Waves on vortices

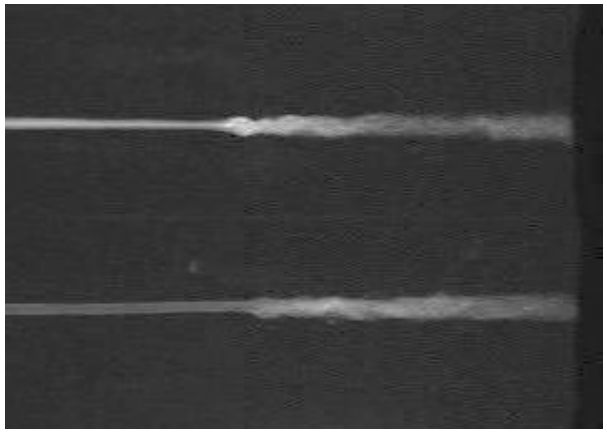
Stéphane Le Dizès, IRPHE, CNRS & Aix-Marseille University

1. Vortex in a homogeneous fluid
   a) Stability equations
   b) WKBJ description
   c) Critical layer damping

2. Vortex in a stably stratified fluid
   a) WKBJ description of the radiative instability
   b) Experimental evidences

Thanks to
L. Lacaze, P. Billant, X. Riedinger, P. Meunier, L. Parras

Waves on (stable) vortices

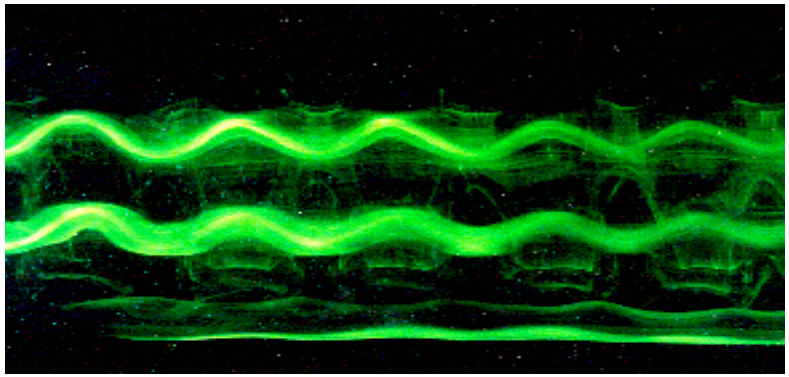


Waves generated by an abrupt modification of the vortex characteristics.

Towing tank experiment (DLR)

Waves generated by background turbulence (Vortex meandering)

Wind tip vortex (Leweke, IRPHE)



Waves excited by resonance (elliptic instability)

Co-rotating vortex pairs (Meunier & Leweke, IRPHE)

Framework

Incompressible, inviscid, constant density

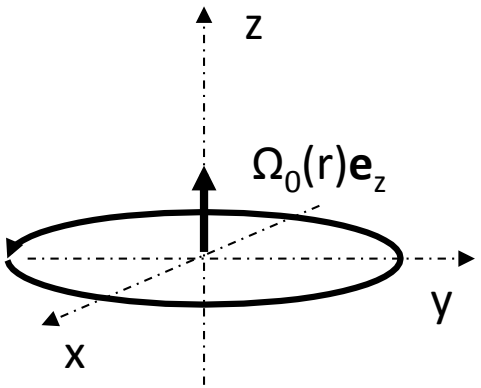
Euler equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} = -\nabla p$$

$$\nabla.\mathbf{u} = 0$$

Base flow: Axisymmetrical vortex

$$\mathbf{u}_0 = (0, r\Omega_0(r), 0) \qquad \frac{dp_0}{dr} = r\Omega_0^2$$



Angular velocity $\Omega_0(r)$

Vorticity $\eta_0(r) = \frac{1}{r}\frac{d(r^2\Omega_0)}{dr}$

Vortex models

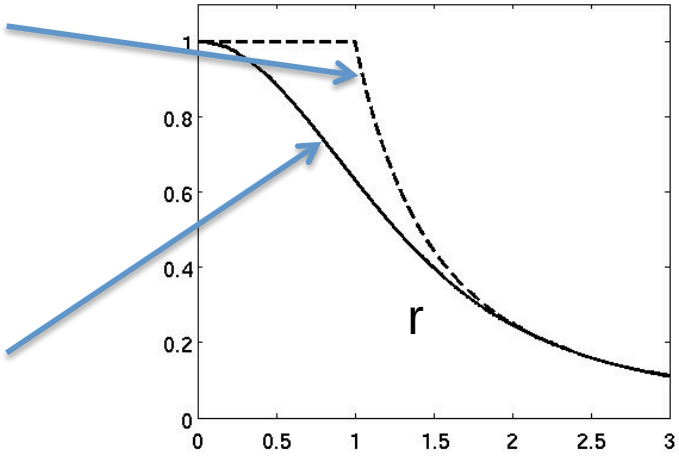
Rankine vortex

$$\Omega_0(r) = 1 \quad r < 1$$

$$\Omega_0(r) = \frac{1}{r^2} \quad r > 1$$

Lamb-Oseen vortex

$$\Omega_0(r) = \frac{1 - e^{-r^2}}{r^2}$$



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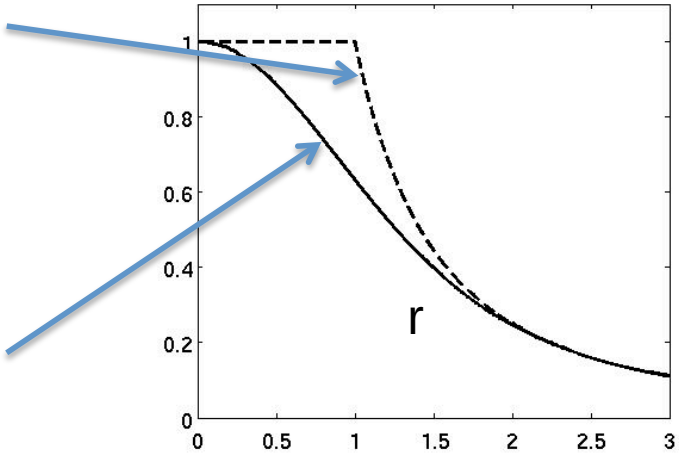


STABLE WITH RESPECT TO THE SHEAR INSTABILITY
STABLE WITH RESPECT TO THE CENTRIFUGAL INSTABILITY

$$(\mathbf{u}_t, p_t) = (0, V_0(r), 0, p_0(r)) + (u, v, w, p)e^{ikz+im\theta-i\omega t}$$

Base flow (Kelvin) Wave

Perturbation equations

$$(-i\omega + im\Omega_0)\, u_r - 2\Omega_0 u_\theta = -\frac{\partial p}{\partial r}$$

$$(-i\omega + im\Omega_0)\, u_\theta + \eta_0 u_r = -\frac{imp}{r}$$

$$(-i\omega + im\Omega_0)\, u_z = -ikp$$

$$\frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{imu_\theta}{r} + iku_z = 0$$

+ Boundary conditions (finite at the origin, vanishing at infinity)

$$(\mathbf{u}_t, p_t) = (0, V_0(r), 0, p_0(r)) + (u, v, w, p)e^{ikz+im\theta-i\omega t}$$

Base flow (Kelvin) Wave

Poincaré equation:

$$\frac{d^2p}{dr^2} + \partial_r \log\left(\frac{r}{\Delta}\right)\frac{dp}{dr} + \left(\frac{2m\Omega_0}{r\Phi}\partial_r \log\left(\frac{\Omega_0}{\Lambda}\right) + \frac{k_z^2\Delta}{\Phi^2} - \frac{m^2}{r^2}\right)p = 0$$

with

$$(\mathbf{u}_t, p_t) = (0, V_0(r), 0, p_0(r)) + (u, v, w, p)e^{ikz+im\theta-i\omega t}$$

Base flow (Kelvin) Wave

« Poincaré » equation:

$$\frac{d^2p}{dr^2} + \partial_r \log\left(\frac{r}{\Delta}\right)\frac{dp}{dr} + \left(\frac{2m\Omega_0}{r\Phi}\partial_r \log\left(\frac{\Omega_0}{\Lambda}\right) + \frac{k_z^2\Delta}{\Phi^2} - \frac{m^2}{r^2}\right)p = 0$$

with

Large axial wavenumber analysis

WKBJ approximation

with

Large axial wavenumber analysis

WKBJ approximation

with

Epicyclic frequencies

Valid far from turning points r_t

i.e.

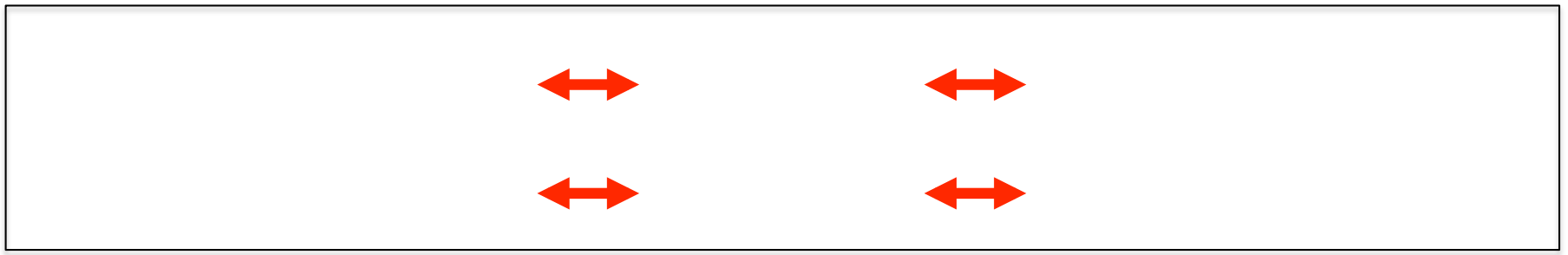
far from critical points r_c

i.e.

Large axial wavenumber analysis

WKBJ approximation

with

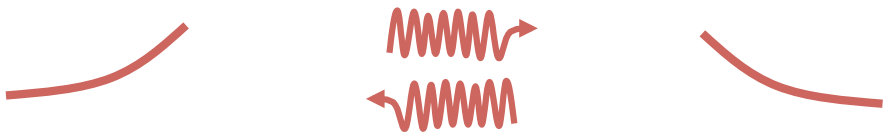


Bounded states in Quantum Mechanics

Schrödinger equation

WKBJ Approximation

with



***Bohr-Sommerfeld* rule**

Critical layers

Critical point= singularity of the linear inviscid normal mode

where

Neutral regular modes ⟹ Damped quasi-modes

However, damped quasi-modes remains regular modes on a contour in the complex plane.

WBKJ description (large k_z) of 3D quasi-modes

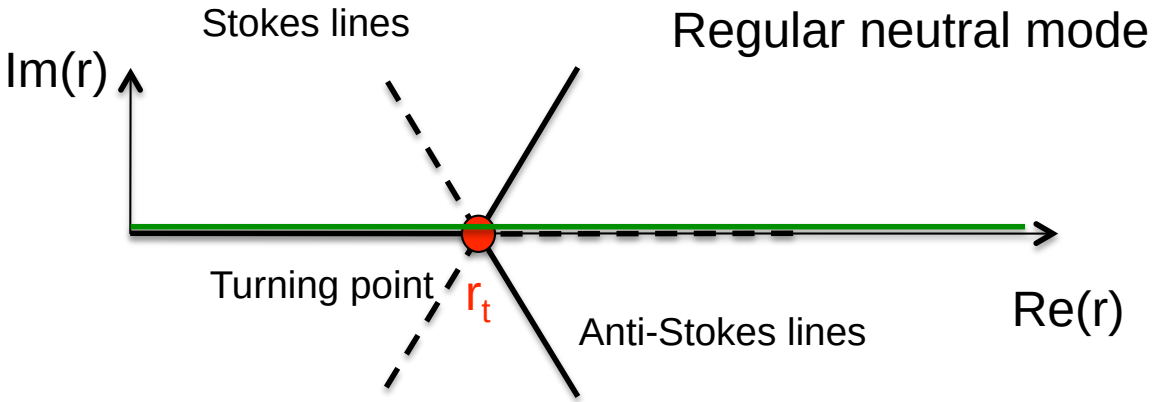
IDEA:

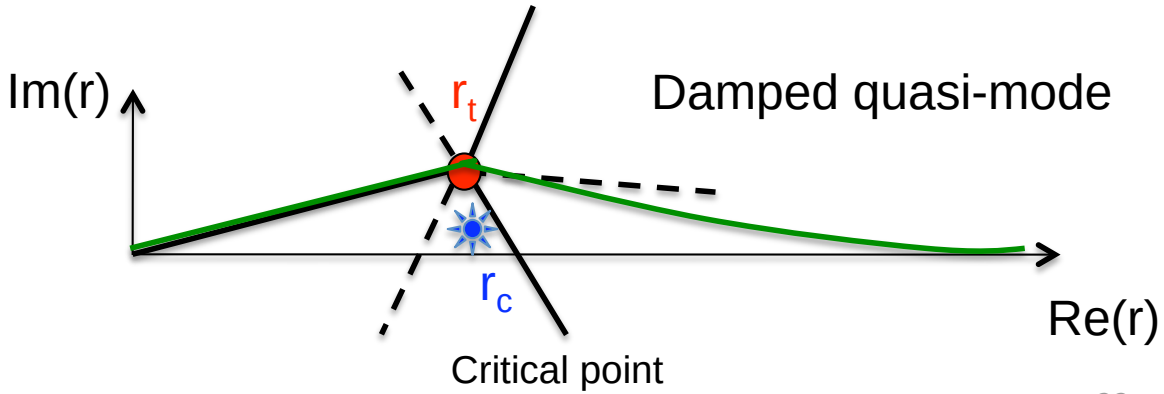
- Apply the discretization rule in the complex plane
- Check that the Stokes line network keeps the same structure

$$\int_0^{r_t} \beta(s)ds = \frac{m\pi}{2} + n\pi$$

$$\Re e\left(\int_0^{r_t} \beta(s)ds\right) = \frac{m\pi}{2} + n\pi$$

$$\Im m\left(\int_0^{r_t} \beta(s)ds\right) = 0$$





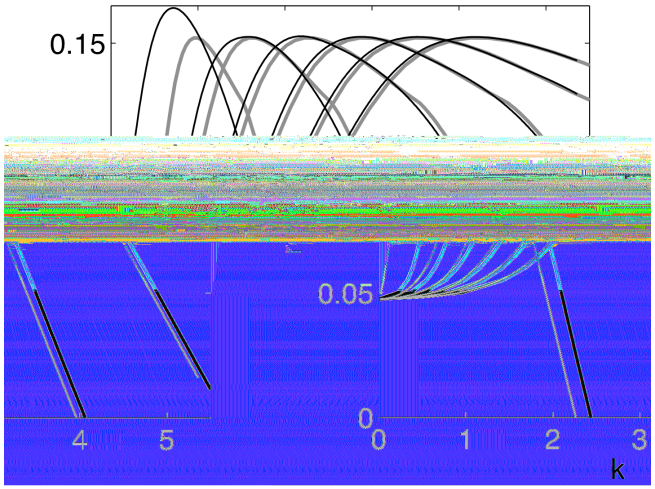
Comparison WKBJ results/Numerical results for quasi-modes

Lamb-Oseen vortex; m=1

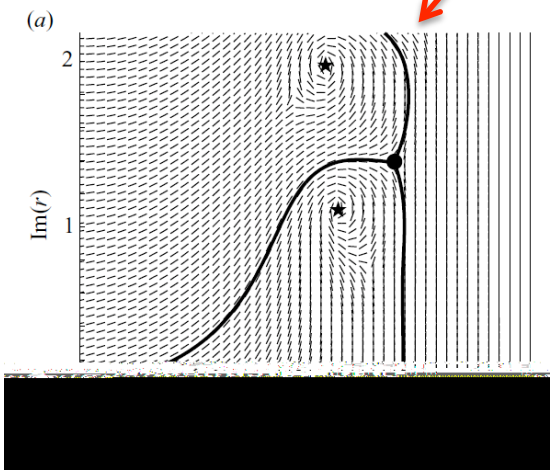
Real part versus k

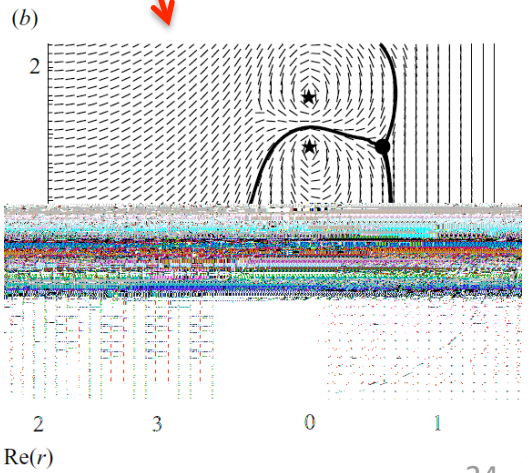
Real part versus imaginary part

Gray: WKBJ
Black: Numerics



Stokes lines network





From Le Dizès & Lacaze, 2005

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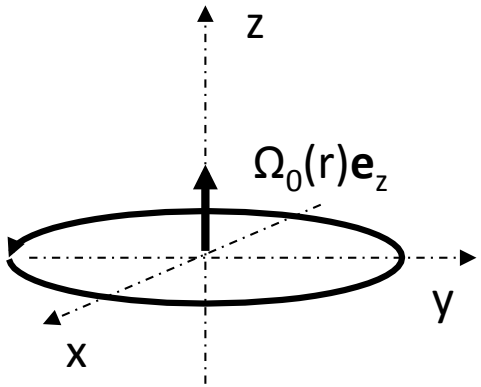
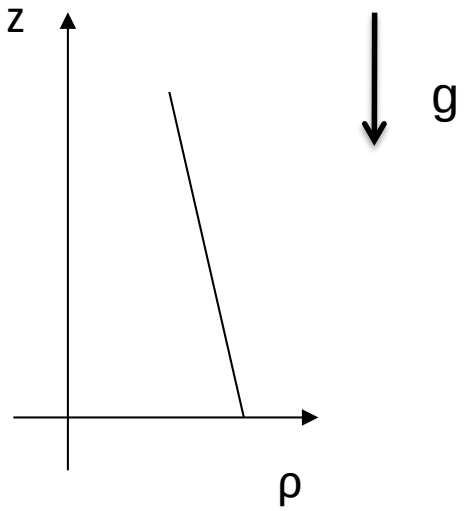
Waves on a vortex in a stratified fluid





$\Omega_0(r)$ Angular velocity

$$\eta_0(r) = \frac{1}{r}\frac{d(r^2\Omega_0)}{dr}$$ Vorticity

Brunt-Väisälä Frequency

Euler + Boussinesq + Linear normal mode perturbation

$$(\mathbf{U}', p', \rho') = (u, v, w, p, \rho) e^{ikz + im\theta - i\omega t}$$

$$\frac{d^2 p}{dr^2} + \partial_r \log\left(\frac{r}{\Delta}\right) \frac{dp}{dr} + \left(\frac{2m\Omega_0}{r\Phi} \partial_r \log\left(\frac{\Omega_0}{\Delta}\right) - \frac{k^2 \Delta}{N^2 - \Phi^2} - \frac{m^2}{r^2} \right) p = 0$$

$$\Phi(r) = -\omega + m\Omega_0(r)$$

$$\Delta(r) = 2\eta_0(r)\Omega_0(r) - \Phi^2(r)$$

Euler + Boussinesq + Linear normal mode perturbation

$$(\mathbf{U}', p', \rho') = (u, v, w, p, \rho) e^{ikz + im\theta - i\omega t}$$

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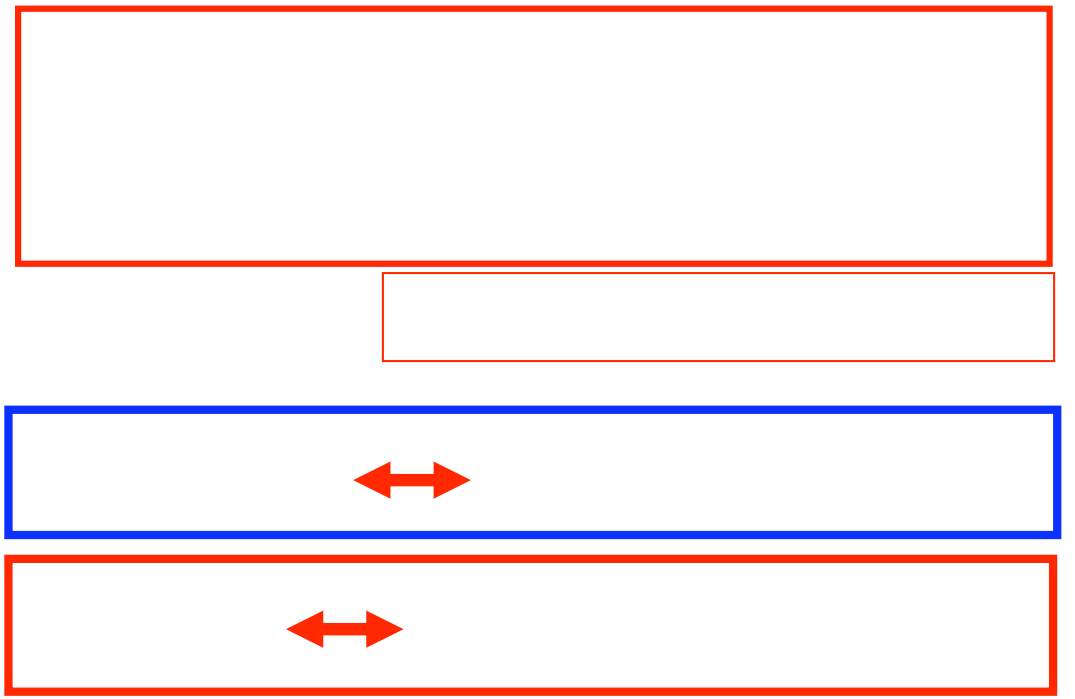


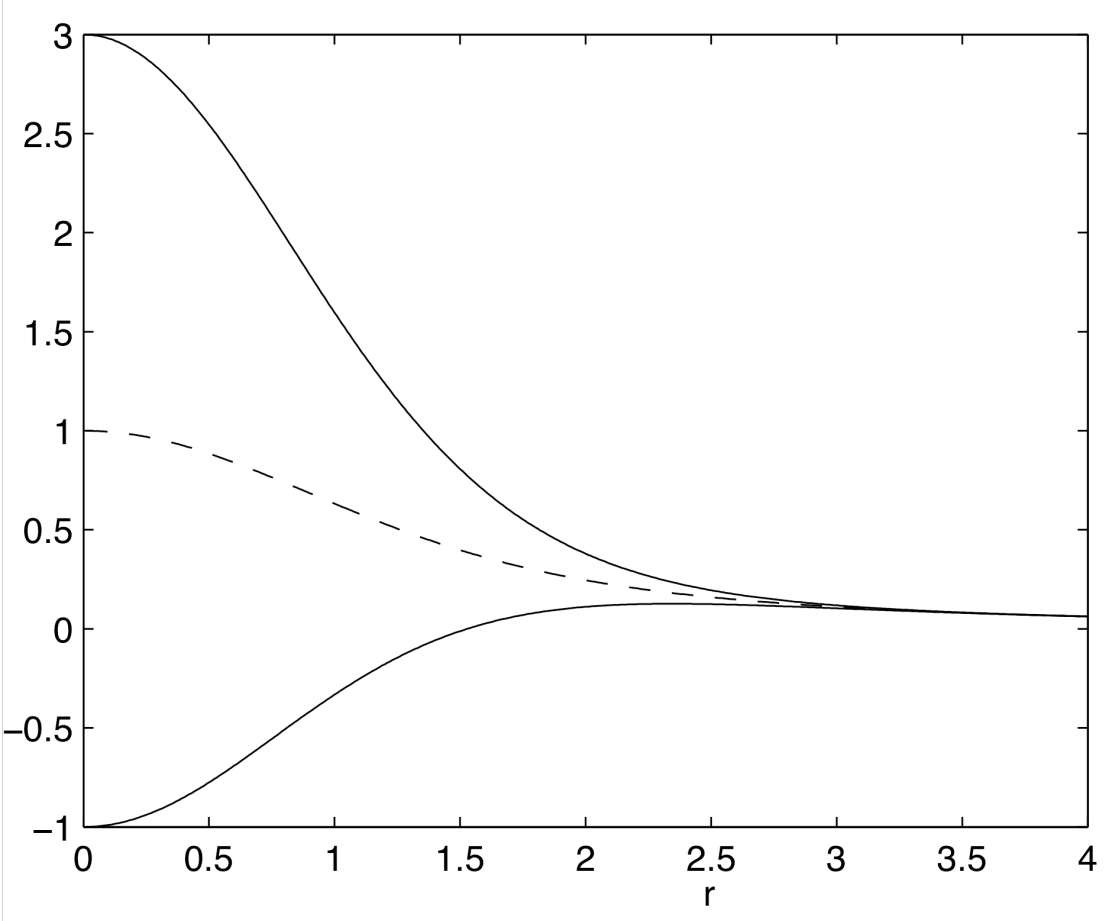
Lamb-Oseen vortex

m=0 **Large N**

→ **NO EIGENMODE**

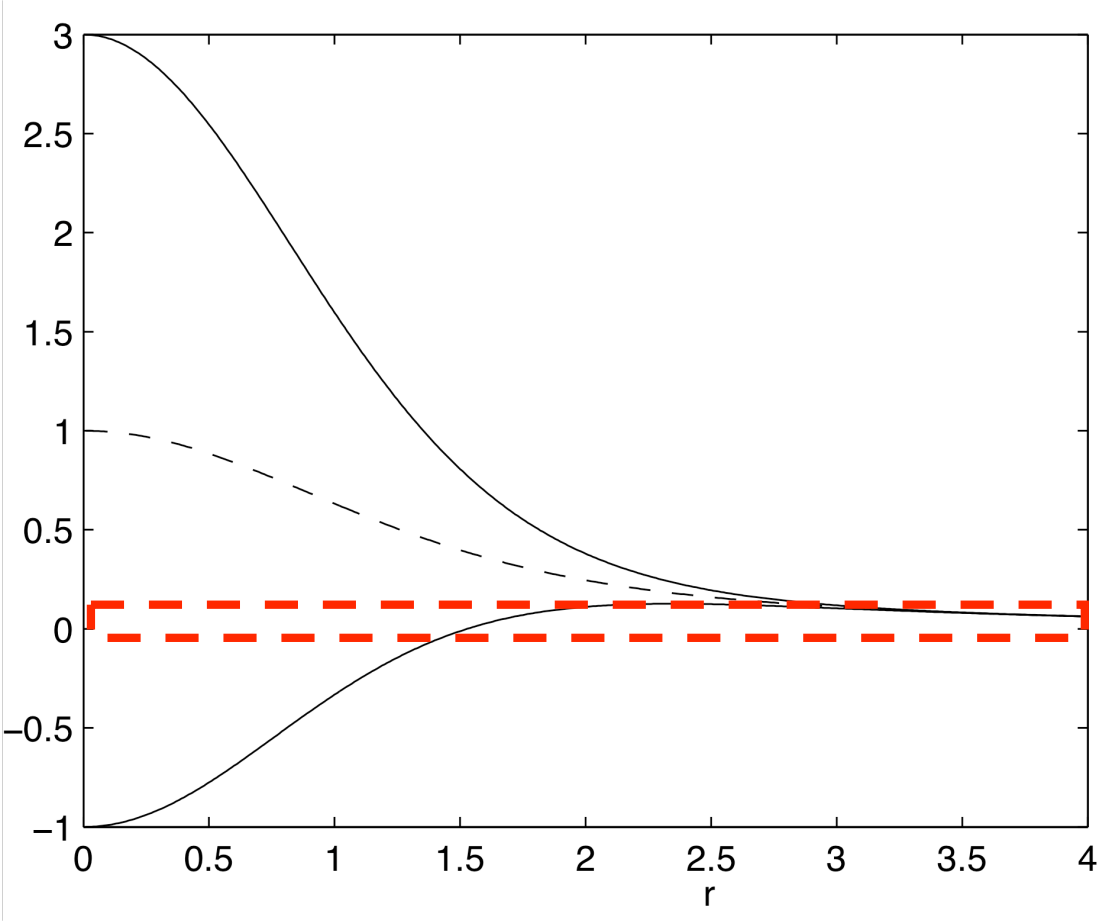
Lamb-Oseen vortex

m=1 **Large N**



Lamb-Oseen vortex

m=1 **Large N**



Lamb-Oseen vortex

m=1

Large N

Ring modes

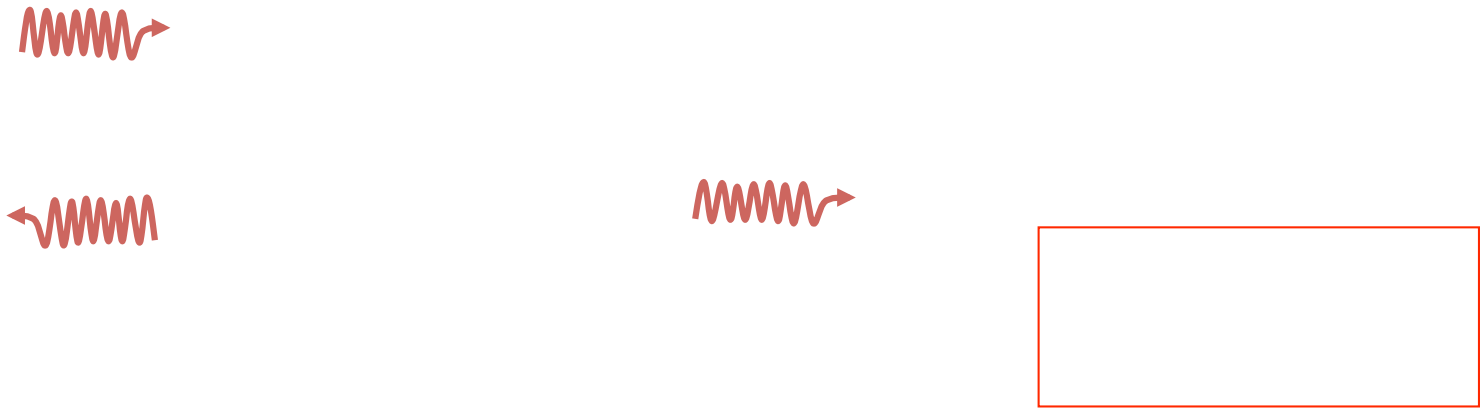
Lamb-Oseen vortex

m=1 **Large N**

Ring modes

BUT THERE IS A RADIATIVE WAVE IN THE REGION (IV)

Quantum Mechanics analogy: radioactivity

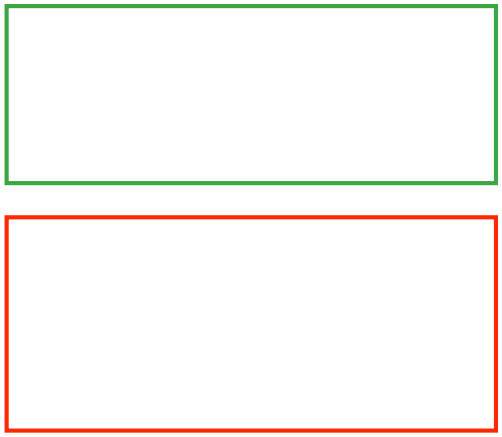


Bounded state decay rate (Gamow 1928)

Lamb-Oseen vortex

m=1

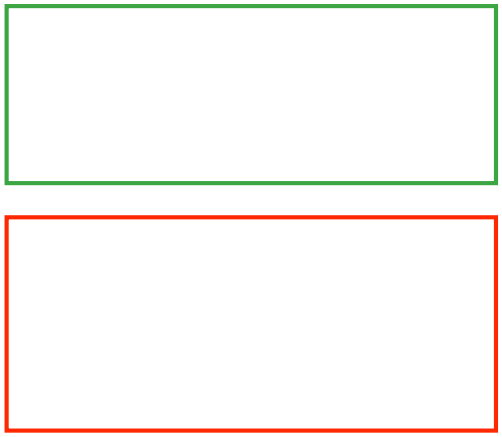
Large N



Ring modes

Lamb-Oseen vortex

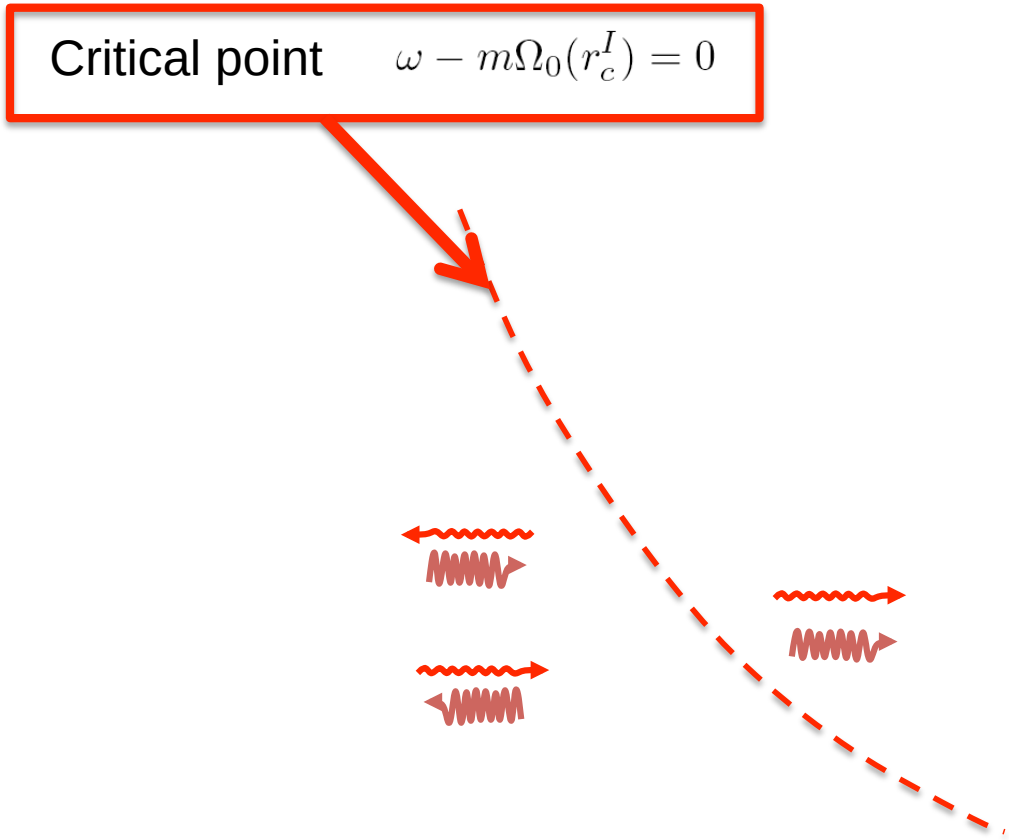
m=1 **Large N**



Ring modes

Here Im(δω)>0 !!

DESTABILIZING RADIATION



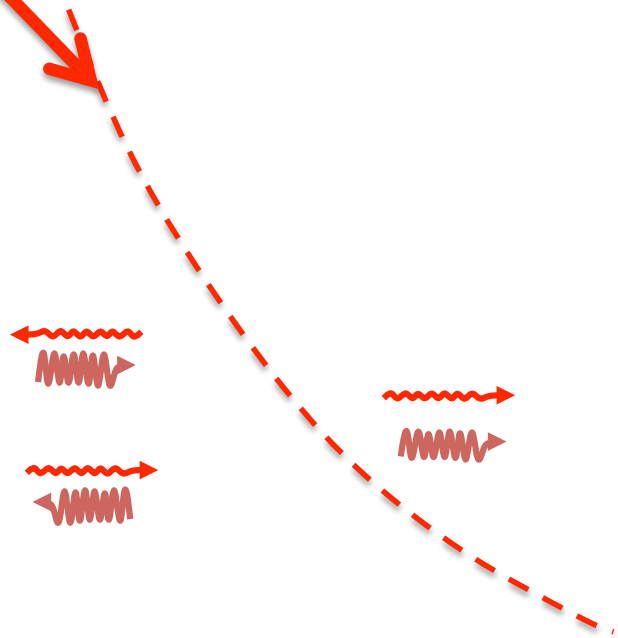
Radial phase velocity $V_\phi = \frac{\omega}{\beta}$

Radial group velocity $V_g = \frac{\partial \omega}{\partial \beta}$

$$\beta(r) = k\sqrt{\frac{2\Omega(r)\eta(r) - (\omega - m\Omega_0(r))^2}{N^2 - (\omega - m\Omega_0(r))^2}}$$

$$\frac{\partial \omega}{\partial \beta} = \frac{(N^2 - 2\Omega_0(r)\eta_0(r))}{(\omega - m\Omega_0(r))} \frac{k^2}{(\beta^2 + k^2)^2}$$

Critical point $\omega - m\Omega_0(r_c^I) = 0$



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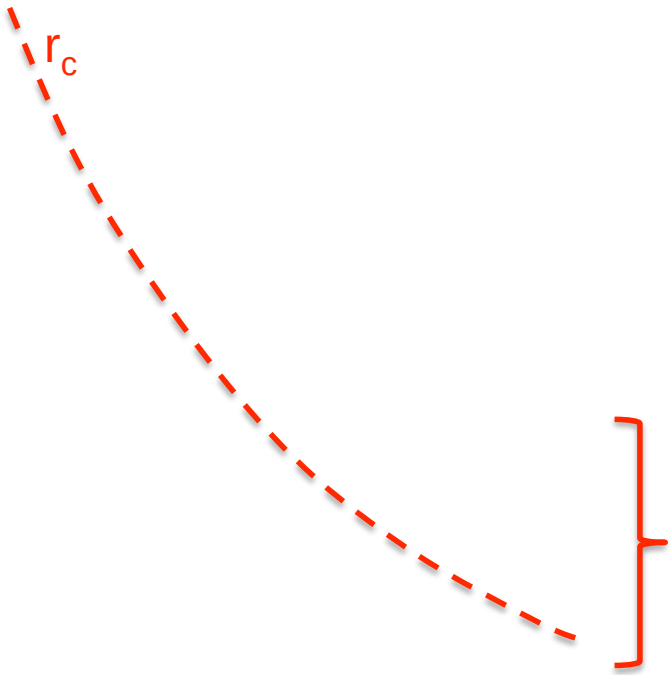
$$\frac{\partial \omega}{\partial \beta} = \frac{(N^2 - 2\Omega_0(r)\eta_0(r))}{(\omega - m\Omega_0(r))}\frac{k^2}{(\beta^2 + k^2)^2}$$

Wave guide + over-reflection = instability



When r_2, r_3 and r_c are close to each other

$$\delta\omega = -\frac{2\pi + i\log(2)}{4\int_{r_1}^{r_2} \partial_\omega \beta}$$

UNSTABLE

Comparison with numerical results

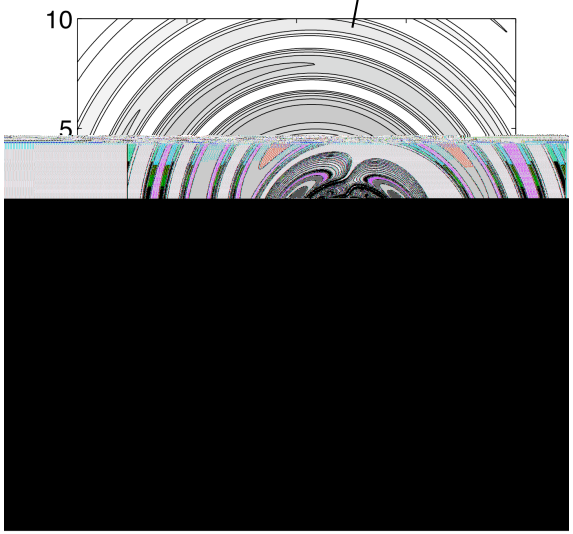
Lamb-Oseen vortex; N=100; m=1

Frequency

Growth rate

Numerics

Theory

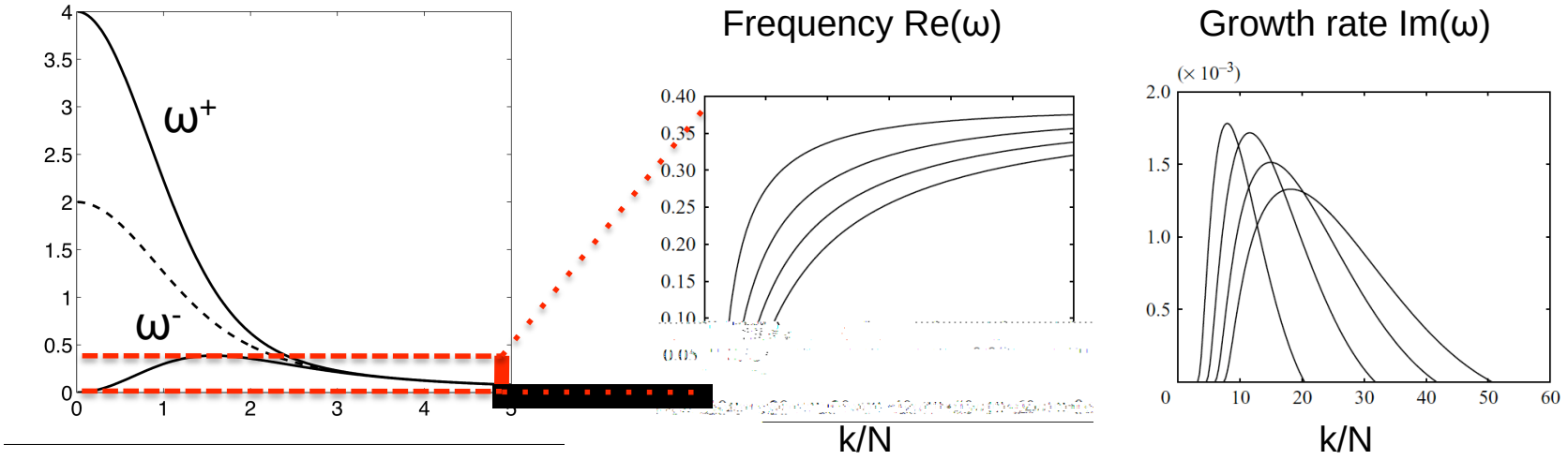


Good agreement, except
for the first branch

Lamb-Oseen vortex

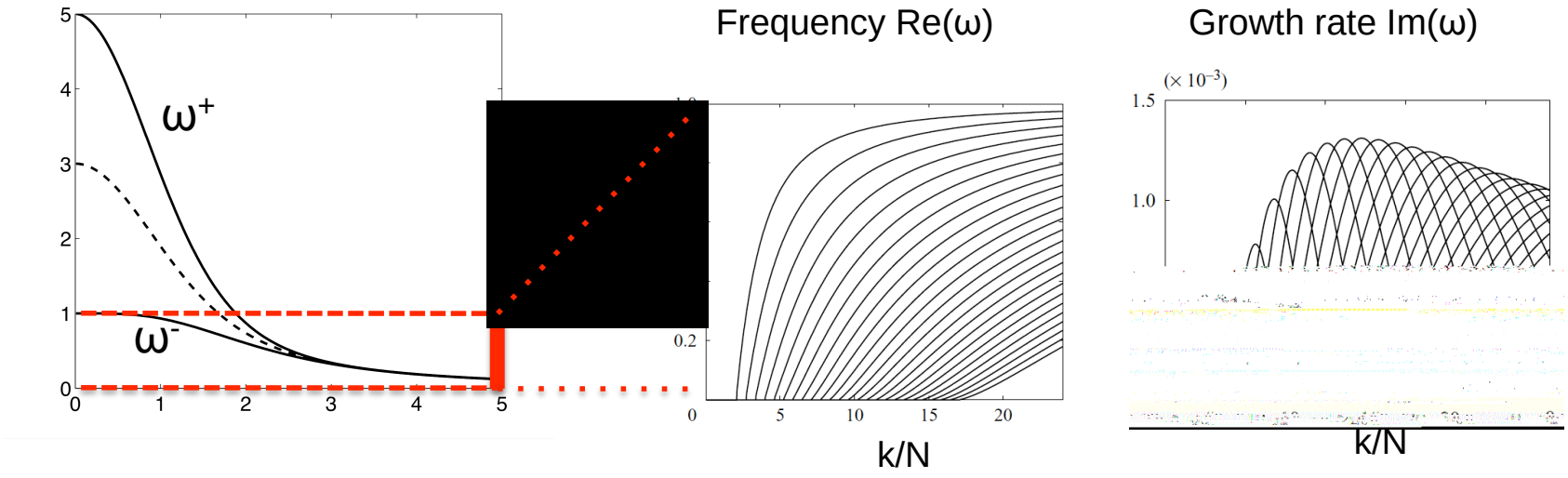
m=2

Frequency of radiative modes



m=3

Frequency of radiative modes



Radiative instability of the Lamb-Oseen vortex

Inviscid results for large N (small Froude)

Unstable modes for all m≠0

m=1 $0<\omega_r/\Omega_{max}<0.17$

m=2 $0<\omega_r/\Omega_{max}<0.4$

m≥3 $0<\omega_r/\Omega_{max}<m-2$

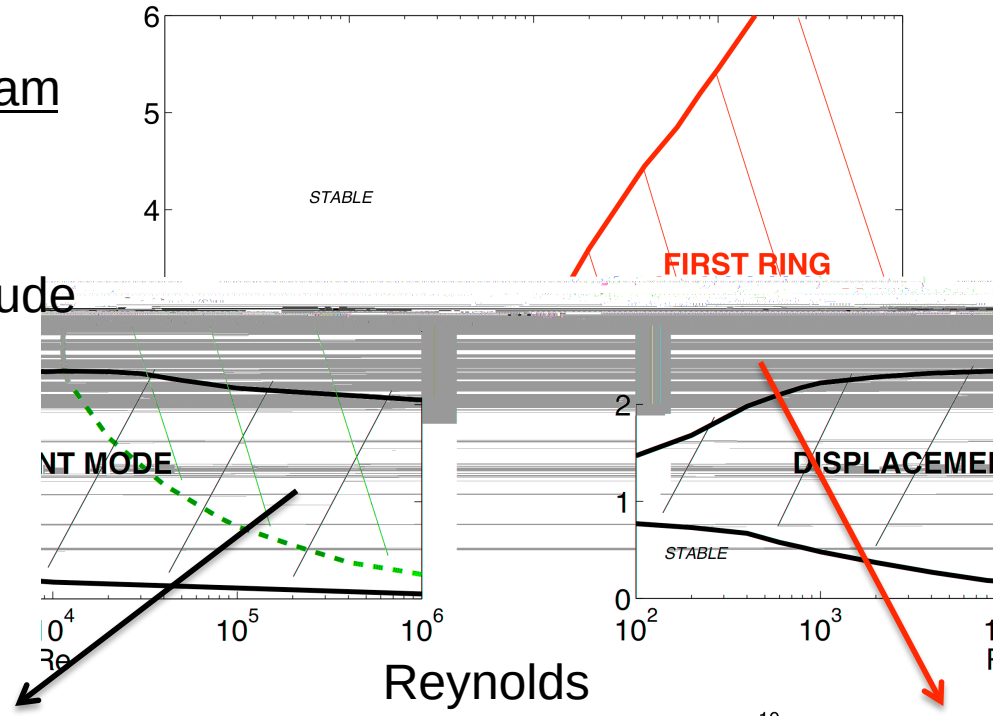
Most unstable mode:

Displacement mode (First branch for m=1) with k/N ≈2.5

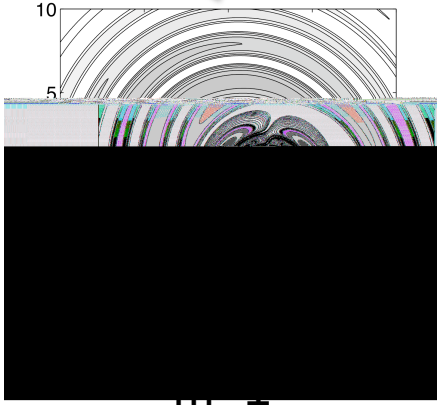
Radiative instability of the Lamb-Oseen vortex

Viscous results for any Froude F=! $_{max}$/N

Stability diagram

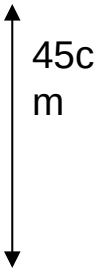


Perturbation pressure field



m=1

Experiment

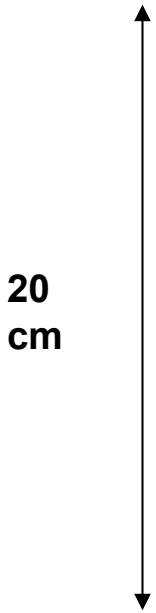


Set up:

- **Lamb-Oseen vortex generated by the rotation of a plate in linearly stratified salted water**
- **PIV measurement, dye and synthetic schlieren visualisation**

Vorticity field (PIV)

Dye visualisation



speed ×5

Conclusion on the radiative instability

- Wave emission can become destabilizing in a stratified vortex
- Same instability mechanism for vortices or boundary layer flows
    - in a compressible fluid: acoustic waves
    - in shallow water: surface waves
    - in a continuously stratified fluid: internal-gravity waves
- Experimental evidence

Conclusion on WKBJ description of vortex spectrum

- Good description of the vortex waves, with and without critical layer damping in a homogeneous fluid.

- Good description of the vortex waves in a stratified fluid, and of the radiative instability mechanism.

Thank you for your attention!