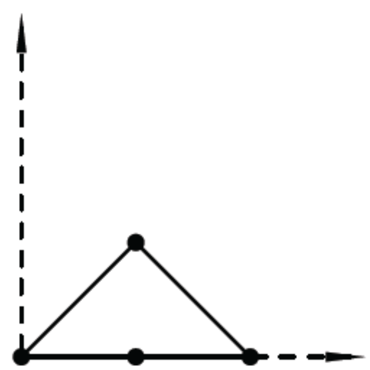

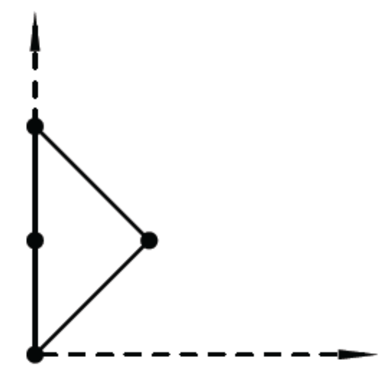

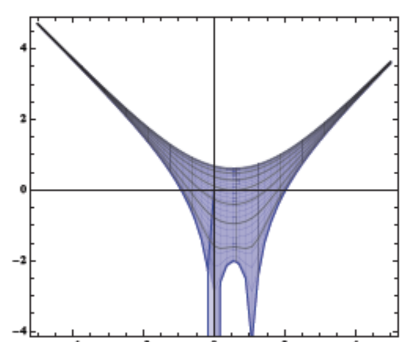

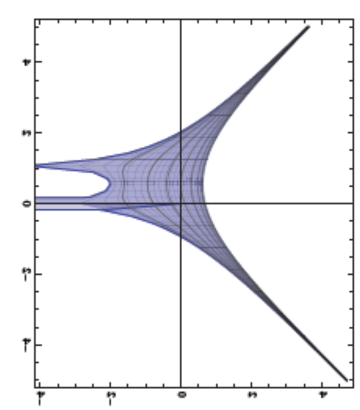

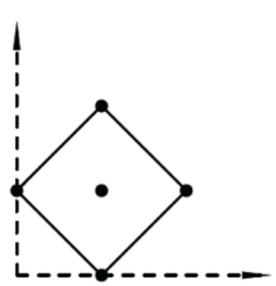

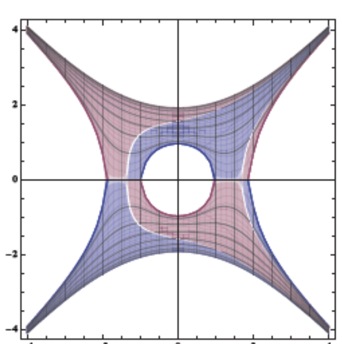

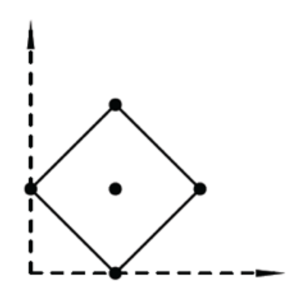

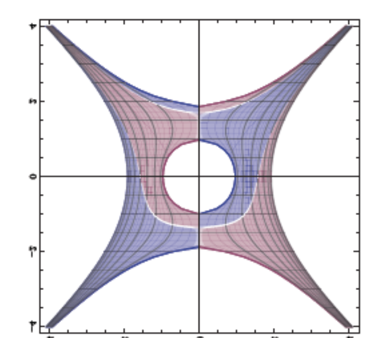

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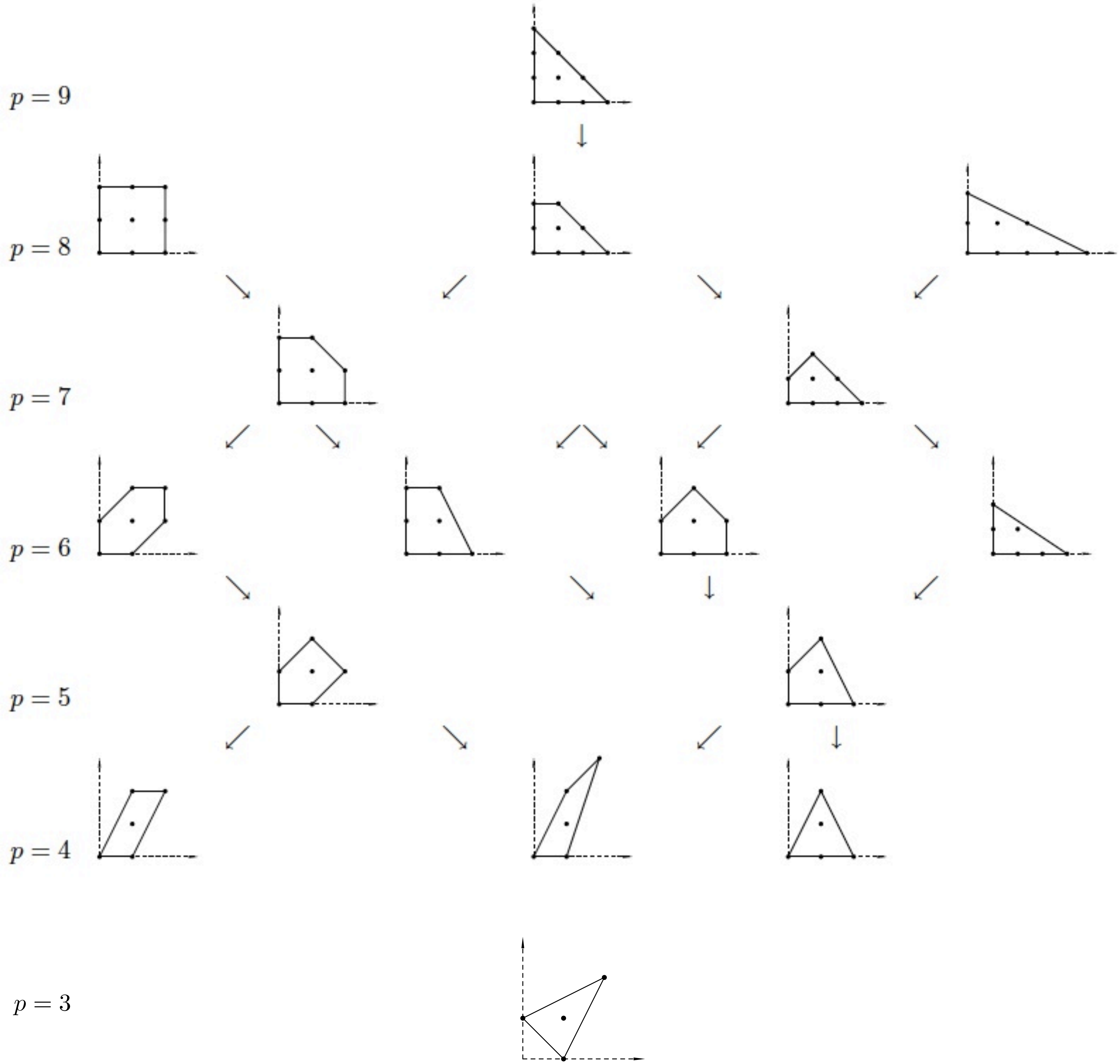
Monowalls with No Moduli (up to SL(2,Z))

based on Khovanskii ’97

- constant energy density monowall,
- Abelian monowall.

Monowalls with Four Moduli (upto SL(2,Z))



Yet another moduli space equivalence

For any given abelian monowall (a,ϕ) with the spectral curve

$$t=P(s)/Q(s),$$

adding this solution to a monowall does not change its moduli space,
So
(A,Φ) has the same moduli space as $(A+a,\Phi+\phi)$.

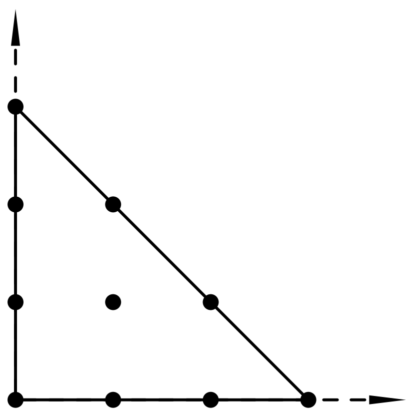
If the spectral curve of (A,Φ) is given by $G(s,t)=0$,
then the spectral curve of $(A+a,\Phi+\phi)$ is given by $G(s,tQ(s)/P(s))=0$

This equivalence reduces the list of monowall moduli spaces to a much shorter list.

Four-dimensional Monowall Moduli Spaces

Number of parameters

8



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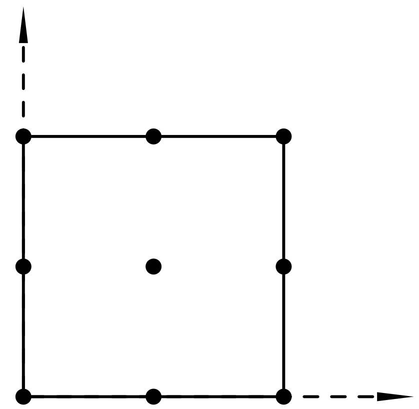

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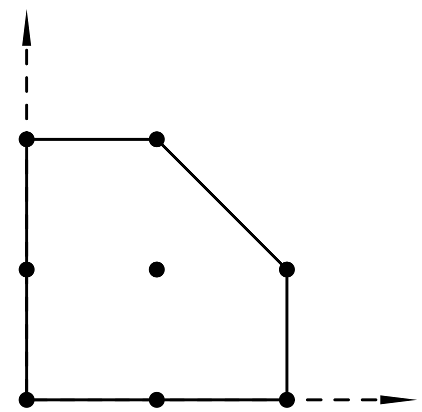

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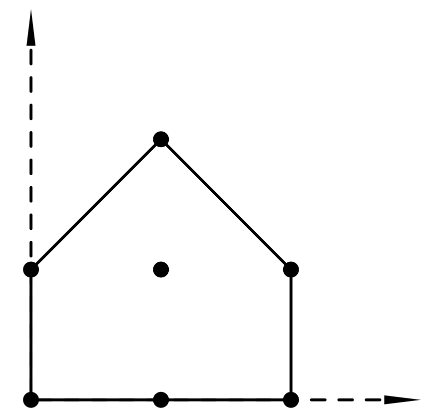

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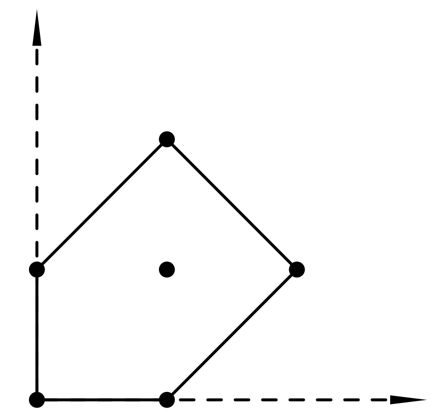

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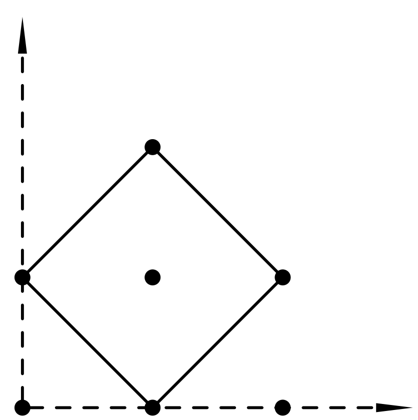

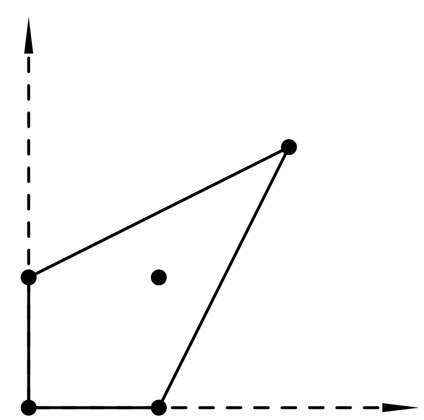

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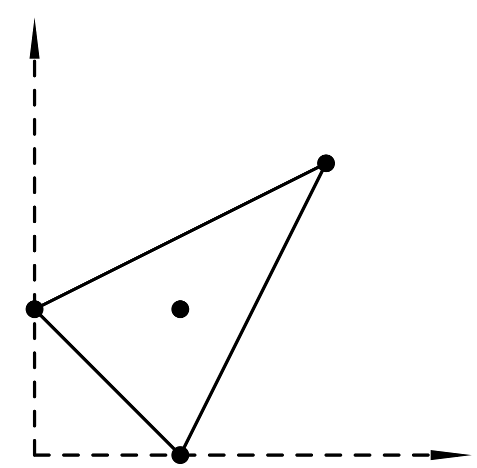

Four-dimensional Monowall Moduli Spaces

Number of parameters

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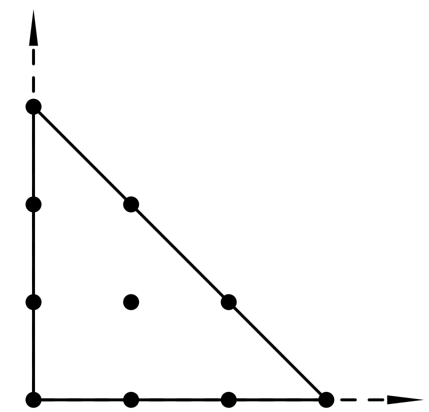

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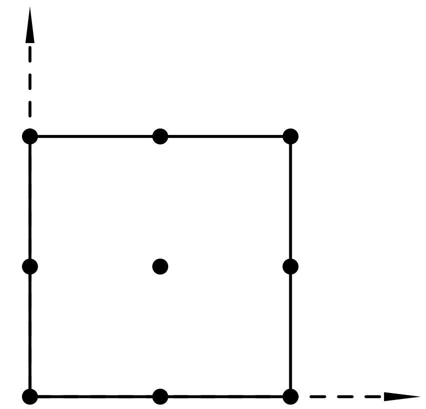

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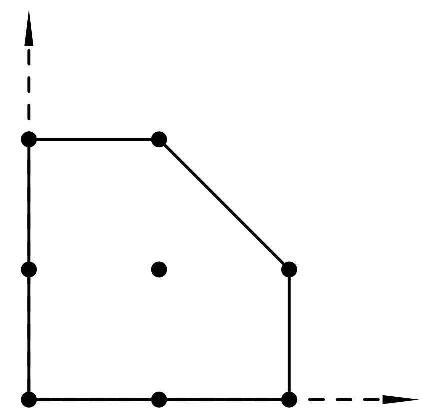

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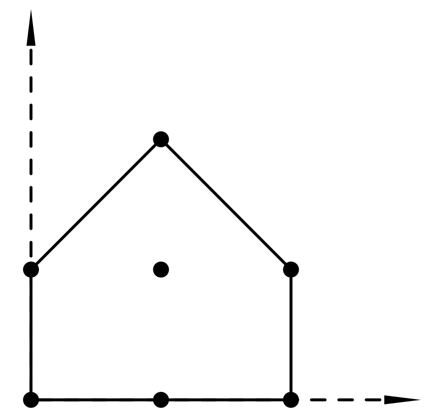

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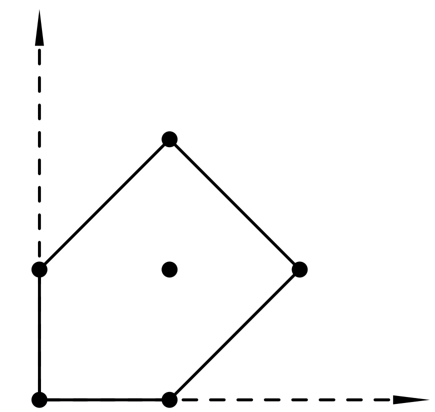

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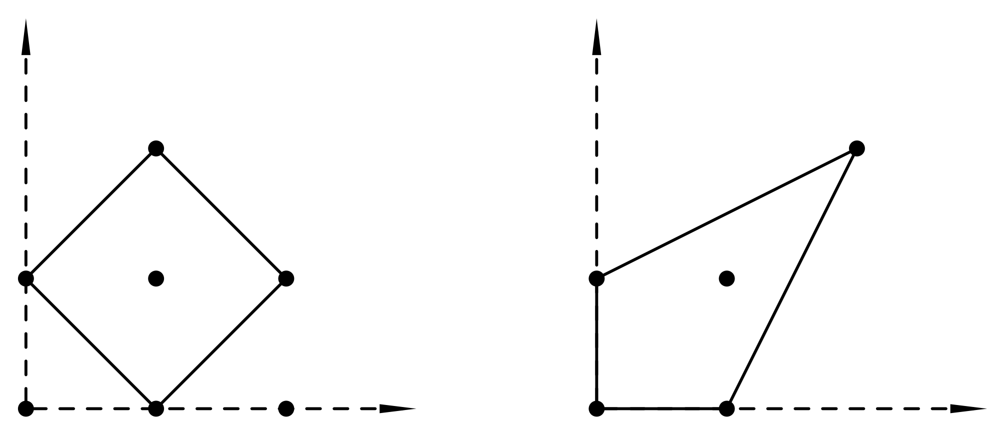

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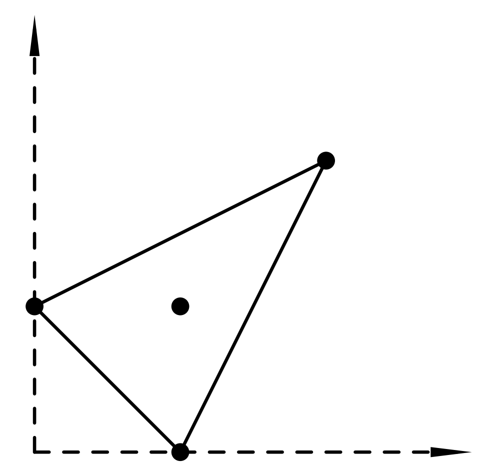
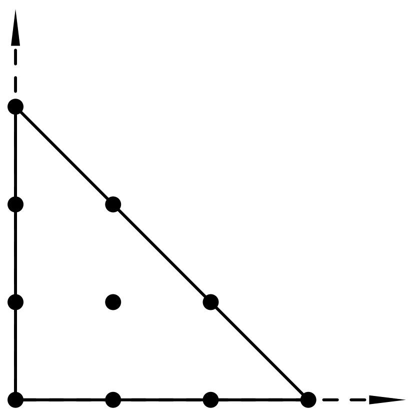


Are these two spaces isometric?

Four-dimensional Monowall Moduli Spaces

Number of parameters

8



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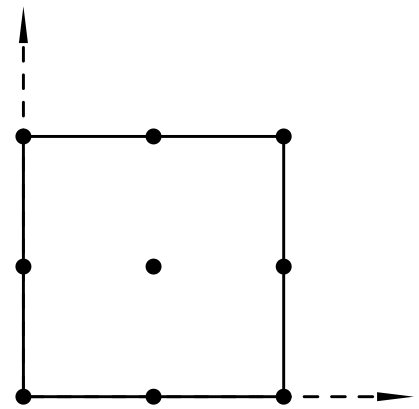

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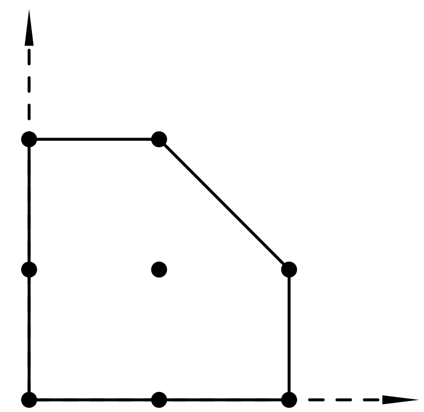

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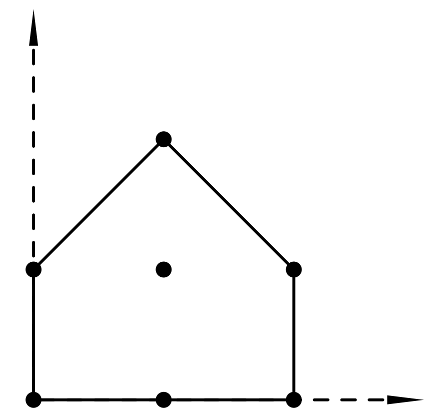

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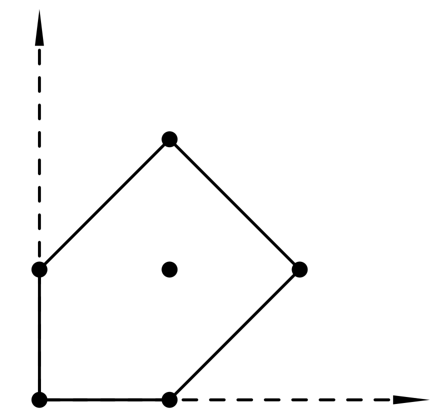
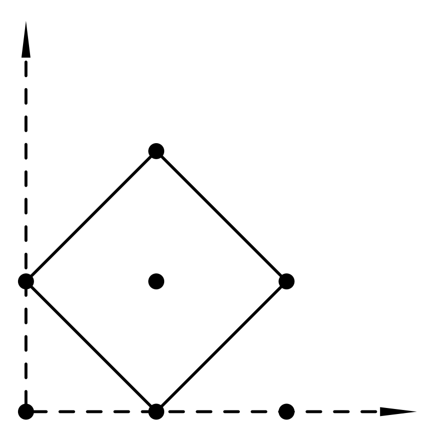
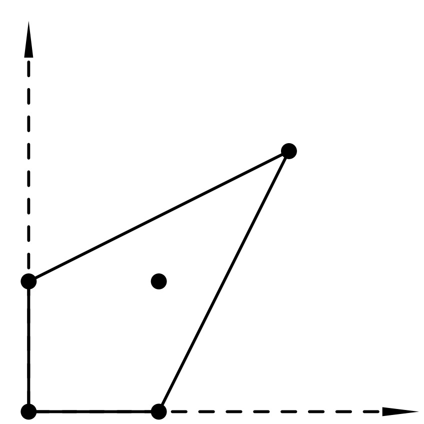


Is this $T^3 x R/Z_2$?

Moduli space of T^*S^3/Z_3 ?

3





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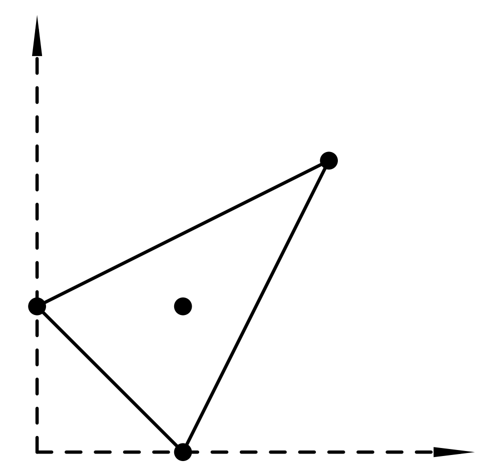


Are these two spaces isometric?

Other Subjects

- Correspondence with toric Calabi-Yau

- Tropical Geometry

- Viro's Patchworking and Monowall Concatenation

- Quantum gauge theory application:
  Monowall moduli spaces are Coulomb branches of 'five-dimensional quantum gauge theories' on T^2

Deformation Theory

Atiyah, Hitchin, Singer '78

Kuranishi complex:

If both (A, Φ) and $(A + a, \Phi + \phi)$ satisfy the Bogomolny Eq. $*D\Phi = -F$, then

$$D_A a + *(D_A \phi - [\Phi, a]) + a \wedge a - *[\phi, a] = 0$$

To simplify notation a bit let $\mathbf{A} = (A, \Phi)$ and $\mathbf{a} = (a, \phi)$

Denote the linearized operator by $\delta_1 \mathbf{a} = \delta_1(a, \phi) = D_A a + *(D_A \phi - [\Phi, a])$

and let $\{\mathbf{a}, \mathbf{a}\} = a \wedge a - *[\phi, a]$

Then the space of linear deformations is given by the middle cohomology of the complex

$$0 \longrightarrow \Lambda^0 \xrightarrow{\delta_0} \Lambda^1 \xrightarrow{\delta_1} \Lambda^2 \longrightarrow 0$$

$$\mathrm{Lin} = \{\mathbf{a} \in \Lambda^1 | \, \delta_1 \mathbf{a} = 0 \text{ and } \delta_0^* \mathbf{a} = 0\}$$

Linearization of Bogomolny Eq.

Gauge fixing condition

On the other hand, the space of all solutions is

$$\mathrm{Sol} = \{\mathbf{a} \in \Lambda^1 | \, \delta_1 \mathbf{a} + \{\mathbf{a}, \mathbf{a}\} = 0 \text{ and } \delta_0^* \mathbf{a} = 0\}$$

Kuranishi map:

The goal is to construct a map $F : \mathrm{Sol} \rightarrow \mathrm{Lin}$ that is invertible (near the origin of Sol) in case of generic $\mathbf{A}$.

Kuranishi map is given by : $F(a) = \mathbf{a} + \delta_1^* G\{\mathbf{a}, \mathbf{a}\}$

where G is the Green's function of the covariant Laplacian $\Delta = D_A^2 + [\Phi, [\Phi, \cdot]]$.

If the vanishing theorem holds and $h^1 = h^3 = 0$ for the linear complex

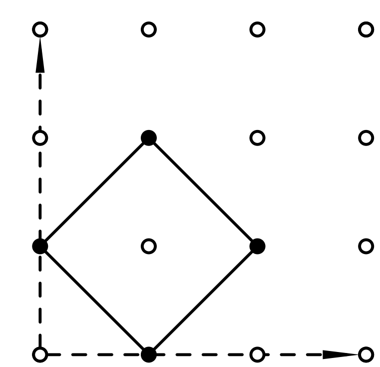
$$0 \rightarrow \Lambda^0 \xrightarrow{\delta_0} \Lambda^1 \xrightarrow{\delta_1} \Lambda^2 \rightarrow 0$$

then F is 1-to-1 and h^1 gives the correct number of moduli.

For a reducible solution (A, Φ) this is not the case.
We overcount by 4x(dimension of Stab(A,Φ)).

Example: for a constant energy reducible solution

$$\Phi = 2\pi i \sigma_3, \qquad A = \pi i (y dx - x dy) \sigma_3$$



the space of first order deformations is 8 dimensional: $h^1 = 8$.

The stabilizer is one dimensional,

so the moduli space has real dimension 4 in agreement with
the Newton polygon having one internal point.

Summary

- Monopole moduli spaces deliver various types of Gravitational Instantons (and higher-dimensional kyperkähler spaces)
- There are seven monowalls (doubly-periodic monopoles) with four-dimensional moduli space.
- Combinatorics of Newton polygons delivers
  ・the moduli space dimension and
  ・the number of parameters.
- It also gives a simple criterium for SL(2,Z) equivalence.