Graphical Krein Signature and its Applications

Richard Kollár
Comenius University Bratislava

Joint work with **Peter Miller** (U Michigan)

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Spectral Stability

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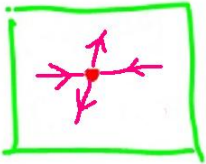
Linearized dynamics identifies possible unstable directions



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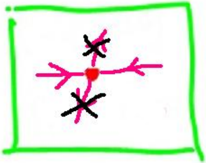
For constrained minimimizers motion in some directions may be prohibited by an additional conserved quantity



Linearized Hamiltonian Problems

Linearized Hamiltonian Problem

A Hamiltonian system linearized about its equilibrium has the form

$$JLu = \nu u\,, \qquad J = -J^*, L = L^*\,.$$

Typically L has a finite number of negative points in its spectrum

$$\sigma(L) = \{\sigma_1 < \sigma_2 < \cdots < \sigma_n < 0 < \sigma_{n+1} < \ldots\}\,.$$

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Linearized Energy

The operator L defines an indefinite linearized energy (u, Lu). The sign of the energy for the (simple) characteristic value ν is called the Krein signature

$$\kappa_L(\nu) = \text{sign}\,(u, Lu)\,.$$

Reformulation

Generalized Characteristic Value Problem

Let assume J is invertible, $K = (iJ)^{-1}$, $\lambda = i\nu$. Then $JLu = \nu u$ reduces to

$$Lu - \lambda K u = 0\,, \qquad \text{and} \quad (u, Lu) = \lambda\,(u, Ku)\,.$$

We define the Krein signature as

$$\kappa(\lambda) = \kappa(\nu) := \kappa_K(\nu) = \text{sign}\,(u, Ku)\,.$$

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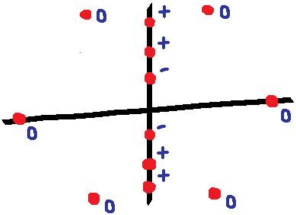
Non-Simple Characteristic Values

If λ is a non-simple characteristic value with the root space U, then the number of positive (negative) eigenvalues of the matrix (U, KU) is the positive (negative) Krein index $\kappa^{\pm}(\lambda)$ of λ. Then the Krein signature of λ can be defined as

$$\kappa(\lambda) = \kappa^{+}(\lambda) - \kappa^{-}(\lambda)\,.$$

Basic Properties of Krein Signature



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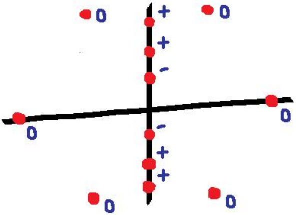

Properties of Krein Signature

Let ν is a simple characteristic value of $JLu = \nu u$. Then

- if $\nu \in i\mathbb{R}$ then $\kappa(\nu) = \pm 1$;
- if $\mathrm{Re}\,\nu \neq 0$ then $\kappa(\nu) = 0$;
- if L is positive definite then $\sigma(JL) \subset i\mathbb{R}$.

If ν is not semi-simple then both $\kappa^{\pm}(\nu)$ are non-zero. For each chain of root vectors the difference $\kappa^{+} - \kappa^{-} \in \{-1, 0, 1\}$.

Operator Pencils

Nonlinear Characteristic Value Problems

$$\mathcal{L}(\lambda)u = 0.$$

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Krein Signature

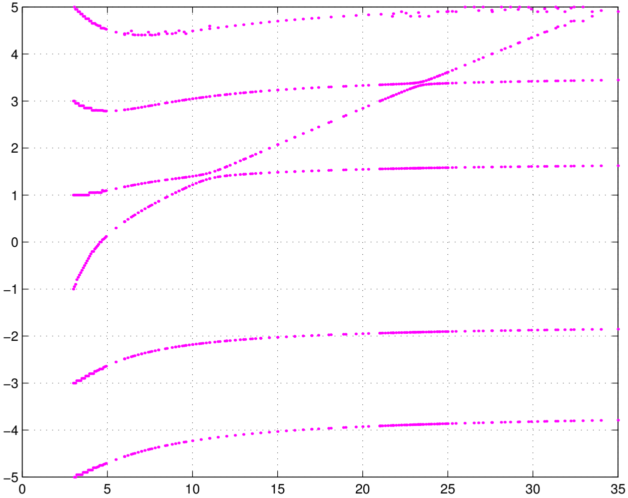
Analogously one can define Krein indeces and signature of polynomial operator pencils (by extention from X to X^n):

$$\mathcal{L}(\lambda)u = (\lambda^n L_n + \lambda^{n-1} L_{n-1} \cdots + L_0)u = 0\,.$$

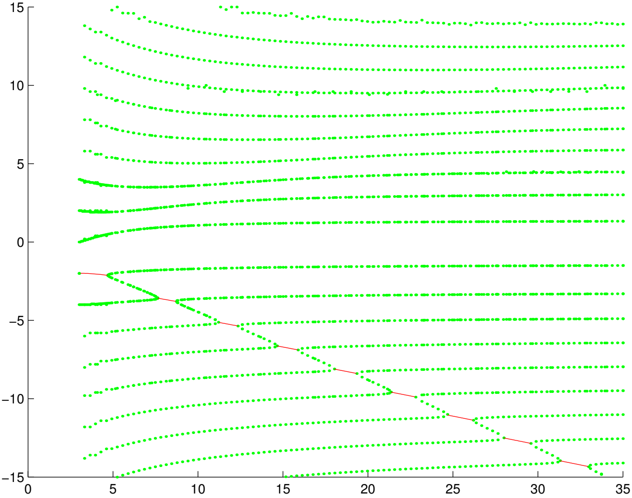
Such a construction fails for nonpolynomial pencils (e.g., stability of solutions of $\dot{x}(t) = Ax(t) + Bx(t-\tau)$)

$$\mathcal{L}(\lambda)u = \left(\lambda - A - e^{-\tau\lambda}B\right)u = 0\,.$$

Example: Avoided Collisions



Example: Hamiltonian-Hopf Bifurcation



Extention of the Problem

$$Lu - \lambda Ku = \mu u\,.$$

If $\mu(\lambda_0) = 0$, then λ_0 is a real characteristic value. The same method also applies to general operator pencils $\mathcal{L}(\lambda)u = \mu u$.

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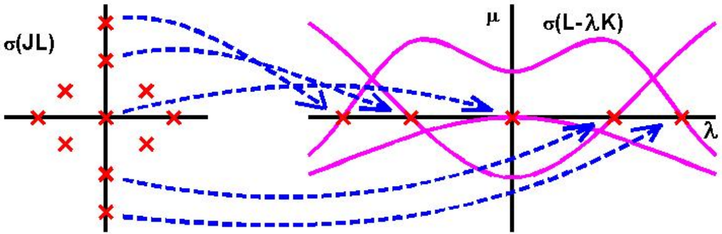

Graphical Interpretation of Multiplicity

Classical Theorem

Let $\mathcal{L}$ be a selfadjoint holomorphic family of type (A) with compact resolvent, and assume that $\mathcal{L}$ has an isolated real characteristic value λ_0. Then the following properties are equivalent:

(a) λ_0 has finite algebraic multiplicity m and geometric multiplicity 1 with a chain of root vectors $\left\{u^{[0]}, \dots, u^{[m-1]}\right\}$.

(b) There exist an analytic eigenvalue branch $\mu = \mu(\lambda)$, vanishing at $\lambda = \lambda_0$ to order m: $\mu^{(k)}(\lambda_0) = 0$ for $0 \leq k < m$, while $\mu^{(m)}(\lambda_0) \neq 0$. The derivatives of the corresponding orthonormal analytic eigenvector branch $u = u(\lambda)$ allow to select the chain of root vectors as

$$u^{[k]} = \frac{1}{k!}\frac{d^k u}{d\lambda^k}(\lambda_0), \quad k = 0, 1, \dots, m-1.$$

Graphical Krein Signature

Differentiation

Differentiate with respect to λ:

$$(L - \lambda K - \mu)u = 0, \qquad \lambda = \lambda_0, \mu = \mu(\lambda_0) = 0\,.$$

$$(L - \lambda K - \mu)'u + (L - \lambda K - \mu)u' = 0\,.$$

$$\big((-K - \mu')u, u\big) + \big((L - \lambda K - \mu)u', u\big) = 0\,.$$

$$\kappa_K(\lambda_0) = \mathrm{sign}(Ku, u) = -\mathrm{sign}\,\mu'(\lambda_0)(u, u)$$
$$= -\mathrm{sign}\,\mu'(\lambda_0)\,.$$

Definition

Let $\mathcal{L}(\lambda)$ be a self-adjoint holomorphic family of type (A) with compact resolvent, and let λ_0 be its isolated real characteristic value of geometric multiplicity 1. Let $\mu = \mu(\lambda)$ be a real analytic eigenvalue branch vanishing on the order *m*, i.e., $\mu^{(m)}(\lambda_0) \neq 0$. Then

$$\kappa_G(\lambda_0) := \begin{cases} -\operatorname{sgn} \mu^{(m)}(\lambda_0) & \text{for } m \text{ odd,} \\ 0 & \text{for } m \text{ even.} \end{cases}$$

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Theorem: Agreement of Signatures

$$\kappa_K(\lambda_0) = \kappa_G(\lambda_0)\,.$$

Evans Function

Spectrum Detecting Function

Let $D(\lambda) : \mathbb{C} \to \mathbb{C}$ is a continuous function such that $D(\lambda_0) = 0$ if and only if λ_0 is a characteristic value of $\mathcal{L}(\lambda)u = 0$ and the multiplicties agree (e.g. $D(\lambda) = \det \mathcal{L}(\lambda)$ for matrices). We call such spectra detecting function the Evans function.

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Typical Construction

$$y' = B(x, \lambda)y \, .$$

where the $n \times n$ system has an asymptotic exponential dichotomy: k-dimensional unstable space at $x = -\infty$ and $(n-k)$-dimensional stable space at $x = \infty$.

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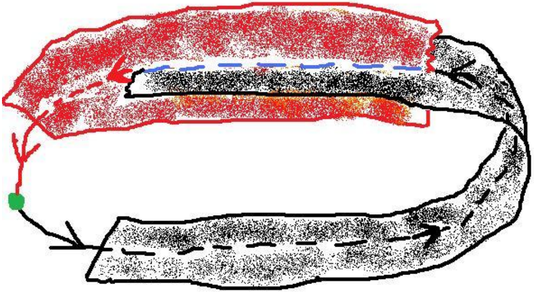
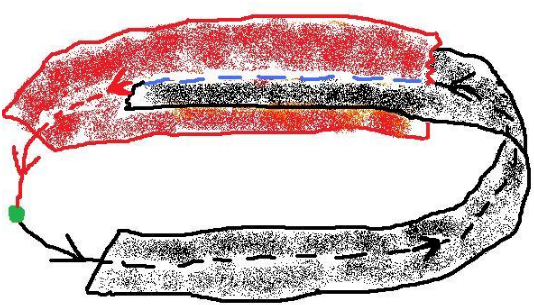
For which λ do these spaces intersect?

Evans Function



Evans Function



Wronskian [Evans (1974), AGJ (1990)]

$$E(\lambda) = a(x)\,\det(W^u_{-\infty}(x,\lambda), W^s_{\infty}(x,\lambda)) = 0\,.$$

Properties of the Evans function

Evans function

- Zeros of $D(\lambda)$ with $\operatorname{Im}\lambda \geq 0$ are the char. values of iJL;
- the symmetry of $iJLu = \lambda u$ implies $D(\lambda) \in \mathbb{R}$ for $\lambda \in \mathbb{R}$.

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Evans-Krein Function

$E(\lambda, \mu)$ is any spectrum detecting function of $(\mathcal{L}\lambda - \mu\mathbb{I})u = 0$.

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Evans-Krein Function

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Mutual Relation (Same Construction)

$$D(\lambda) = E(\lambda, 0)\,.$$

Krein Signature from Evans Function

Formula for Krein Signature

By differentiating $E(\lambda, \mu(\lambda))$ by λ at a simple characteristic value $\lambda = \lambda_0$ and the eigenvalue $\mu(\lambda) = 0$ along a particular branch $\mu(\lambda)$ we obtain

$$E_\lambda(\lambda_0, 0) + E_\mu(\lambda_0, 0)\mu'(\lambda_0) = 0\,.$$

For a simple characteristic value λ_0 also $E_\mu(\lambda_0, 0) \neq 0$:

$$\kappa(\lambda_0) = -\mathrm{sign}\,\mu'(\lambda_0) = \mathrm{sign}\,\frac{E_\lambda(\lambda_0, 0)}{E_\mu(\lambda_0, 0)}\,.$$

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Krein Signature Forumula

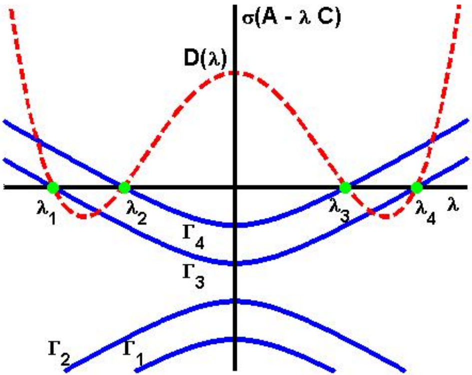
$$\kappa(\lambda_0) = \mathrm{sign}\,\frac{D'(\lambda)}{E_\mu(\lambda_0, 0)}\,.$$

Krein Signature from Evans Function

Advantages

- Preserved dichotomy;
- The same construction as the traditional Evans function;
- Minimal changes to existing codes;
- Easy to calculate;
- Only continutity of spectrum necessary (for simple eigenvalues).

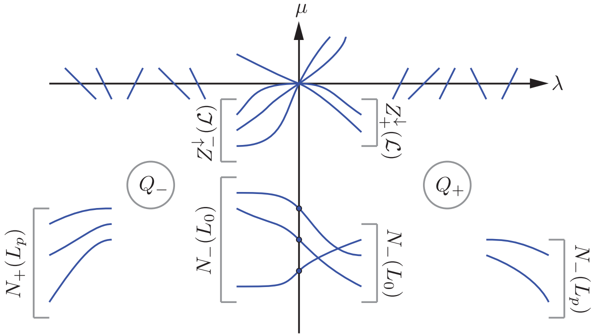
Comparison of Evans functions



Graphical Count

Let $\mathcal{L}(\lambda)$ be a selfadjoint polynomial matrix pencil of odd degree $p = 2\ell + 1$ acting on $X = \mathbb{C}^N$. Then

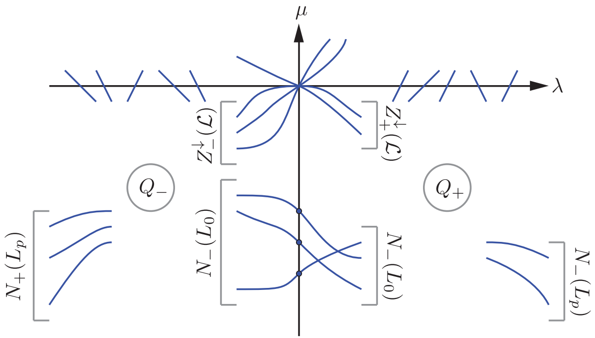
$$N - 2N_-(L_0) - Z_+^{\downarrow}(\mathcal{L}) - Z_-^{\downarrow}(\mathcal{L}) - \sum_{\lambda>0} \kappa(\lambda) + \sum_{\lambda<0} \kappa(\lambda) = 0.$$



Graphical Count

Also, the following inequalities hold true:

$$N_{\pm}(\mathcal{L}) \geq \left| N_{-}(L_0) + Z_{\pm}^{\downarrow}(\mathcal{L}) - N_{\mp}(L_p) \right| .$$



Corollaries of Graphical Index Theorem

Corollaries

The generalization for unbounded operators is sometimes straightfoward but sometimes requires technical tricks.

- Vakhitov-Kolokolov ['73], Grillakis-Shatah-Strauss ['87], Binding-Browne ['88], Kapitula-Kevrekidis-Sandstede ['04], Pelinovsky ['04]:

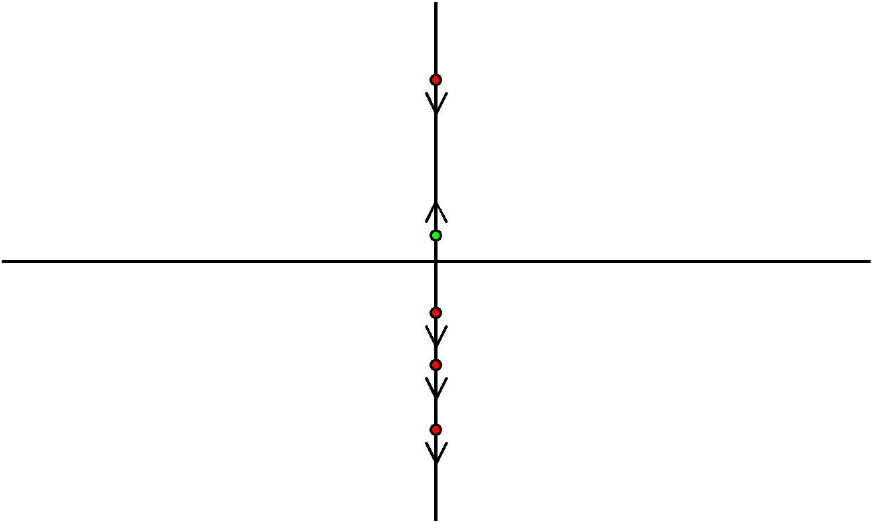
$$N_r + 2N_c + 2N_i^- = n(L) - n(D)\,,$$

- Grillakis ['88], Jones ['88]:

$$\frac{1}{2} N_{\mathbb{R}}(JL) \geq |N_-(M_+) - N_-(M_-)|\,, \qquad M_\pm := PL_\pm P.$$

- Various counts for quadratic eigenvalue pencils (Chugunova & Pelinovsky ['10]).

Hamiltonian-Hopf Bifurcations



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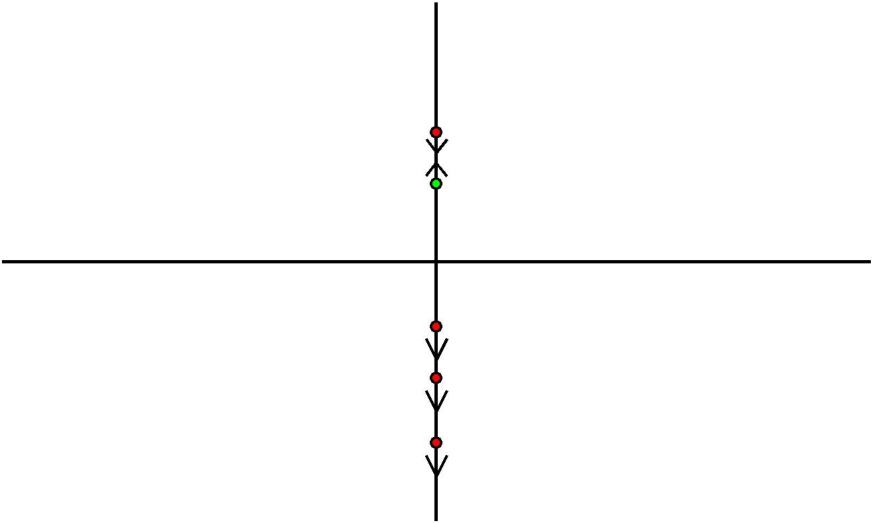

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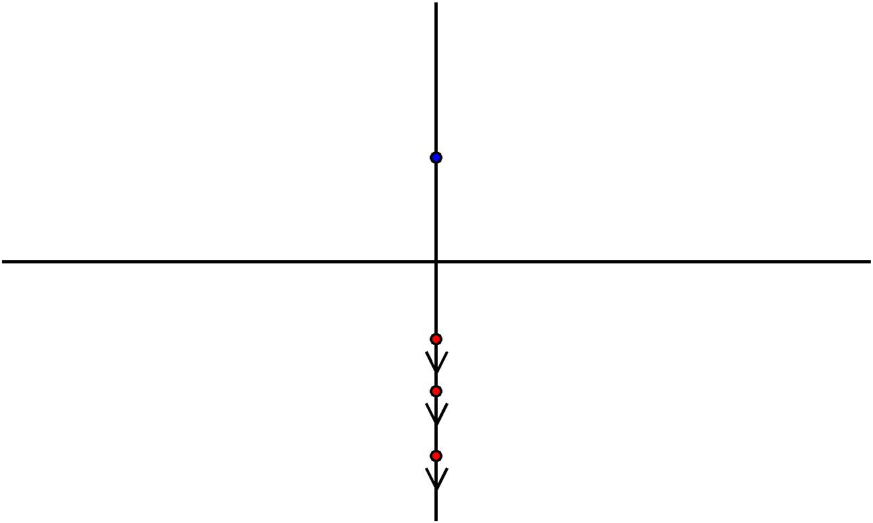

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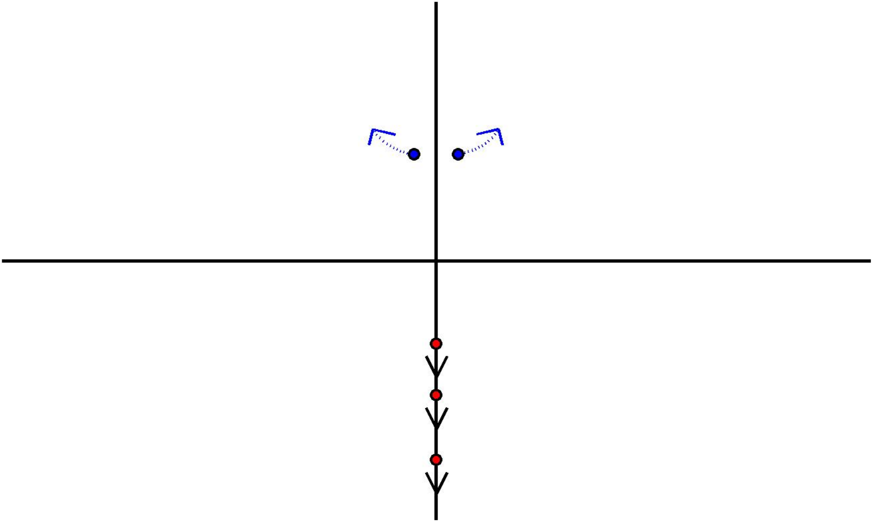

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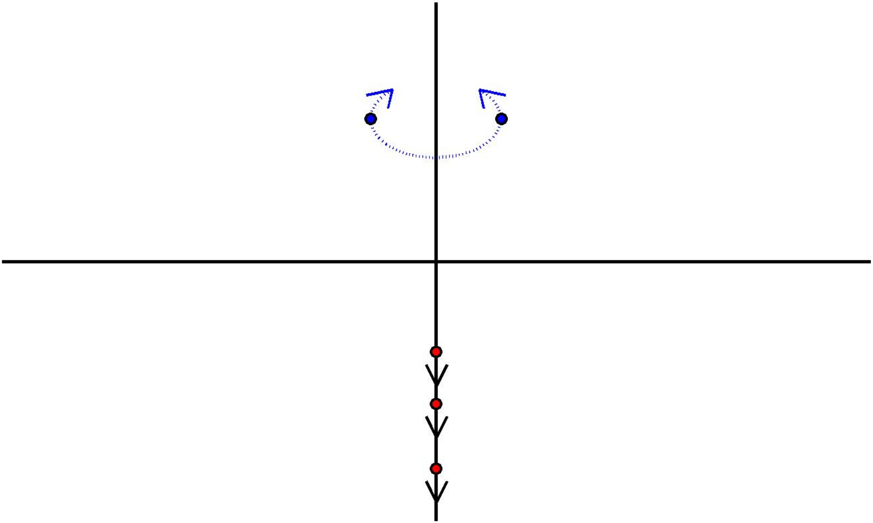

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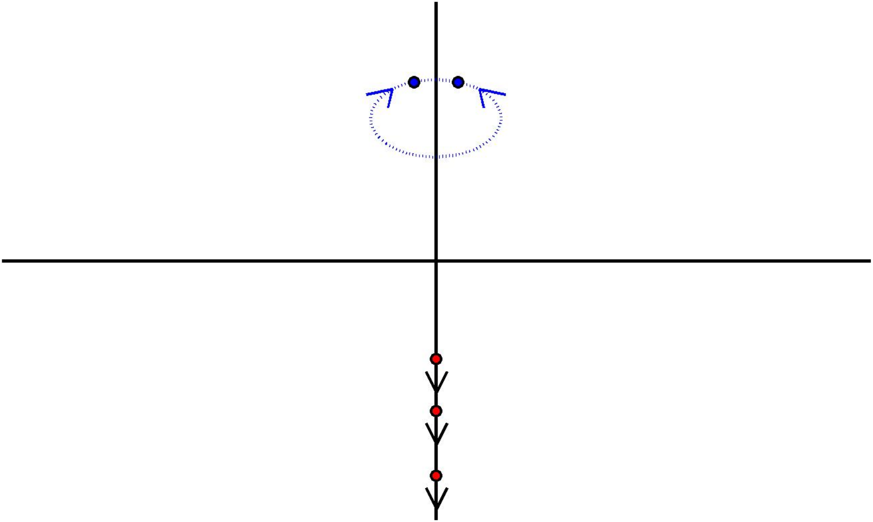

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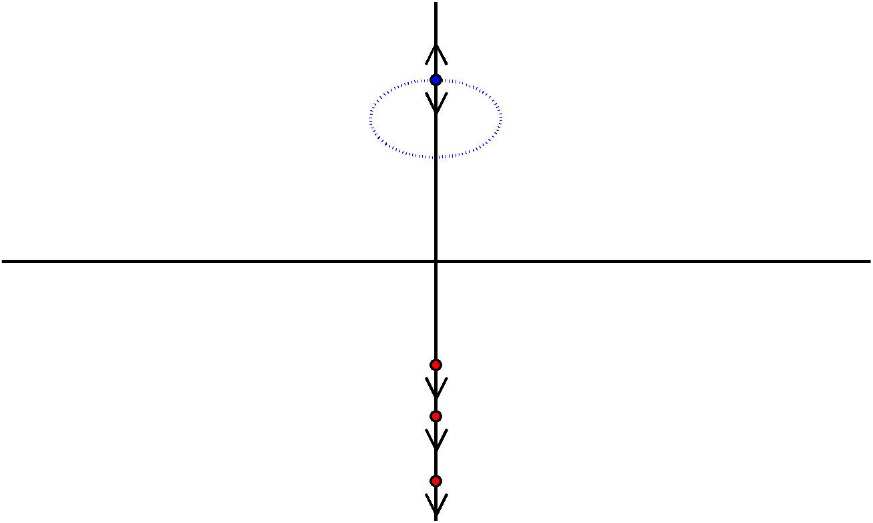

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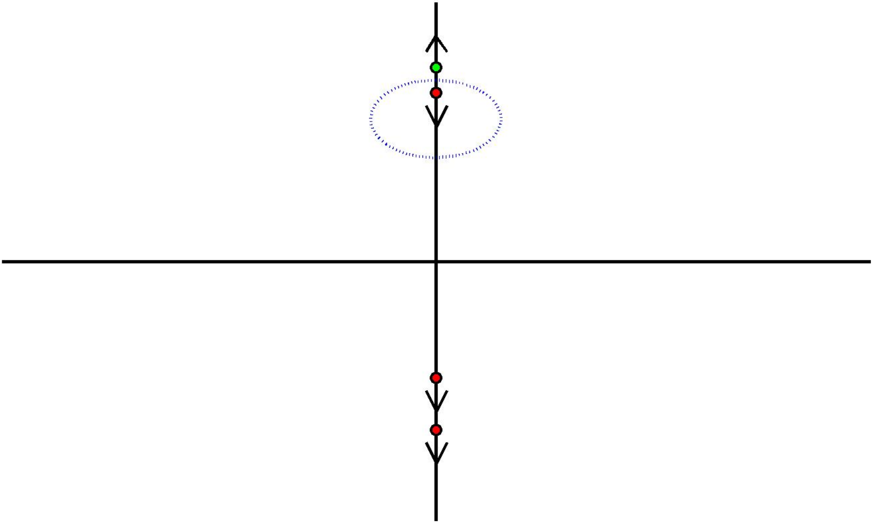

Hamiltonian-Hopf Bifurcations

Main Question

Can the extra information on Krein signature help to predict Hamiltonian-Hopf bifurcations?

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Necessary Condition

Mixed signature of eigenvalues is a necessary condition for a Hamiltonian-Hopf bifurcation (a Krein collision). [Gelfand & Lidskii (1955), Arnold & Avez (1968), Yakubovitch & Starzhinskii (1975)]

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Sufficient Condition

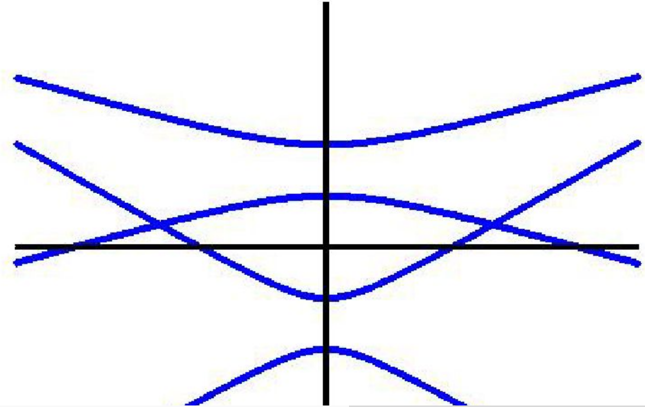
What is the sufficient condition?

Preservation of Branch Crossings

Perturbed system

Is there a Hamiltonian-Hopf bifurcation if one perturbes the problem

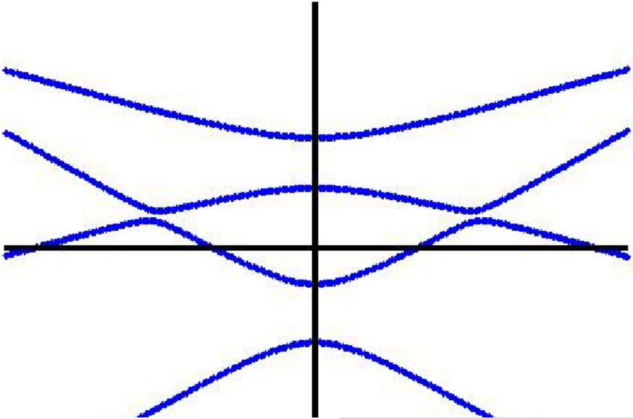
$$(L + tL_1 - \lambda K)u = 0 = \mu(\lambda)\,?$$



Arbitrary Perturbations

Generic Case

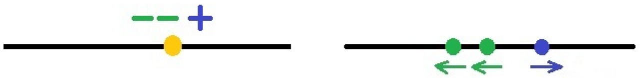
Two close eigenvalues of opposite Krein signature generically undergo an Hamiltonian-Hopf bifurcation. [MacKay & Saffman (1986)]



Positive Perturbations

Periodic Systems

If L_1 is positive (or negative) definite then an Hamiltonian-Hopf bifurcation is avoided, i.e,. an eigenvalue of any higher multiplicity unfolds according to Krein signatures of colliding eigenvalues [Krein & Ljubarskii (1970)].



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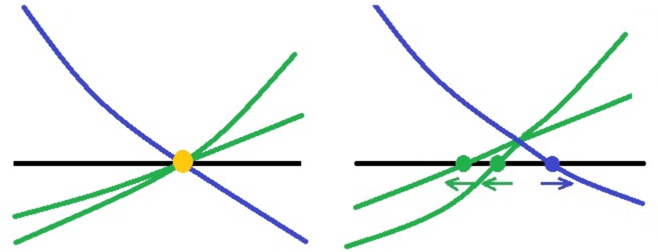
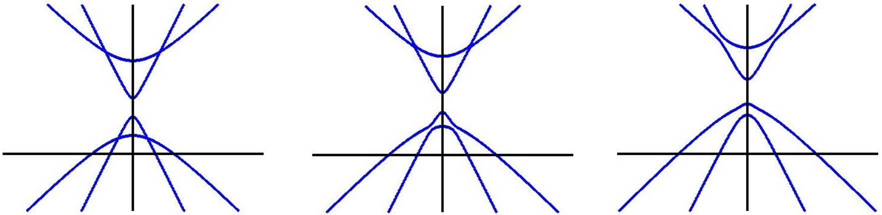


Preservation of Intersections

Surprise

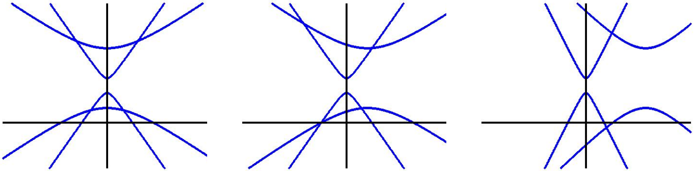
Crossings of eigenvalue branches under a positive perturbation do not need to be preserved!



Preservation of Intersections

Preservation of an intersection of eigenvalue branches $\mu(\lambda)$

- a very singular case of implicit function theory;
- it requires an infinite set of conditions to be met;
- but it is common in simple examples.



Sufficient Condition

Sparse Matrices

The intersection of two eigenvalue branches $\mu(\lambda)$ of

$$\mathcal{L}(\lambda, t) = L + tL_1 - \lambda K \qquad \text{at } t = 0, \mu = \mu_0,\ \lambda = \lambda_0$$

is preserved for small $t \neq 0$ **if**

$$L - \mu_0 \mathbb{I} = UDU^\dagger, \qquad D \text{ is a diagonal matrix,}$$

and

$$U^\dagger K U = \begin{pmatrix} * & 0 & 0 & * & 0 & 0 \\ 0 & * & 0 & 0 & 0 & * \\ 0 & 0 & * & 0 & * & 0 \\ * & 0 & 0 & * & 0 & 0 \\ 0 & 0 & * & 0 & * & 0 \\ 0 & * & 0 & 0 & 0 & * \end{pmatrix}, \ U^\dagger L_1 U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & * & 0 \\ 0 & * & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & * \\ 0 & 0 & * & 0 & * & 0 \end{pmatrix}$$

Preservation of Intersections

Necessary Condition

The intersection of two eigenvalue branches $\mu(\lambda)$ of $\mathcal{L}(\lambda, t)$ at $t = 0$ is preserved for small $t \neq 0$ **only if**

$$k_{12}(\ell_{11} - \ell_{22}) = \ell_{12}(k_{11} - k_{22}) \,,$$

where

$$k_{ij} = u_i^\dagger K u_j \,, \qquad \ell_{ij} = u_i^\dagger L_1 u_j \,,$$

where Ker $(L_0 - \lambda_0 K) = \text{span}\,\{u_1, u_2\}$.

The condition is equivalent to vanishing of the Hessian:

$$\det\left(D_t^2 \det(L_0 - \lambda K + tL_1 - \mu \mathbb{I})\right) = 0, \quad t = 0, \lambda = \lambda_0, \mu = \mu_0 \,.$$

Conclusions

- A geometric interpretation of Krein signature — graphical Krein signature (generalizes beyond the scope of polynomial pencils).
- Introduction of the Evans-Krein function: allows to calculate Krein signature directly.
- Unified geometric interpretation of index theorems.
- A new mechanism for avoidance of Hamiltonian-Hopf bifurcations (necessary, necessary and sufficient, and various typical classes of sufficient conditions).

Quadratic Characteristic Value Problem

Quadratic Characteristic Value Problem

Find $\lambda \in \mathbb{C}$ such that

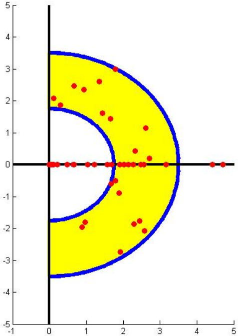
$$x = \lambda Bx + \frac{1}{\lambda} Cx$$

admits a nonzero solution on a Hilbert space X.

Assumptions

- B and C are compact self-adjoint operators on X;
- B is positive;
- C is non-negative with both infinitely dimensional kernel and range.

Main Theorem



(i) $\mathrm{Re}\,\lambda > 0$.

(ii) $\mathrm{Im}\,\lambda \neq 0$, then $\frac{1}{2\|B\|} < |\lambda| < 2\|C\|$.

(iii) Zero is the only possible accumulation point.

(iv) Infinite sequence of real $\lambda \to 0$, and an infinite sequence of real $\lambda \to \infty$.

Intuition

$$x = \lambda Bx + (1/\lambda)Cx\,.$$

If $0 < \lambda \ll 1$ then λBx is very small

$$x \approx (1/\lambda)Cx\,, \qquad \text{or} \qquad \lambda x \approx Cx\,.$$

Similarly for $\lambda \gg 1$ the term $(1/\lambda)Cx$ is small

$$(1/\lambda)x \approx Bx\,.$$

Hence one expects

- a sequence $\lambda \to 0$ due to the spectrum of the operator C (stratification);
- a sequence $\lambda \to \infty$ due to the spectrum of the operator B (dissipation).

Previous Results

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- Extend the problem to a space $X \times X$, substitute $\mu = \lambda - \frac{1}{\lambda}$ and reformulate the problem as

$$Az = \mu z\,.$$

- The operator A is not self-adjoint, only if it is considered in an appropriate indefinite metric space (similar to linearized Hamiltonian systems $JLu = \nu u$).
- One needs a theory on spectra of self-adjoint operators in indefinite metric spaces.
- To relate the spectrum of A to spectrum of non-linear characteristic problem, mini-max estimates were used.

Continuation of Characteristic Values

Perturbation Argument

Consider $\lambda \ll 1$:

$$\lambda u = Cu + \lambda^2 Bu = (C + \lambda^2 B)u \,.$$

The operator

$$C + \lambda^2 B \approx C \,.$$

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Solution: Introduce a new small parameter ε into a problem.

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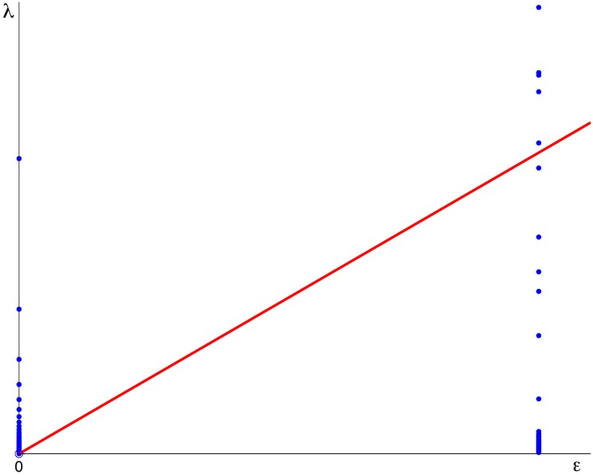
Spectrum of $C + \varepsilon^2 B$

- Spectrum $\sigma(C) = \{0, \lambda_1^0, \lambda_2^0, \dots;\ \ \lambda_1^0 \geq \lambda_2^0 > \dots > 0\}$;
- Spectrum $\sigma(C + \varepsilon^2 B) = \{\lambda_1^\varepsilon, \lambda_2^\varepsilon, \dots;\ \ \lambda_1^\varepsilon > \lambda_2^\varepsilon > \dots > 0\}$;
- Individual eigenvalues of $C + \varepsilon^2 B$ (a compact self-adjoint family) are continuous in ε [Kato 1976].
- Real eigenvalues $\lambda = \varepsilon$ correspond to real characteristic values of non-linear problem.

Spectrum of $C + \varepsilon^2 B$



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