Linear Algebra meets Mechanics in Gyroscopic Systems

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The references

A general criterion concerning stability of gyroscopic systems has been widely discussed since the work of Thomson and Tait in the nineteenth century (see [4]). There is a recent formulation in a paper of Krechetnikov and Marsden [2].

Our main objective is to construct gyroscopic systems which appear to contradict this criterion.

In the process we provide new algebraic arguments for some general propositions to be found in the standard reference work of Merkin, [3].

REFERENCES:

[1] Gohberg I, Lancaster P, Rodman L, Matrix Polynomials, *Academic Press*, 1982, and SIAM, 2009.

[2] Krechetnikov R, and Marsden J E, *Dissipation-induced instabilities in finite dimensions*, Reviews of Modern Physics, **79**, (2007), 519-553.

[3] Merkin D R, Introduction to the Theory of Stability, *Springer*, New York, 1997.

[4] Thomson W (Lord Kelvin), and Tait P G, Treatise on Natural Philosophy, Part 1, *Cambridge University Press*, Cambridge, 1879 and 1921.

Algebraic/Mechanical systems:

A **system** is a set of constant coefficient differential equations:

$$M\ddot{q}(t) + (D+G)\dot{q}(t) + (K+L)q(t) = f(t),$$

where $t \in \mathbb{R}$ is the time, $f(t)$ and $q(t)$ take values in $\mathbb{R}^n$.
M, D, $K \in \mathbb{R}^{n\times n}$ are *symmetric*, $M > 0$. G, $L \in \mathbb{R}^{n\times n}$ are *skew-symmetric*:

$$D^T = D, \quad G^T = -G^T, \quad K^T = K, \quad L^T = -L.$$

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In general, $M > 0$, $D \geq 0$, $L = 0$. The system is **gyroscopic** if $G \neq 0$.
Associated *quadratic eigenvalue problem*:

$$\big(M\lambda^2 + (D+G)\lambda + (K+L)\big)x = 0$$

associated with the matrix function

$$\mathcal{L}(\lambda) := M\lambda^2 + (D+G)\lambda + (K+L), \qquad \lambda \in \mathbb{C}.$$

The spectrum of $\mathcal{L}(\lambda)$ is the set of all e.v. of $\mathcal{L}(\lambda)$, i.e.

$$\{\lambda \in \mathbb{C} : \det \mathcal{L}(\lambda) = 0\}.$$

$M > 0 \rightarrow$ spectrum a bounded set in the complex plane.

Definitions:

(a) $\mathcal{L}(\lambda)$ is stable if all e.v. are in the open left half of the complex plane. (This is sometimes referred to as "strong stability".)

(b) $\mathcal{L}(\lambda)$ is weakly stable if all e.v. are in the closed left half of the complex plane, there is at least one pure-imaginary e.v, and all such pure-imaginary e.v. are semisimple.

The term marginal stability refers to the extreme case of weak stability in which *all* e.v. are pure-imaginary and semisimple.

(c) $\mathcal{L}(\lambda)$ is unstable if it is neither stable nor weakly stable - so that there is at least one e.v. which is either in the open right half-plane, or is pure-imaginary and defective (and there may be several such "unstable" e.v.).

Semisimple?

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TEST: Given ev λ_0 we have $\mathcal{L}(\lambda_0)v = 0$ for some $v \neq 0$ (v is an *eigenvector*).

Define $\mathcal{L}^{(1)}(\lambda) = 2M\lambda + (D+G)$. If we have nonzero solutions $v,\ w$ for both

$$\mathcal{L}(\lambda_0)v = 0, \quad \text{and} \quad \mathcal{L}(\lambda_0)w = -\mathcal{L}^{(1)}(\lambda_0)v,$$

then λ_0 is not semisimple - and conversely.

Special case - no damping.

Define

$$\mathcal{L}(\lambda) := M\lambda^2 + G\lambda + K, \quad M > 0.$$

(a)**Hamiltonian symmetry:** The spectrum of $\mathcal{L}(\lambda)$ is symmetric with respect to both the real and imaginary axes. To verify this observe that

$$\mathcal{L}(\lambda)^* = \mathcal{L}(-\bar{\lambda}), \qquad \mathcal{L}(\lambda)^T = \mathcal{L}(-\lambda).$$

Thus, if λ is an ev so are $\bar{\lambda}$, $-\lambda$, and $-\bar{\lambda}$.
(b) A matrix A in $\mathbb{R}^{2n\times 2n}$ is said to be Hamiltonian if $(JA)^T = JA$ where

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

(c) A matrix A in $\mathbb{C}^{2n\times 2n}$ is said to be Hamiltonian if $(JA)^* = JA$.
The spectrum of a real Hamiltonian matrix is symmetric wrt *both* axes.

No damping - contd.

Now consider the sytem:

$$\mathcal{L}(\lambda) := M\lambda^2 + G\lambda + K, \quad M > 0, \quad K \geq 0.$$

Briefly, the following result says that this system is *marginally stable* if $K > 0$ and is *unstable* if $K \geq 0$ **and is singular**.

Theorem

For the spectrum of $\mathcal{L}(\lambda)$:
(a) *Nonzero evs are pure-imaginary and semisimple.*
(b) *There is an ev equal to zero iff $\det K = 0$ and, in this case, the zero ev is defective.*

Apply the TEST: The system is *marginally stable* if $K > 0$ and will be *unstable* only when $K \geq 0$ and is singular. Then the zero ev of $\mathcal{L}(\lambda)$ is defective.

No damping - contd.

Relax the (negative semidef.) condition on K and consider

$$\mathcal{L}(\lambda) := M\lambda^2 + G\lambda + K, \quad M > 0, \quad \det K < 0.$$

In mechanics (see [3]) the *degree of instability* of a system is the number of negative ev. of K. So $\det K < 0$ is equivalent to K *has an odd number of negative ev.*. We have Theorem 6.3 of [3]:

Theorem

If $\det K < 0$ then $\mathcal{L}(\lambda)$ is unstable.

Proof: Use the Hamiltonian symmetry: The product of conjugate pure imaginary ev is positive. So $\det K < 0$ implies that there is at least one ev which is not pure imag.. The Ham'n symmetry implies there must be at least one unstable ev.. $\square$

Relative sizes of G and K: admit $K < 0$

Theorem

(Hagedorn 1975) If $4K < -GG^T$ then there are no pure-imaginary ev.

Define $|G| = (G^T G)^{1/2} \geq 0$.

Theorem

(Barkwell/Lancaster, 1992) If $K < 0$ and

$$|G| \geq kI - k^{-1}K \quad \text{for some} \quad k > 0$$

then the system $I\lambda^2 + G\lambda + K$ is marginally stable.

Notice the double negative - the system is unstable if $G = 0$.

Gyroscopic systems with damping

$$\mathcal{L}(\lambda) := M\lambda^2 + (D+G)\lambda + K.$$

Recall result of p.10: When $D = 0$ system is *marginally stable* if $K > 0$ and is *unstable* if $K \geq 0$ and is singular.
What is the effect of adding damping term $D \geq 0$?

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(Theorems 6.4 and 6.5 of [3].)

Theorem

If $M > 0$, $K > 0$ and $D \geq 0$ then $\mathcal{L}(\lambda)$ above is either stable or weakly stable (whatever $G = -G^T$ may be).

Rather delicate proof uses our TEST to show that, when $K > 0$, all ev are in the closed left half-plane and pure-imaginary ev are semisimple.
NB: Can start with stable gyro system $M\lambda^2 + G\lambda + K$ - add damping, and stability is retained.

Getting ev off the imaginary axis

Theorem

(Wimmer, 1974, Lancaster/Tismenetsky, 1983) Let M, K, D be $n \times n$ Hermitian matrices with M and K nonsingular, $D \geq 0$ real, and

$$\mathcal{L}(\lambda) := M\lambda^2 + (D+G)\lambda + K.$$

Then $\pi(\mathcal{L}) \leq \nu(M) + \nu(K)$ *and* $\nu(\mathcal{L}) \leq \pi(M) + \pi(K)$. *If also*

$$\begin{bmatrix} 0 & KM^{-1} \\ I & GM^{-1} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ D \end{bmatrix}$$

is controllable, then $\mathcal{L}$ has no pure imaginary eigenvalues and

$$\pi(\mathcal{L}) = \nu(K) + \nu(M), \quad \nu(\mathcal{L}) = \pi(K) + \pi(M).$$

Corollary

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If $M > 0$, $K > 0$ and $D \geq 0$ then $\mathcal{L}$ is stable iff the controllability condition holds.

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A "MIXED" SYSTEM: Form a direct sum of two systems. THE FIRST:

$$\mathcal{L}_1(\lambda) = I_1\lambda^2 + G_1\lambda + K_1, \qquad K_1 < 0$$

with $G_1^T = -G_1$ and

$$|G_1| \geq kI_1 - k^{-1}K_1, \quad k > 0.$$

so that $\mathcal{L}_1(\lambda)$ is marginally stable.

The second system:

$$\mathcal{L}_2(\lambda) = I_2\lambda^2 + D_2\lambda + K_2,$$

with $D_2 > 0$, and $K_2 > 0$.
Now form the monic system

$$\mathcal{L}_0(\lambda) = I\lambda^2 + (D+G)\lambda + K := \begin{bmatrix} \mathcal{L}_1(\lambda) & 0 \\ 0 & \mathcal{L}_2(\lambda) \end{bmatrix}.$$

Since the component systems are at least weakly stable, so is the direct sum. For the coefficients we have:

$$D = \begin{bmatrix} 0 & 0 \\ 0 & D_2 \end{bmatrix} \geq 0, \quad G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} = -G^T, \quad K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$$

and recall $K_1 < 0$, $K_2 > 0$: we have "unstable potential energy".

A mixed system

By applying real congruence transformations to $\mathcal{L}_0(\lambda)$. we generate a family of "gyroscopically stabilized" *weakly stable systems* (all with *unstable potential energy*, $K_1 < 0$).

The damping $\begin{bmatrix} 0 & 0 \\ 0 & D_2 \end{bmatrix}$ can be "increased" without disturbing the stability (as long as the increase is confined to D_2).

An inconsistent statement?

A statement of Kretchetnikov and Marsden [2] (2007) which they associate with Thomson, Tait, and Chetayev:

A TTC Proposition: *If a system with an unstable potential energy is stabilized with gyroscopic forces, then this stability is lost after the addition of arbitrarily small dissipation*.

A counter-example

Example: The system

$$\mathcal{L}(\lambda) = I\lambda^2 + \begin{bmatrix} 0 & 3 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \lambda + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the direct sum of a 2×2 and a 1×1 system and is weakly stable. The system

$$\mathcal{L}_2(\lambda) = \begin{bmatrix} 6 & 4 & 6 \\ 4 & 3 & 5 \\ 6 & 5 & 11 \end{bmatrix} \lambda^2 + \begin{bmatrix} 2 & 5 & 9 \\ -1 & 2 & 6 \\ 3 & 6 & 18 \end{bmatrix} \lambda + \begin{bmatrix} -4 & -2 & 0 \\ -2 & -1 & 1 \\ 0 & 1 & 7 \end{bmatrix}$$

(with unstable P.E.) is *congruent* to $\mathcal{L}(\lambda)$. Increases in damping of the form

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 3 & 3 & 9 \end{bmatrix} \varepsilon \geq 0$$

do not disturb the equilibrium - i.e. the weak stability.

Time to go!