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Additive noise does not destroy a pitchfork bifurcation

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- What is a "bifurcation" for a random dynamical system?
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- Lyapunov exponent and finite-time Lyapunov exponents.
- Dichotomy spectrum.
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Bifurcation

- Bifurcation $\sim$ "Qualitative change in the dynamics."
- What signifies such a change and can we develop a mathematical theory (in analogy to deterministic setting)?
- History: [Arnold98] distinguishes *Phenomenological* (P) bifurcations, characterized by a change in the shape of the stationary measure, from *Dynamic* (D) bifurcations, characterized by a change in the Lyapunov exponent spectrum.

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Case study.

$$dx = (\alpha x - x^3)dt + \sigma dW_t \text{ with Wiener process } W_t$$

- If $\sigma = 0$: classical pitchfork bifurcation with exchange of stability from $x = 0$ to branches $x = \pm\sqrt{\alpha}$ when $\alpha > 0$.
- If $\sigma \neq 0$, a stationary distribution arises that changes shape when α increases through 0. ([Arnold] "P-bifurcation")
- [Crauel & Flandoli 1998] for all α
  - Strictly negative Lyapunov exponent ([Arnold] no "D-bifurcation")
  - Unique attracting random fixed point: "Additive noise destroys a pitchfork bifurcation."

SDE as Random Dynamical System

- $(\Omega, \mathcal{F}, \mathbb{P})$ probability space, with $\Omega = C_0(\mathbb{R}, \mathbb{R})$
- $\theta : \mathbb{R} \times \Omega \to \Omega$, $\theta_0 \omega = \omega$, $\theta_{t+s}\omega = \theta_t \theta_s \omega$.
- $\mathbb{P}\theta_t A = \mathbb{P}A$ (measure preserving)
- $\theta_t A = A \; \forall t \implies \mathbb{P}A \in \{0, 1\}$ (ergodicity)
- Skew product flow: $\Theta : \mathbb{R} \times \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}$ with $\Theta_t(\omega, x) = (\theta_t \omega, \phi(t, \omega)x)$.
- Invariant probability measure μ on $(\Omega \times \mathbb{R}, \mathcal{F} \times \mathcal{B})$; (i) $\mu(\Theta_t A) = \mu(A)$ and (ii) $\pi_\Omega \mu = \mathbb{P}$.
- Disintegration of μ: $\exists \{\mu_\omega\}_{\omega \in \Omega}$ prob meas on $(\mathbb{R}, \mathcal{B})$ such that $\mu(A) = \int_\Omega \int_\mathbb{R} \mathbb{1}_A(\omega, x) d\mu_\omega(x) d\mathbb{P}(\omega)$.

Random pitchfork analysis.

$$dx = (\alpha x - x^3)dt + \sigma dW_t$$

Arnold98, CF98: SDE has unique stationary measure $\rho(B) = \int_{\mathbb{R}} T(x, B) d\rho(x) \ \forall B \in \mathcal{B}(\mathbb{R})$ (where $T(x, B)$ denotes the transition probability of the induced Markov semi-group) with density $p_{\alpha,\sigma}(x) = N_{\alpha,\sigma} \exp(\frac{1}{\sigma^2}(\alpha x^2 - \frac{1}{4}x^4))$ corresponding to global random attractor $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ with invariant measure μ of the RDS with disintegration $\mu_\omega = \lim_{t\to\infty} \phi(t, \theta_{-t}\omega)\rho = \delta_{a_\alpha(\omega)}$ (random Dirac measure, i.e. random fixed point) and Lyapunov exponent $\lambda(\mu) = -\frac{2}{\sigma^2} \int_{\mathbb{R}} (\alpha x - x^3)^2 p_{\alpha,\sigma}(x) dx < 1$.[1]

[1] Lyapunov exponent: $\lambda = \lim_{t\to\pm\infty} \ln ||\Phi(t, \omega)x||$.

Qualitative change in the attractivity.

- $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ is locally uniformly attractive if $\exists\delta > 0$ such that

$$\lim_{t\to\infty} \sup_{x\in(-\delta,\delta)} \text{ess sup}_{\omega\in\Omega} |\phi(t,\omega)(a_\alpha(\omega) - x) - a_\alpha(\omega))| = 0$$

- **Theorem:** (i) If $\alpha < 0$, the random attractor $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ is locally uniformly attractive (even globally uniformly exponential attractive),
(ii) if If $\alpha > 0$, this is no longer the case.
- In fact $|\phi(t,\omega)(a_\alpha(\omega) - x) - a_\alpha(\omega))| \leq K(\omega)\exp(-\lambda t)x$, where $K(\omega) < \hat{K} < \infty$ iff $\alpha < 0$.

Finite-time Lyapunov exponents.

- $\lambda_\alpha(T,\omega) := \frac{1}{T} \ln \left| \frac{\partial \phi_\alpha}{\partial x}(T, \omega, a_\alpha(\omega)) \right|$. (random variable!)
- Lyapunov exponent is $\lambda_\alpha := \lim_{T\to\infty} \lambda_\alpha(T,\omega)$.
- **Theorem:** (i) If $\alpha < 0$, the random attractor is finite-time attractive: $\lambda_\alpha(\omega) \leq \alpha < 0$. (ii) If $\alpha < 0$, the random attractor is not finite-time attractive and $\mathbb{P}\{\omega \in \Omega : \lambda_\alpha(T,\omega) > 0\} > 0$.
- **Corollary:** The (negative) Lyapunov exponent can only be observed "almost surely" in finite time, if $\alpha < 0$.

Lyapunov spectrum

- Linear RDS in $\mathbb{R}^N$:
  $\phi(t,\omega)(ax_1 + bx_2) = a\phi(t,\omega)x_1 + b\phi(t,\omega)x_2$.
  Denoted as $\Phi : \mathbb{R} \times \Omega \to \mathbb{R}^{N\times N}$.
- Osceledets: (under mild assumptions) $\exists k$ Lyapunov exponents $\lambda_1 < \lambda_2 < \ldots < \lambda_k$ and $\mathbb{R}^N = W_1(\omega) \oplus \ldots W_k(\omega)$ so that $\lambda_i := \lim_{t\to\pm\infty} \frac{1}{|t|} \ln ||\Phi(t,\omega)||$ for $0 \neq x \in W_i(\omega)$.
- But we have just seen that "bifurcation" is not necessarily associated with a change of stability in the Lyapunov spectrum.
- We claim that a better concept for this purpose is the Dichotomy spectrum

Dichotomy spectrum

- Definition: (θ, Φ) has an exponential dichotomy wrt growth rate $\gamma \in \mathbb{R}$ if there exists a splitting $\mathbb{R}^N = S(\omega) \oplus U(\omega)$, measurable and invariant ($\Phi(t,\omega)S(\omega) = S(\theta_t \omega)$, etc), satisfying for some $K, \varepsilon > 0$
  $||\Phi(t,\omega)x|| \leq Ke^{(\gamma-\varepsilon)t}||x||$, for all $t \geq 0$ n $x \in S(\omega)$.
  $||\Phi(t,\omega)x|| \geq K^{-1}e^{(\gamma+\varepsilon)t}||x||$, for all $t \geq 0$, $x \in U(\omega)$.
- Dichotomy spectrum $\Sigma := \mathbb{R} \setminus \bigcup_{\text{growth rates } \gamma} \{\gamma\}$.
- **Spectral Theorem:** $\Sigma = I_1 \cup \ldots \cup I_k$ with $I_i = \{W_i(\omega)\}_{\omega \in \Omega}$ and corresponding decomposition $\mathbb{R}^N = W_1(\omega) \oplus \ldots \cup W_k(\omega)$.
- In the pitchfork example, $\Sigma = (-\infty, \alpha]$, so that the random pitchfork bifurcation corresponds to a loss of hyperbolicity of the Dichotomy spectrum.

Topological versus uniform topological equivalence.

- RDSs $\phi_1(t,\omega)$ and $\phi_2(t,\omega)$ are topologically conjugate iff $\exists$ homeomorphism $h : \Omega \times \mathbb{R} \to \mathbb{R}$ so that for all $\omega \in \Omega$, $\phi_2(t,\omega)h(\omega,x) = h(\theta_t\omega, \phi_1(t,\omega)x)$ for all t, x.
- **Theorem:** For the pitchfork example all ϕ_α are topologically equivalent.
- **Theorem:** A topological conjugacy h from ϕ_α to $\phi_{\alpha'}$ with $\text{sgn}(\alpha) = -\text{sgn}(\alpha')$ cannot be uniformly continuous.
  Proof: uniformly continuous conjugacies preserve local uniform attractivity.

Main result and some questions:

- Additive noise does not destroy a random pitchfork bifurcation. (cf [CF98])

- Is a change in the signature of the Dichotomy Spectrum a good indicator for bifurcation of RDS?
- Is uniform topological equivalence a suitable equivalence relation to define the notion of bifurcation in RDS?

References

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