On the Continuum Hamiltonian Hopf Bifurcation I*

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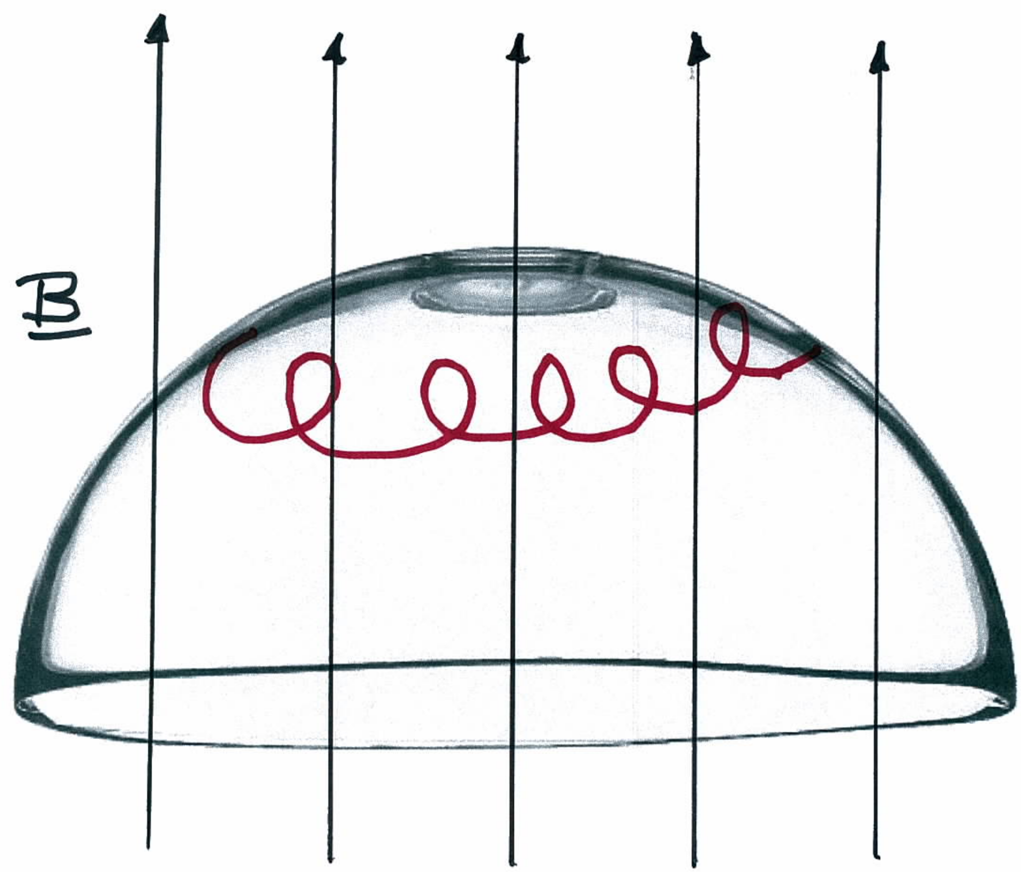
Goal: Prove a Krein-like theorem for instabilities that emerge from the continuous spectrum in a large class of Hamiltonian Eulerian matter models (CHH). Motivate a nonlinear normal form (pde).

*With G. Hagstrom

Charged Particle on Slick Mountain



Falls and Rotates $\Rightarrow$ Precession

Charged Particle on Quadratic Mountain

Simple model of FLR stabilization $\rightarrow$ plasma mirror machine.

Lagrangian:

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{eB}{2}\left(\dot{y}x - \dot{x}y\right) + \frac{K}{2}\left(x^2 + y^2\right)$$

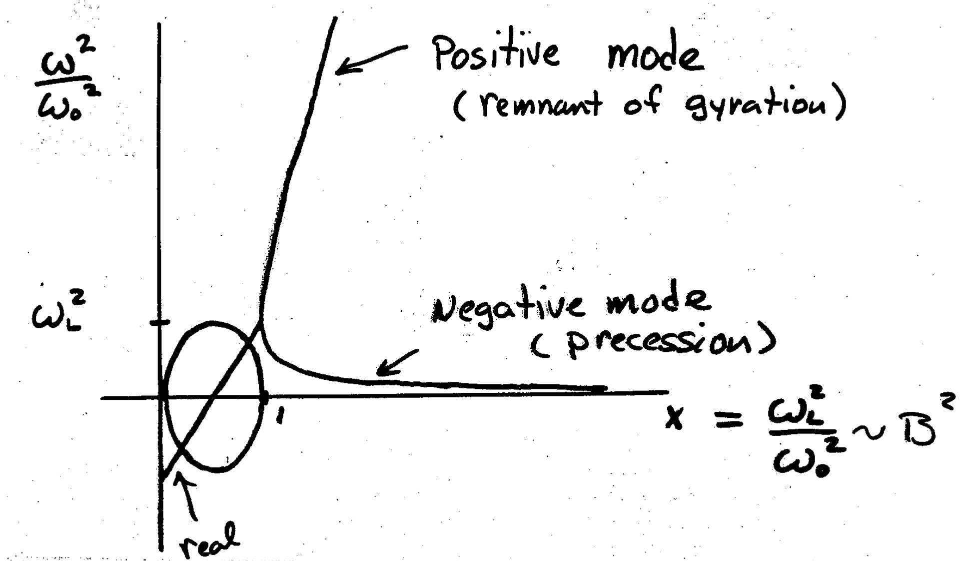
Hamiltonian:

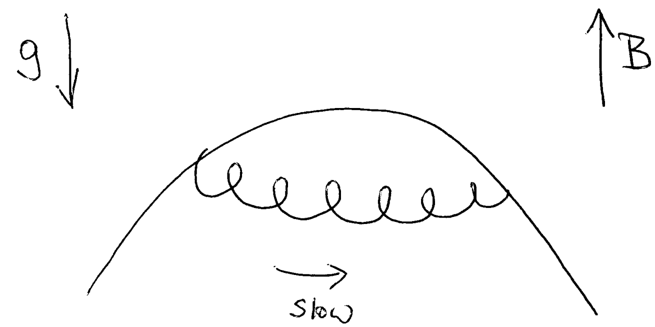
$$H = \frac{m}{2}\left(p_x^2 + p_y^2\right) + \omega_L\left(yp_x - xp_y\right) - \frac{m}{2}\left(\omega_L^2 - \omega_0^2\right)\left(x^2 + y^2\right)$$

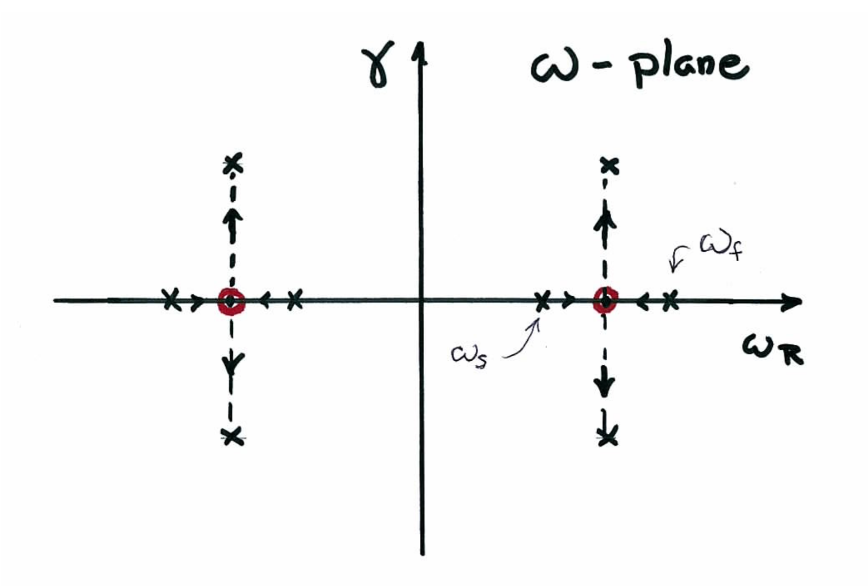
Two frequencies:

$$\omega_L = \frac{eB}{2m} \qquad \text{and} \qquad \omega_0 = \sqrt{\frac{K}{m}}$$

Quadratic Mountain - Krein Crash







$$x, y \sim e^{i\omega t} = e^{\lambda t}$$

Quadratic Mountain Stable Normal Form

For large enough B system is stable and $\exists$ a coordinate change, a canonical transformation $(q,p) \rightarrow (Q,P)$, to

$$H = \frac{|\omega_f|}{2}\left(P_f^2 + Q_f^2\right) - \frac{|\omega_s|}{2}\left(P_s^2 + Q_s^2\right)$$

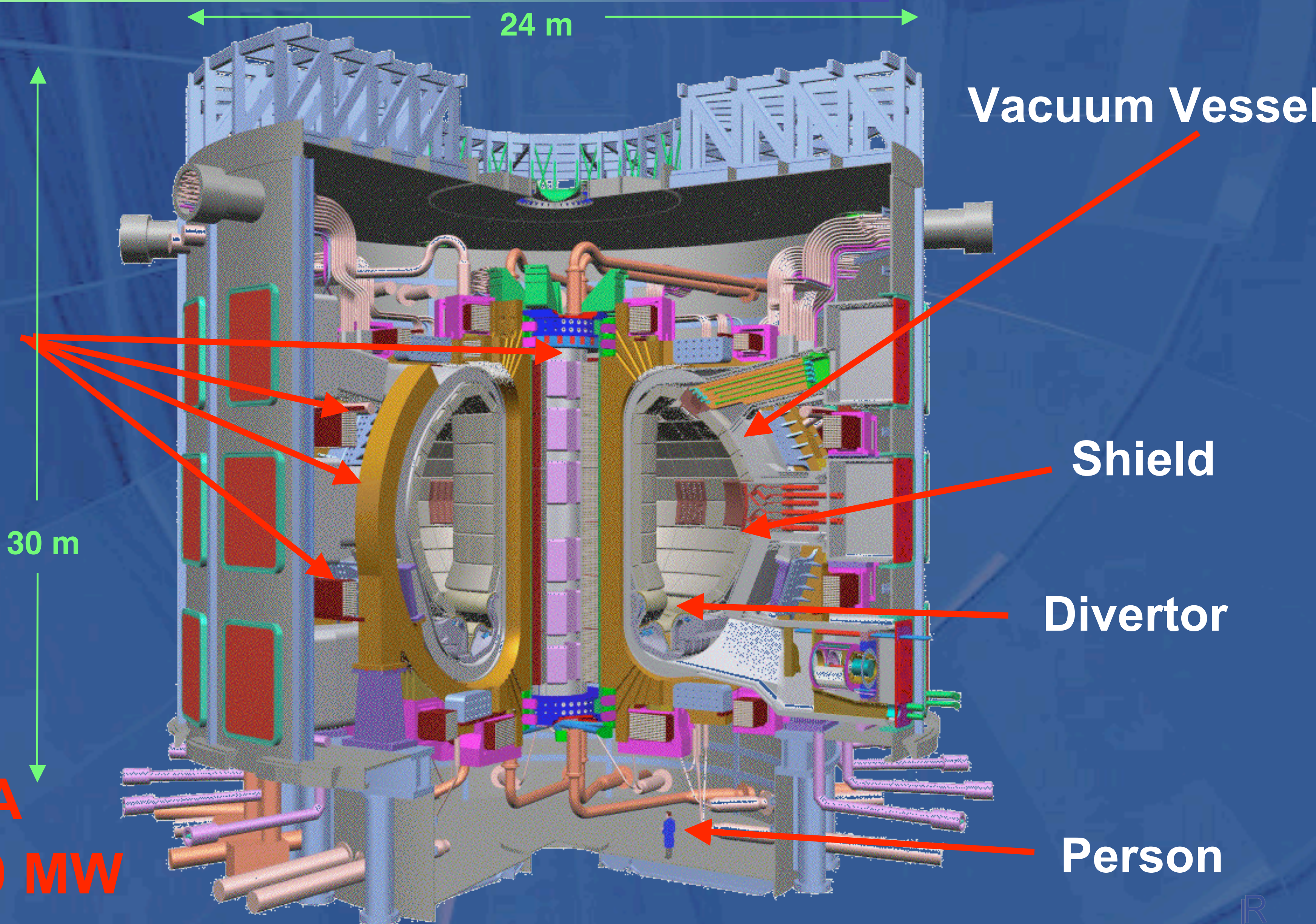
Slow mode is a negative energy mode – a stable oscillation that lowers the energy relative to the equilibrium state.

Weierstrass (1894), Williamson (1936), ...

.

$\longrightarrow$ Later will do analog of this for continuous spectrum.

Tokamak



$R = 6.2$ m

$I_p = 15$ MA

$P_{fus} = 500$ MW

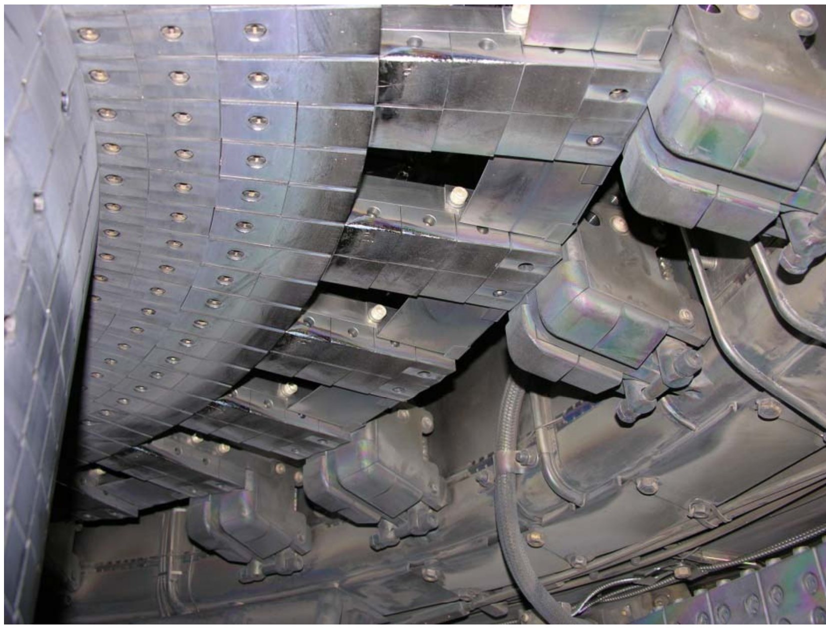
10^{10}

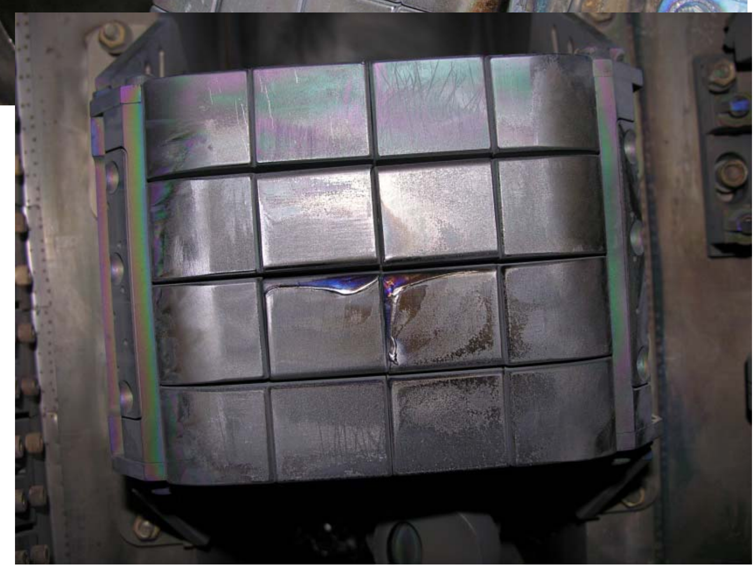
$\$10^{10}$

$\$10^{10} \uparrow$

June 2008 Alcator C-Mod, in-vessel inspection

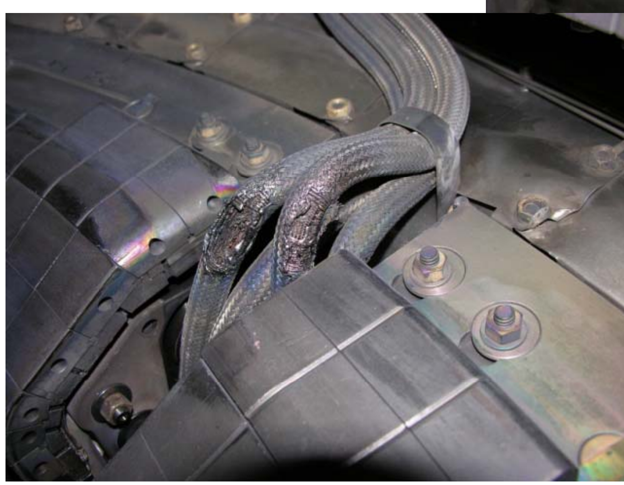
localized melt damage most likely due to runaways





Melt damage at upper edges

“Far away” diagnostic harness burned/melted by runaways



Tokamak Issues

All interesting plasma magnetic confinement equilibria are either spectrally unstable or spectrally stable with indefinite linearized energy, i.e. have negative energy modes. Both can be dangerous.

PJM and D. Pfirsch (1989)

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PJM and D. Pfirsch (1989)

USDOE DE-FG05-80ET-53088 for 31 years!

Maxwell-Vlasov System

Vlasov Equation:

$$\frac{\partial f_\alpha(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\nabla f_\alpha + \frac{e_\alpha}{m_\alpha}\Big(\mathbf{E} + \mathbf{v}\times\mathbf{B}\Big)\cdot\nabla_{\mathbf{v}} f_\alpha = \frac{\partial f_\alpha}{\partial t}\Big)_c \approx 0$$

where f is phase space density, $\alpha = e, i$ is species index, and the sources, charge density and current density, are given by

$$\rho(x,t) = \sum_\alpha e_\alpha \int_{\mathbb{R}^3} d^3v\; f_\alpha\,, \qquad \mathbf{J}(x,t) = \sum_\alpha e_\alpha \int_{\mathbb{R}^3} d^3v\; \mathbf{v} f_\alpha\,,$$

which couple into

Maxwell's Equations:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}\,, \qquad \nabla\cdot\mathbf{B} = 0$$

$$\epsilon_0\frac{\partial \mathbf{E}}{\partial t} = \nabla\times\mathbf{B} - \mu_0\mathbf{J}\,, \qquad \epsilon_0\nabla\cdot\mathbf{E} = \rho$$

Maxwell-Vlasov Regularity

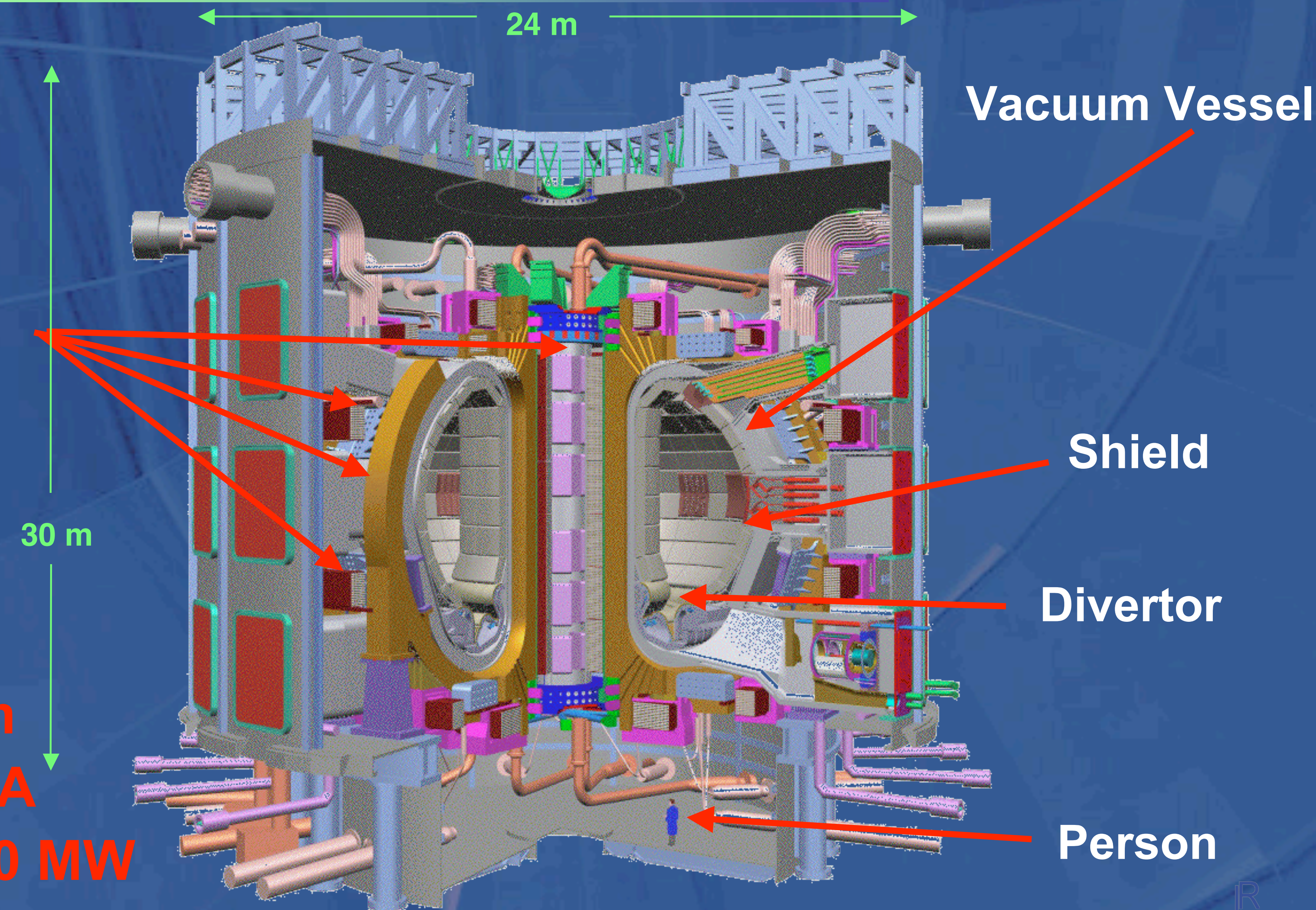
Maxwell-Vlasov global existence: Open!

R. Glassey, J. Schaeffer, ...

> "After 40 years we have precious little to show for it."

Computation? age of solar system $<$ age of universe

Tokamak



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Maxwell-Vlasov System (to scale)

$$\frac{\partial f_\alpha(x,v,t)}{\partial t} + v \cdot \frac{\partial f_\alpha}{\partial x} + \cdots = 0 \qquad (1)$$

 $\longleftarrow$ person

Vlasov-Poisson System

Phase space density (1 + 1 + 1 field theory):

$$f : \Pi \times \mathbb{R}^2 \to \mathbb{R}^+ , \qquad f(x,v,t) \geq 0$$

Conservation of phase space density:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}\frac{\partial \phi[x,t;f]}{\partial x}\frac{\partial f}{\partial v} = 0$$

Poisson's equation:

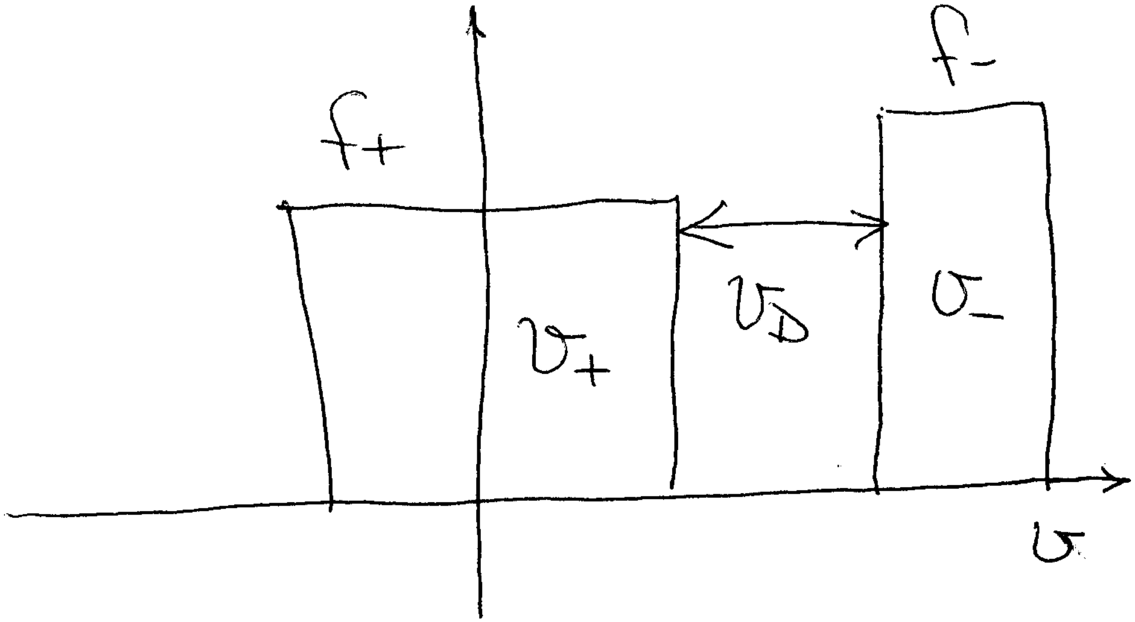
$$\phi_{xx} = 4\pi \left[e \int_{\mathbb{R}} f(x,v,t)\, dv - \rho_B \right]$$

Energy:

$$H = \frac{m}{2}\int_{\Pi}\int_{\mathbb{R}} v^2 f \, dx dv + \frac{1}{8\pi}\int_{\Pi} (\phi_x)^2 \, dx$$

Fluid Two-Stream

Waterbag distribution function:



Two-Stream Instability (warm ions & electrons)

$$\frac{\partial v_\alpha}{\partial t} + v_\alpha \frac{\partial v_\alpha}{\partial x} = \frac{e_\alpha}{m_\alpha} E - \frac{1}{\rho_\alpha}\frac{\partial P_\alpha}{\partial x}$$

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x}(n_\alpha v_\alpha) = 0 \qquad\qquad \frac{\partial E}{\partial x} = 4\pi e (n_i - n_e)$$

equil. n_{0i}, n_{0e}, v_D ← drifting electrons

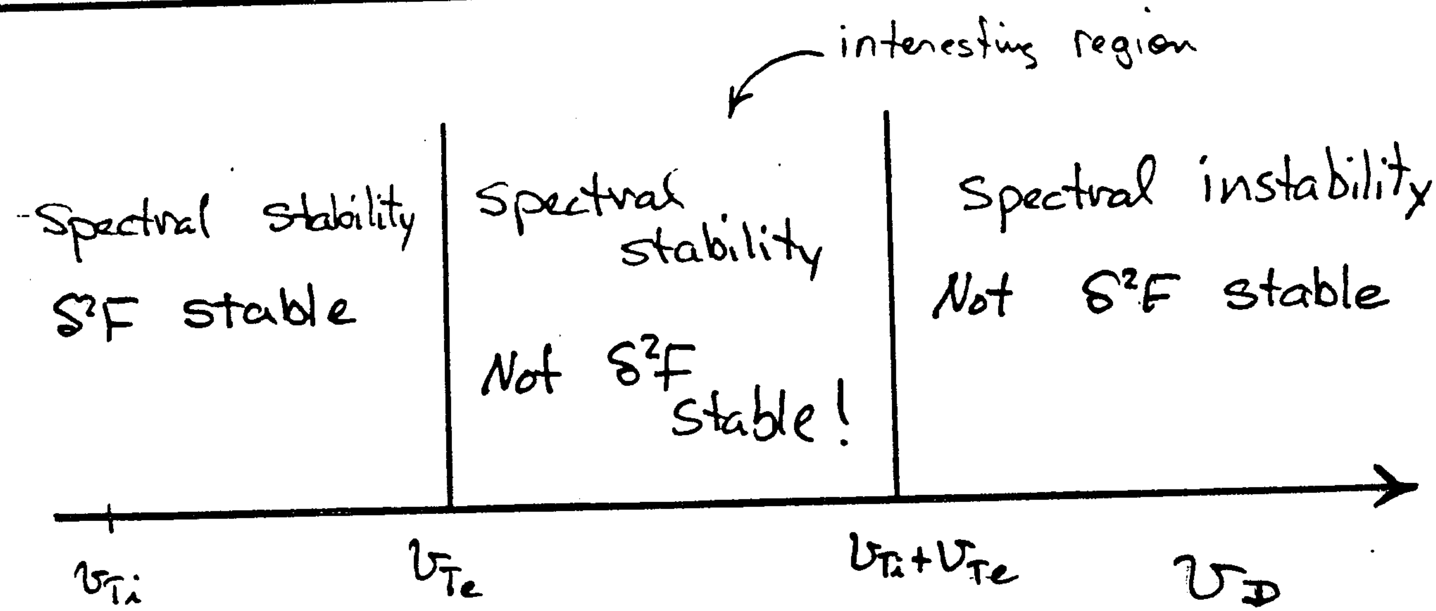
spectral stability condition given via

$$0 = 1 - \frac{\omega_{pi}^2}{\omega^2 - k^2 v_{Ti}^2} - \frac{\omega_{pe}^2}{(\omega - k v_0)^2 - k^2 v_{Te}^2} = \varepsilon(k,\omega)$$

Threshold: $v_D > v_{Ti} + v_{Te}$ ⇒ instab.

$\delta^2 F$:

Threshold: $v_D < v_{Te}$ ⇒ $\delta^2 F$ positive definite



Two-Stream Instability $\leftrightarrow$ Hamiltonian Hopf

Three equivalent definitions of negative energy modes:

- Von Laue 1905:

$$\mathrm{sgn}\left(\omega(k)\frac{\partial\varepsilon(k,\omega(k))}{\partial\omega}\right)$$

- Energy Casmir: $\delta^2 F = \delta^2(H+C)$

- Symplectic signature: H_L on eigenvector or two-form

Krein (1950) – Moser (1958) – Sturrock (1958)
Avoidance crossing etc. Sturrock $\rightarrow$ Cairns ....

Von Laue (wave) energy incorrect for continuous spectrum
pjm and Pfirsch (1992)

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Poisson's equation:

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Energy:

$$H = \frac{m}{2}\int_{\Pi}\int_{\mathbb{R}} v^2 f\, dx dv + \frac{1}{8\pi}\int_{\Pi} (\phi_x)^2\, dx$$

Class of Hamiltonian Systems

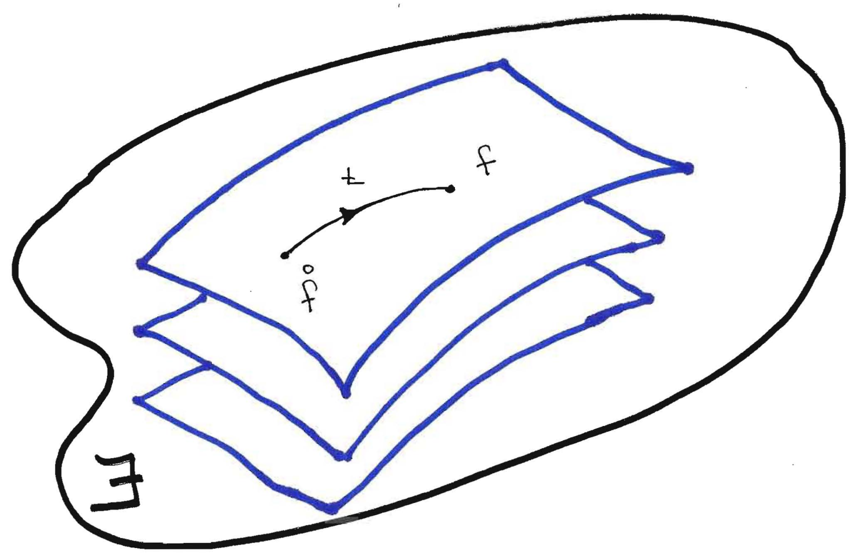
- plasma physics (charged particles-electrostatic)
- vortex dynamics, QG, shear flow
- stellar dynamics
- statistical physics (XY-interaction)
- ...
- general transport via mean field theory

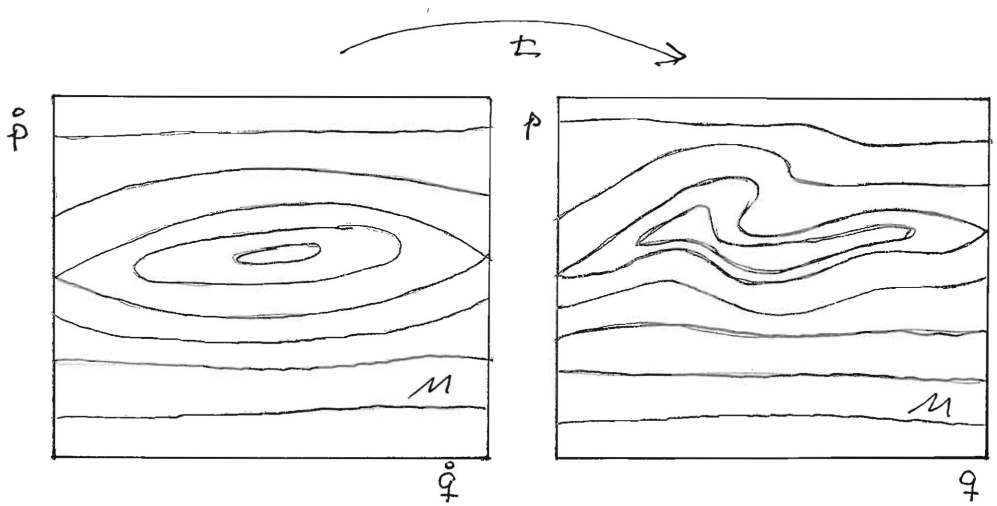
VP Cartoon– Symplectic Rearrangement

$f(x,v,t) = \mathring{f} \circ \mathring{z}$

$f \sim g$ if $f = g \circ z$

with z symplectomorphism





$p = mv$

μ volume measure

$$f(x,v,t) = \mathring{f}(\mathring{x}(x,v,t), \mathring{v}(x,v,t))$$

Natural Hamiltonian Structure of Matter

Noncanonical Poisson Bracket:

$$\{F,G\} = \int_{\mathcal{Z}} dq dp\, f \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right] = \int_{\mathcal{Z}} dq dp\, F_f \mathcal{J} G_f = \Big\langle f, [F_f, G_f] \Big\rangle$$

Cosymplectic Operator:

$$\mathcal{J} \;\cdot = \frac{\partial f}{\partial p}\frac{\partial \;\cdot}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial \;\cdot}{\partial p}$$

Vlasov:

$$\frac{\partial f}{\partial t} = \{f, H\} = \mathcal{J}\frac{\delta H}{\delta f} = -[f, \mathcal{E}].$$

Casimir Degeneracy:

$$\{C, F\} = 0 \qquad \forall F \qquad \text{for} \qquad C[f] = \int_{\mathcal{Z}} dq dp\, \mathcal{C}(f)$$

Too many variables and not canonical.

Recall Cartoon – Hamiltonian on leaf.

Linear Vlasov-Poisson System

Expand about Stable Homogeneous Equilibrium:

$$f = f_0(v) + \delta f(x, v, t)$$

Linearized EOM:

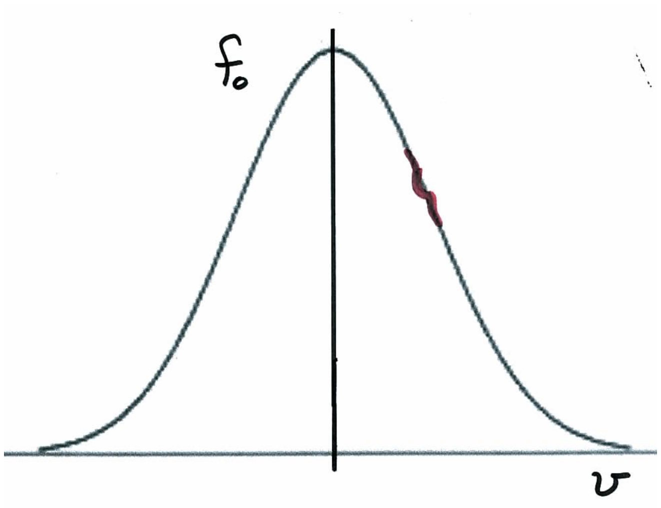
$$\frac{\partial \delta f}{\partial t} + v \frac{\partial \delta f}{\partial x} + \frac{e}{m} \frac{\partial \delta\phi[x, t; \delta f]}{\partial x} \frac{\partial f_0}{\partial v} = 0$$

$$\delta\phi_{xx} = 4\pi e \int_{\mathbb{R}} \delta f(x, v, t)\, dv$$

Linearized Energy (Kruskal-Oberman 1958):

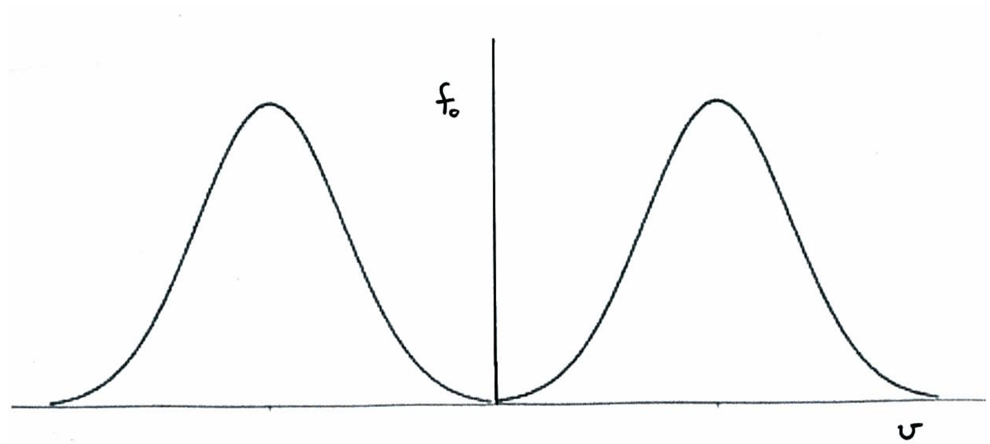
$$H_L = -\frac{m}{2} \int_{\Pi} \int_{\mathbb{R}} \frac{v\,(\delta f)^2}{f_0'}\, dv dx + \frac{1}{8\pi} \int_{\Pi} (\delta\phi_x)^2\, dx$$

Sample Homogeneous Equilibria



← Maxwellian

BiMaxwellian →



Linear Hamiltonian Theory

Expand f-dependent Poisson bracket and Hamiltonian $\Rightarrow$

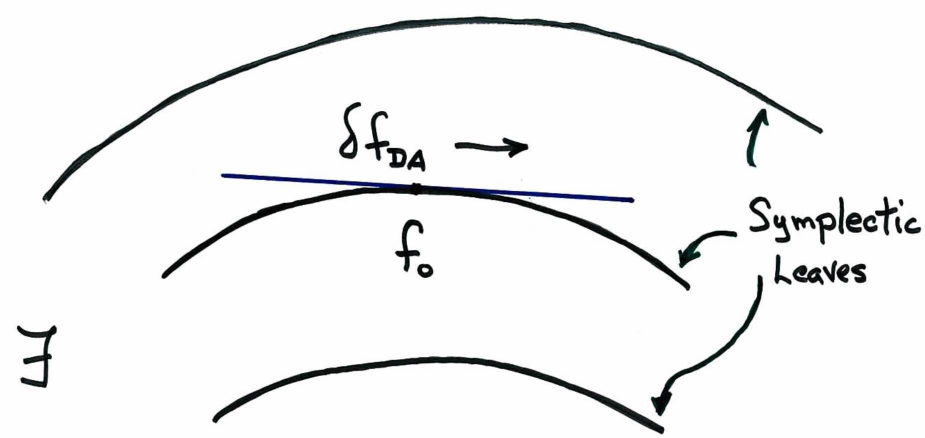
$$\frac{\partial \delta f}{\partial t} = \{\delta f, H_L\}_L\,,$$

where quadratic Hamiltonian H_L is the Kruskal-Oberman energy and linear Poisson bracket is $\{\,,\,\}_L = \{\,,\,\}_{f_0}$.

Note:

δf not canonical

H_L not diagonal



Landau's Problem

Assume

$$\delta f = \sum_k f_k(v,t)e^{ikx}\,, \qquad \delta\phi = \sum_k \phi_k(t)e^{ikx}$$

Linearized EOM:

$$\frac{\partial f_k}{\partial t} + ikvf_k + ik\phi_k\frac{e}{m}\frac{\partial f_0}{\partial v} = 0\,, \qquad k^2\phi_k = -4\pi e\int_{\mathbb{R}} f_k(v,t)\,dv$$

Three methods:

1. Laplace Transforms (Landau and others 1946)
2. Normal Modes (Van Kampen, Case,... 1955)
3. Coordinate Change $\Longleftrightarrow$ Integral Transform (PJM, Pfirsch, Shadwick, ... 1992)

Canonization & Diagonalization

Fourier Linear Poisson Bracket:

$$\{F,G\}_L = \sum_{k=1}^{\infty} \frac{ik}{m} \int_{\mathbb{R}} f_0' \left(\frac{\delta F}{\delta f_k} \frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k} \frac{\delta F}{\delta f_{-k}} \right) dv$$

Linear Hamiltonian:

$$H_L = -\frac{m}{2} \sum_k \int_{\mathbb{R}} \frac{v}{f_0'} |f_k|^2 \, dv \;+\; \frac{1}{8\pi} \sum_k k^2 |\phi_k|^2$$

$$= \sum_{k,k'} \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(v) \, \mathcal{O}_{k,k'}(v|v') \, f_{k'}(v') \, dv dv'$$

Canonization:

$$q_k(v,t) = f_k(v,t) \,, \qquad p_k(v,t) = \frac{m}{ik f_0'} f_{-k}(v,t) \qquad \Longrightarrow$$

$$\{F,G\}_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(\frac{\delta F}{\delta q_k} \frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k} \frac{\delta F}{\delta p_k} \right) dv$$

Integral Transform

Definintion:

$$\boxed{f(v) = \mathcal{G}[g](v) \;:= \varepsilon_R(v)\, g(v) + \varepsilon_I(v)\, H[g](v)\,,}$$

where

$$\varepsilon_I(v) = -\pi \frac{\omega_p^2}{k^2} \frac{\partial f_0(v)}{\partial v}\,, \qquad \varepsilon_R(v) = 1 + H[\varepsilon_I](v)\,,$$

and the Hilbert transform

$$H[g](v) := \frac{1}{\pi} \fint \frac{g(u)}{u - v}\, du\,,$$

with $\fint$ denoting Cauchy principal value of $\int_{\mathbb{R}}$.

Theorem (G1) *$\mathcal{G}\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$, $1 < p < \infty$, is a bounded linear operator; i.e.*

$$\|\mathcal{G}[g]\|_p \le B_p \, \|g\|_p \,,$$

where B_p depends only on p.

Theorem (G2) *If $f_0' \in L^q(\mathbb{R})$, stable, Hölder decay, then $\mathcal{G}[g]$ has a bounded inverse,*

$$\mathcal{G}^{-1}\colon L^p(\mathbb{R}) \to L^p(\mathbb{R}) \,,$$

for $1/p + 1/q < 1$, given by

$$\begin{aligned} g(u) &= \mathcal{G}^{-1}[f](u) \\ &:= \frac{\varepsilon_R(u)}{|\varepsilon(u)|^2} \, f(u) - \frac{\varepsilon_I(u)}{|\varepsilon(u)|^2} \, H[f](u) \,. \end{aligned}$$

where $|\varepsilon|^2 := \varepsilon_R^2 + \varepsilon_I^2$.

Diagonalization

Mixed Variable Generating Functional:

$$\mathcal{F}[q, P'] = \sum_{k=1}^{\infty} \int_{\mathbb{R}} q_k(v)\, \mathcal{G}[P'_k](v)\, dv$$

Canonical Coordinate Change $(q, p) \longleftrightarrow (Q', P')$:

$$p_k(v) = \frac{\delta \mathcal{F}[q, P']}{\delta q_k(v)} = \mathcal{G}[P_k](v)\,, \qquad Q'_k(u) = \frac{\delta \mathcal{F}[q, P']}{\delta P_k(u)} = \mathcal{G}^\dagger[q_k](u)$$

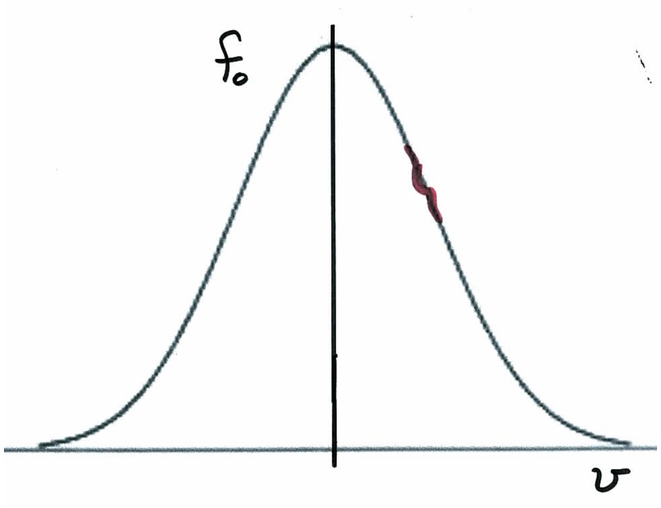
New Hamiltonian:

$$H_L = \frac{1}{2} \sum_{k=1}^{\infty} \int_{\mathbb{R}} du\, \sigma_k(u) \omega_k(u) \left[Q_k^2(u) + P_k^2(u) \right]$$

where $\omega_k(u) = |ku|$ and the signature is

$$\boxed{\sigma_k(v) := -\mathrm{sgn}(v f_0'(v))}$$

Sample Homogeneous Equilibria



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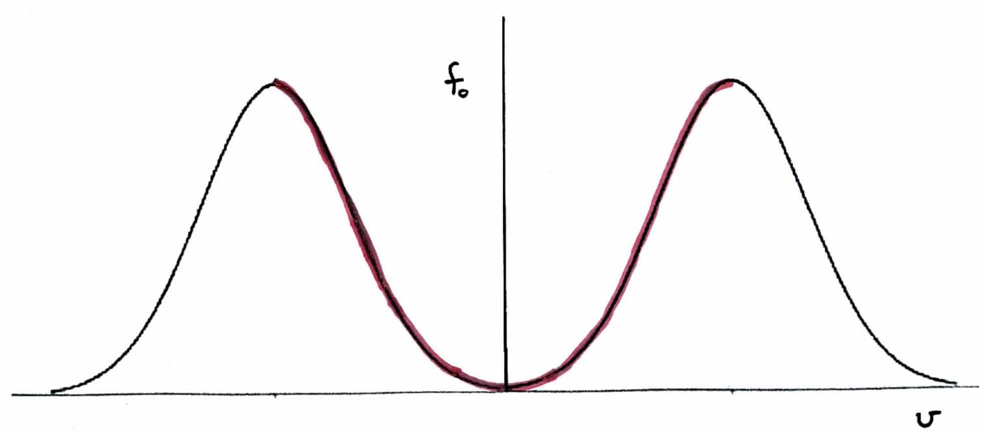


Hamiltonian Spectrum

Hamiltonian Operator:

$$f_{kt} = -ikvf_k + \frac{if_0'}{k}\int_{\mathbb{R}} d\bar{v}\, f_k(\bar{v},t) =: T_k f_k\,,$$

Complete System:

$$f_{kt} = T_k f_k \qquad \text{and} \qquad f_{-kt} = T_{-k} f_{-k}\,, \qquad\qquad k \in \mathbb{R}^+$$

Lemma *If λ is an eigenvalue of the Vlasov equation linearized about the equilibrium $f_0'(v)$, then so are $-\lambda$ and λ^* . Thus if $\lambda = \gamma + i\omega$, then eigenvalues occur in the pairs, $\pm\gamma$ and $\pm i\omega$, for purely real and imaginary cases, respectively, or quartets, $\lambda = \pm\gamma \pm i\omega$, for complex eigenvalues.*

Spectral Stability

Definition The dynamics of a Hamiltonian system linearized around some equilibrium solution, with the phase space of solutions in some Banach space $\mathcal{B}$, is spectrally stable if the spectrum $\sigma(T)$ of the time evolution operator T is purely imaginary.

Theorem *If for some $k \in \mathbb{R}^+$ and $u = \omega/k$ in the upper half plane the plasma dispersion relation,*

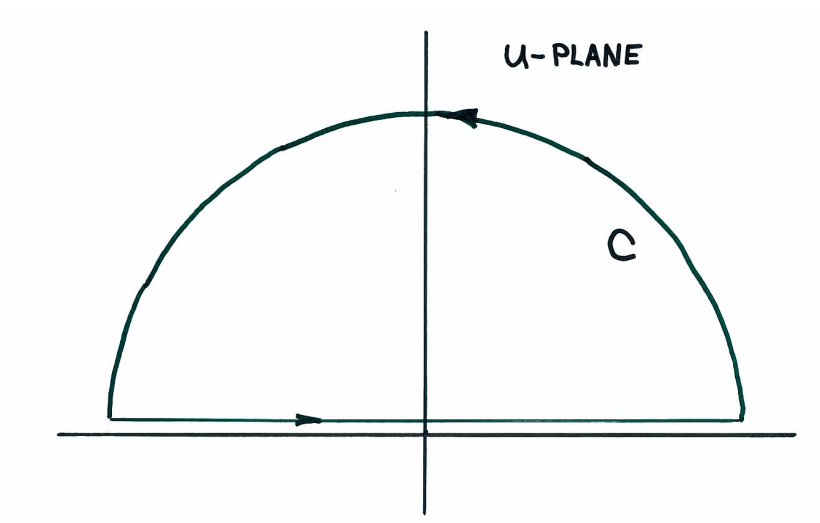
$$\varepsilon(k,u) := 1 - k^{-2} \int_{\mathbb{R}} dv \frac{f_0'}{u-v} = 0\,,$$

then the system with equilibrium f_0 is spectrally unstable. Otherwise it is spectrally stable.

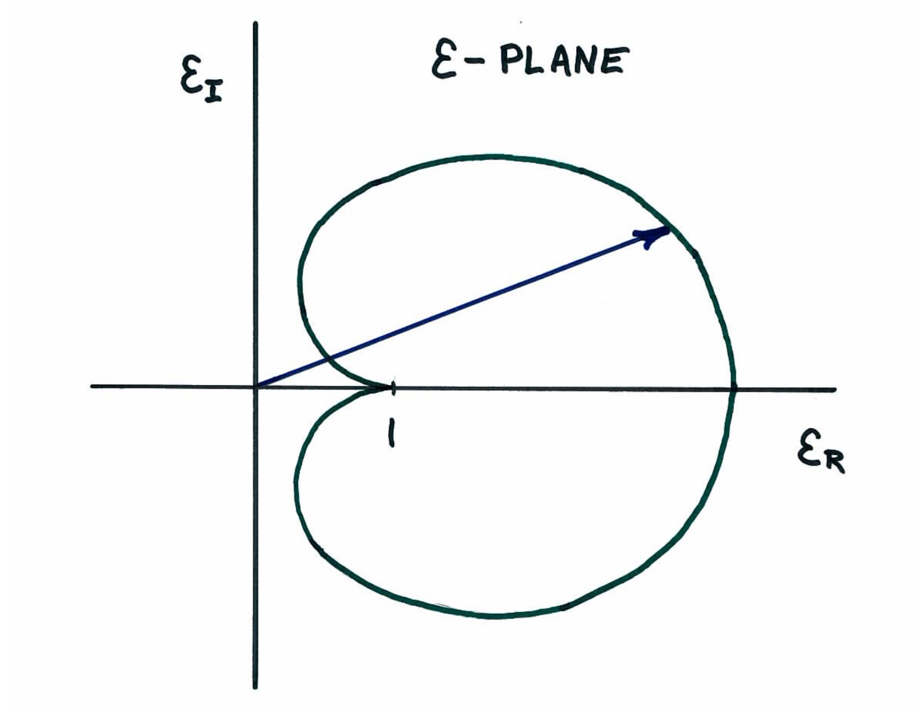
Nyquist Method

$$f_0' \in C^{0,\alpha}(\mathbb{R}) \Rightarrow \varepsilon \in C^{\omega}(uhp).$$

Therefore, Argument Principle $\Rightarrow$ winding # = # zeros of ε



Stable $\rightarrow$



Spectral Theorem

Set $k = 1$ and consider $T\colon f \mapsto ivf - if_0' \int f$ in the space $W^{1,1}(\mathbb{R})$.

$W^{1,1}(\mathbb{R})$ is Sobolev space containing closure of functions

$$\|f\|_{1,1} = \|f\|_1 + \|f'\|_1 = \int_{\mathbb{R}} dv(|f| + |f'|)$$

Definition Resolvent of T is $R(T,\lambda) = (T - \lambda I)^{-1}$ and $\lambda \in \sigma(T)$. (i) λ in point spectrum, $\sigma_p(T)$, if $R(T,\lambda)$ not injective. (ii) λ in residual spectrum, $\sigma_r(T)$, if $R(T,\lambda)$ exists but not densely defined. (iii) λ in continuous spectrum, $\sigma_c(T)$, if $R(T,\lambda)$ exists, densely defined but not bounded.

Theorem *Let* $\lambda = iu$. *(i)* $\sigma_p(T)$ *consists of all points* $iu \in \mathbb{C}$, *where* $\varepsilon = 1 - k^{-2} \int_{\mathbb{R}} dv\, f_0'/(u - v) = 0$. *(ii)* $\sigma_c(T)$ *consists of all* $\lambda = iu$ *with* $u \in \mathbb{R} \setminus (-i\sigma_p(T) \cap \mathbb{R})$. *(iii)* $\sigma_r(T)$ *contains all the points* $\lambda = iu$ *in the complement of* $\sigma_p(T) \cup \sigma_c(T)$ *that satisfy* $f_0'(u) = 0$.

cf. e.g. P. Degond (1986). Similar but different.

The CHH Bifurcation

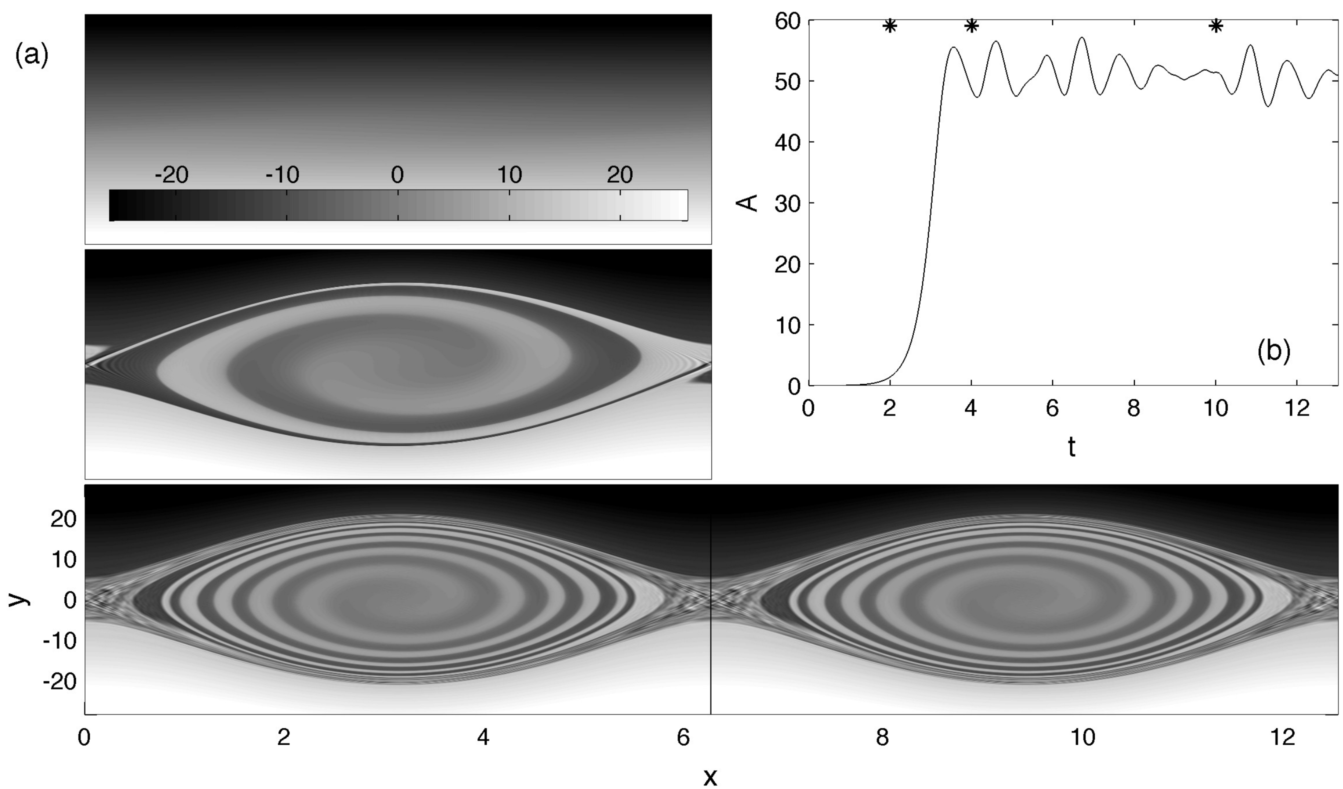
- Usual case: $f_0(v, v_d)$ one-parameter family of equilibria. Vary v_d, embedded mode appears in continuous spectrum, then $\varepsilon(k, \omega)$ has a root in uhp.

- But all equilibria infinitesmally close to instability in $L^p(\mathbb{R})$. Need measure of distance to bifurcation.

- Waterbag 'onion' replacement for f_0 has ordinary Hamiltonian Hopf bifurcation. Thus, gives a discretization of the continuous spectrum.

Single-Wave Behavior- Nonlinear

Behavior near marginality in many simulations in various physical contexts



Single-Wave Model

Asymptotics with trapping scaling ... $\Rightarrow$

$$Q_t + [Q, \mathcal{E}] = 0\,, \qquad \mathcal{E} = y^2/2 - \varphi$$

$$\mathrm{i}A_t = \left\langle Q\,\mathrm{e}^{-\mathrm{i}x}\right\rangle\,, \qquad \varphi = A\mathrm{e}^{\mathrm{i}x} + A^*\mathrm{e}^{-\mathrm{i}x}\,,$$

where

$$[f, g] := f_x g_y - f_y g_x\,, \qquad \langle\ \cdot\ \rangle := \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \int_0^{2\pi} dx\ \ \cdot \qquad (2)$$

and

$Q(x, y, t)$ = density (vorticity), $\varphi(x, t)$=potential (streamfunction), $A(t)$=single-wave of amplitude, $\mathcal{E}$= particle energy

Model has continuous spectrum with embedded mode that can be pushed into instability and then tracked nonlinearly.

Summary – Conclusions

For large class of Hamiltonian pdes with continuous spectrum:

- Diagonalization by $\mathcal{G}$-transform defines signature for cont. spec.
- Variety of Krein-like theorems, e.g. valley theorem of next talk

Single-wave model is nonlinear normal form:

- Read all about it in Balmforth, PJM, and Thiffeault RMP soon.