ZIEGLER-BOTTEMA DISSIPATION-INDUCED INSTABILITY AND RELATED TOPICS

BIRS, Banff, Canada. November 7, 2012

Oleg Kirillov

1878 Kelvin

stability of rotating ellipsoidal shells containing fluid

1883 Greenhill

buckling of a screw-shaft of a steamer

1927 Nicolai

buckling and flutter of shafts under compression and (also follower) torque

1952 Ziegler

flutter of rods under follower force, **destabilization paradox due to small dissipation**

1880 Greenhill

prolate shells unstable, oblate stable

1942 Sobolev

instability of chemical artillery shells

1944 Pontryagin, 1950 Krein

Hilbert space with indefinite metric, Krein collision, Krein signature

1960 Sturrock

Dissipation-induced instability of negative energy modes

?

Mitglied der Helmholtz-Gemeinschaft

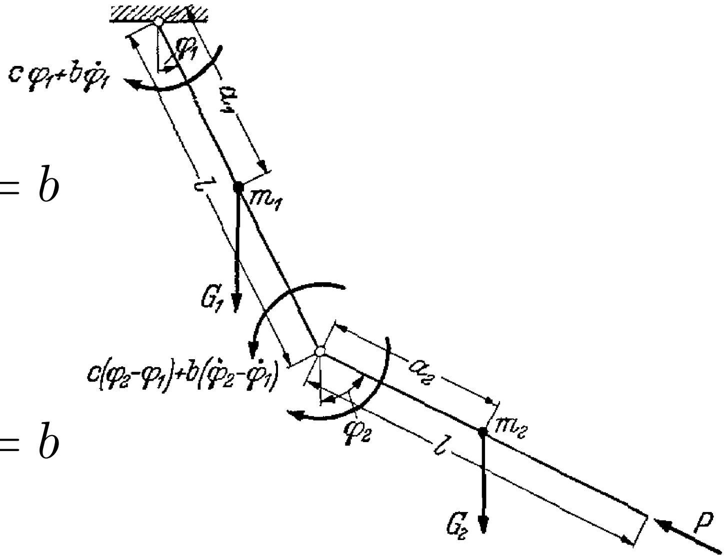
Dr. Oleg N. Kirillov | Magneto-Hydrodynamics (FWDH) | http://www.hzdr.de

1952 Ziegler
destabilization paradox

- Ziegler's pendulum

$$b_1 = b$$

$$b_2 = b$$



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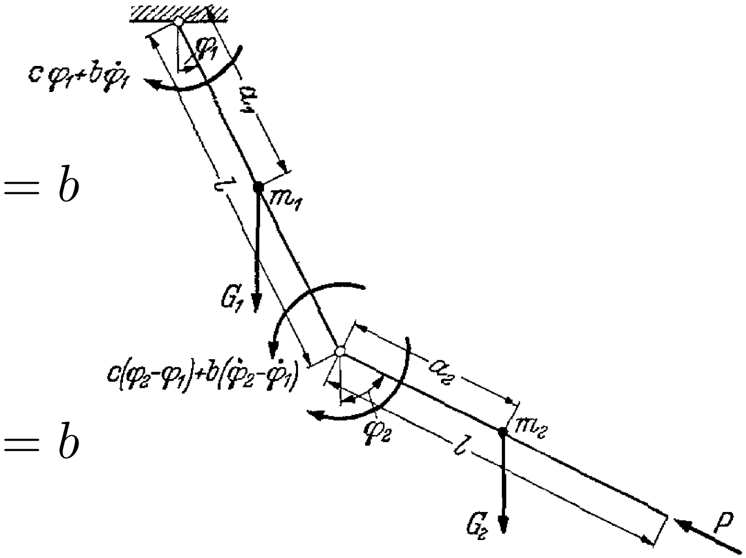

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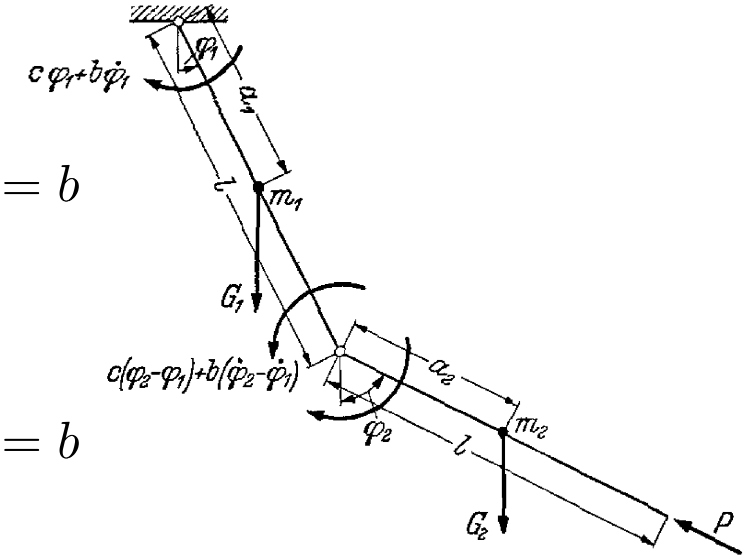

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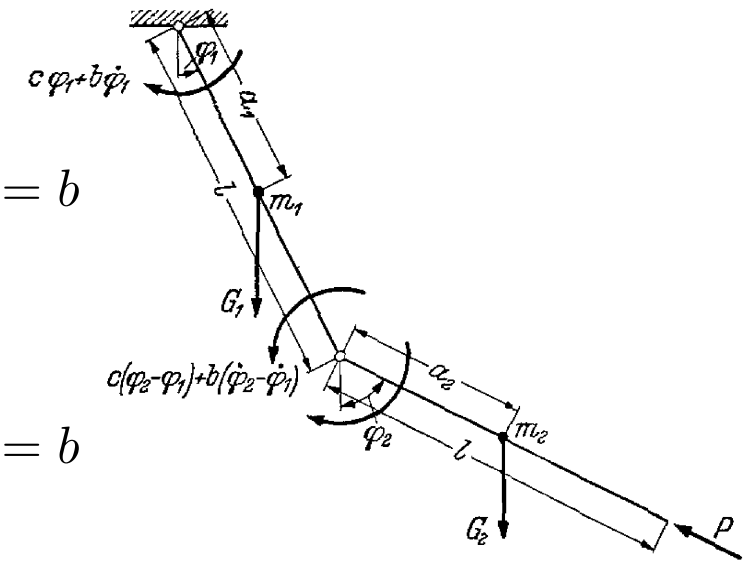

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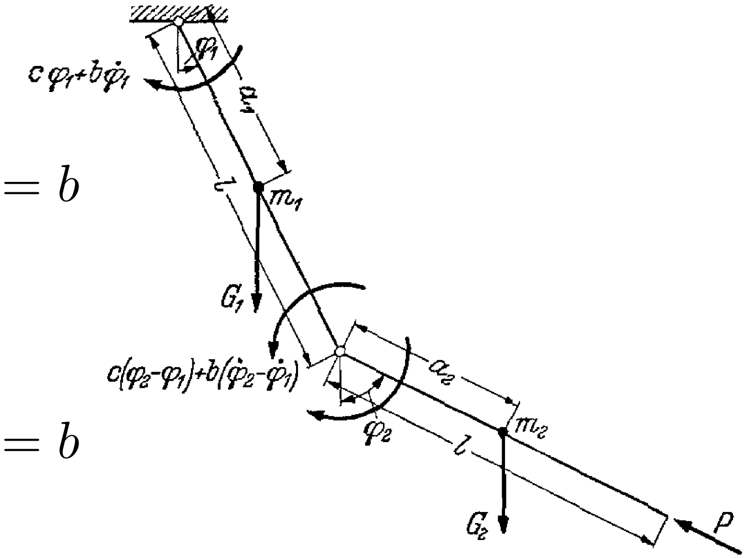

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- Stability of the vertical equilibrium

b=0: Lyapunov stability for $P<P_k$

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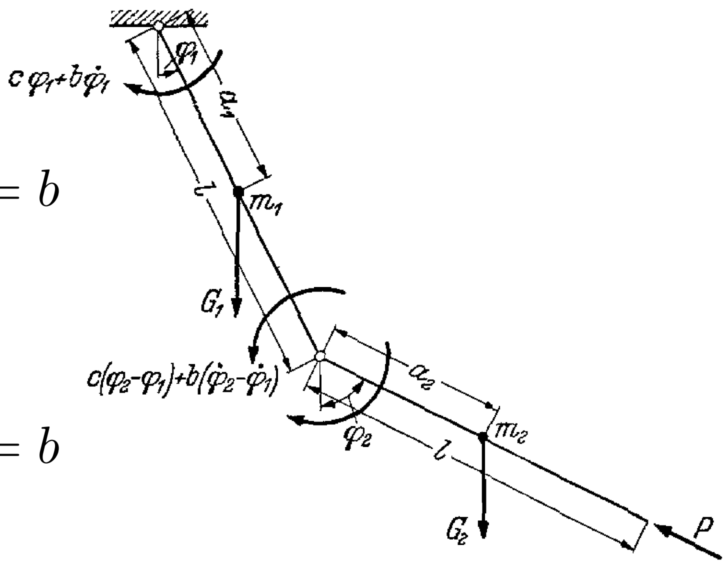

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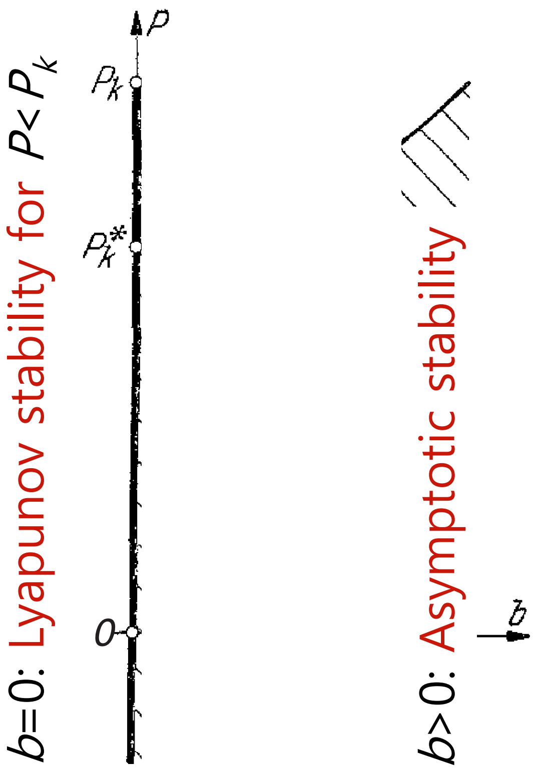

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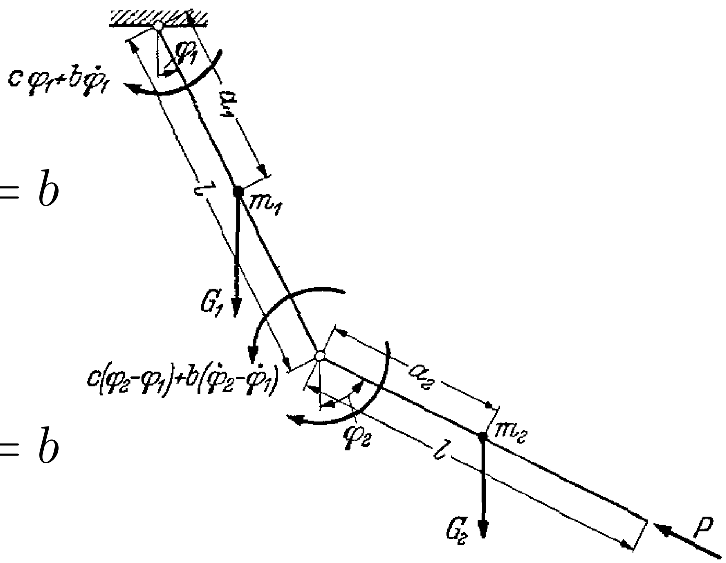

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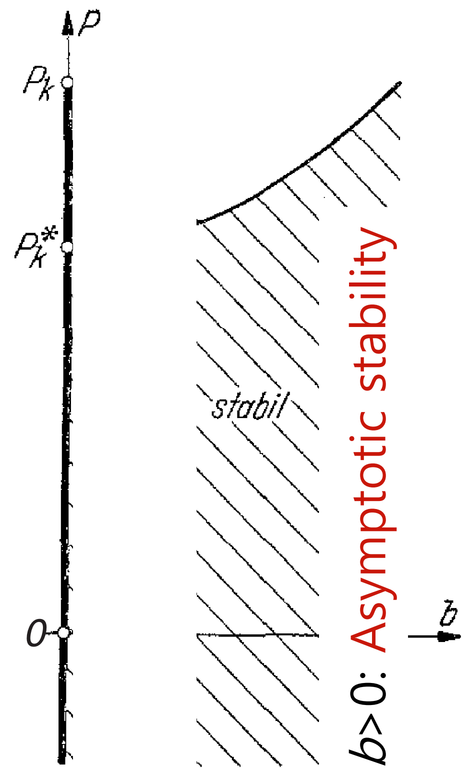

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b>0: Asymptotic stability

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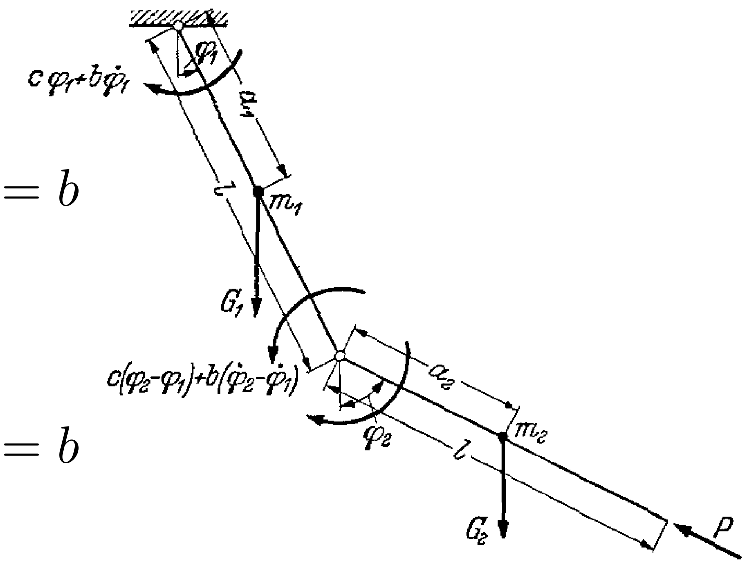

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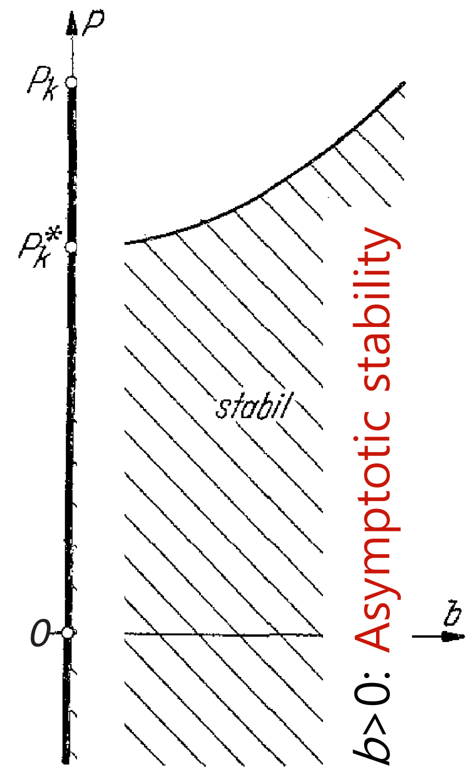

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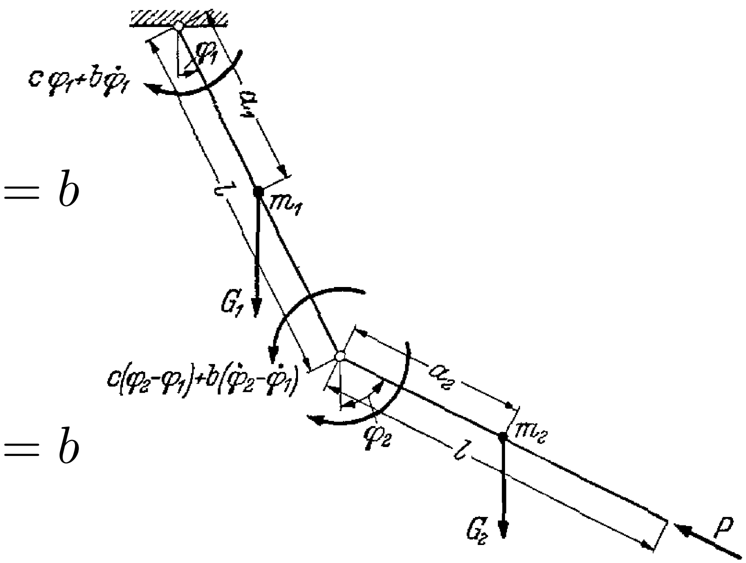

$b_2 = b$

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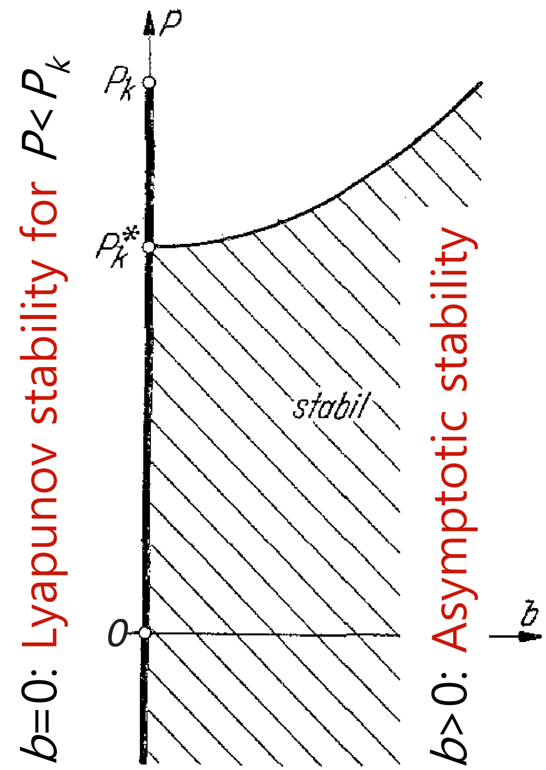

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$$m_1 = 2m, \quad m_2 = m$$

$$a_1 = a_2 = l, \quad b_1 = b_2 = b$$

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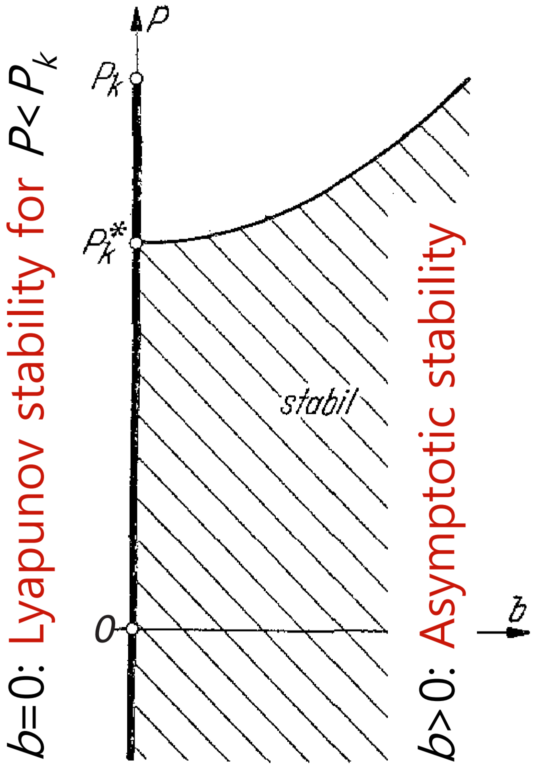

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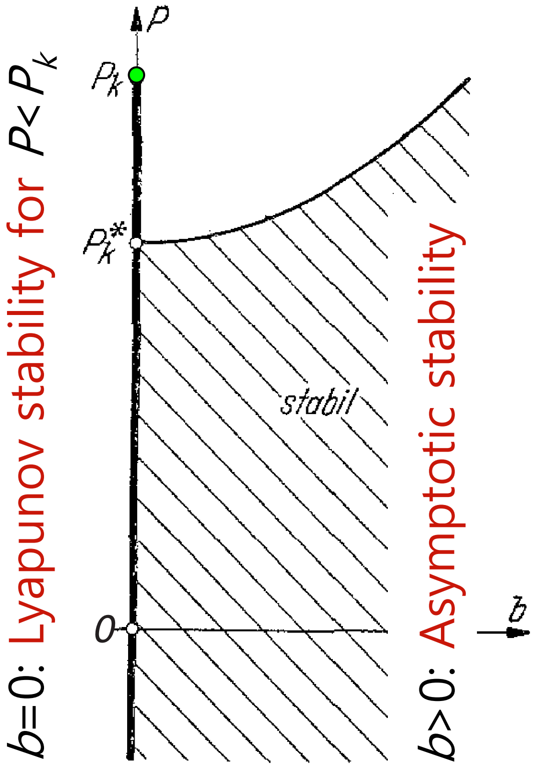
- Stability of the vertical equilibrium



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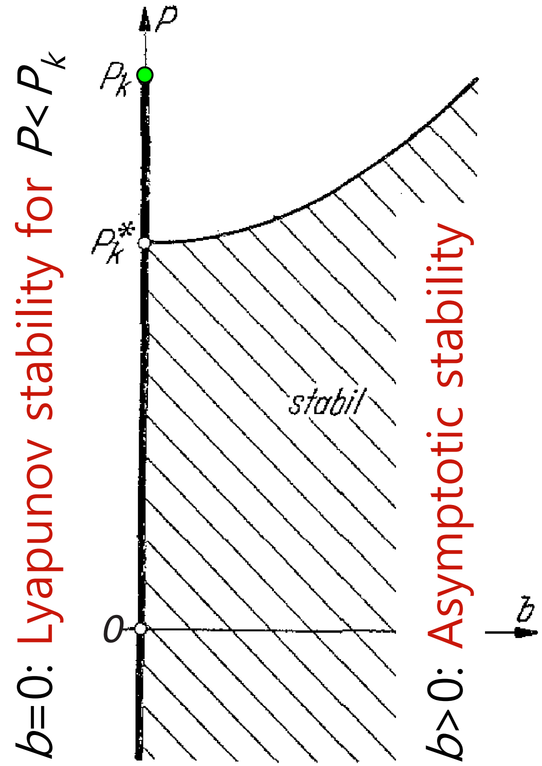

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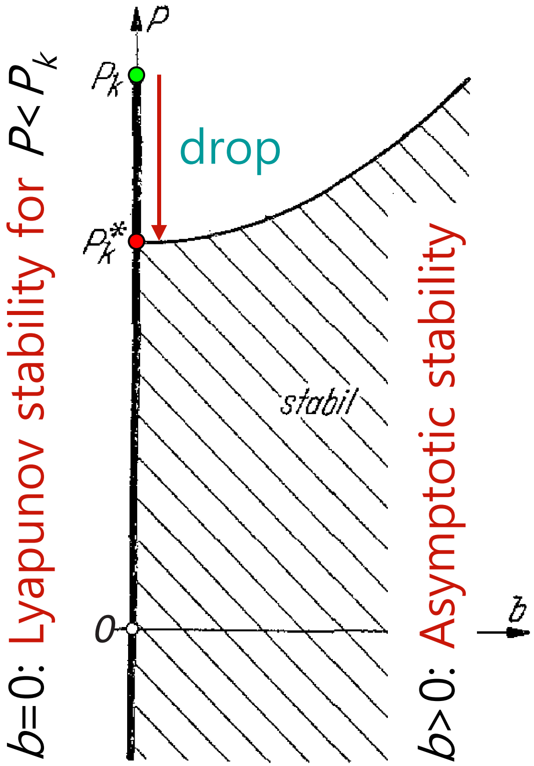
$$P_k^* = \lim_{b\to 0} P(b) = \frac{41}{28}\frac{c}{l} \approx 1.464\frac{c}{l} < P_k$$

- Stability of the vertical equilibrium



1952 Ziegler

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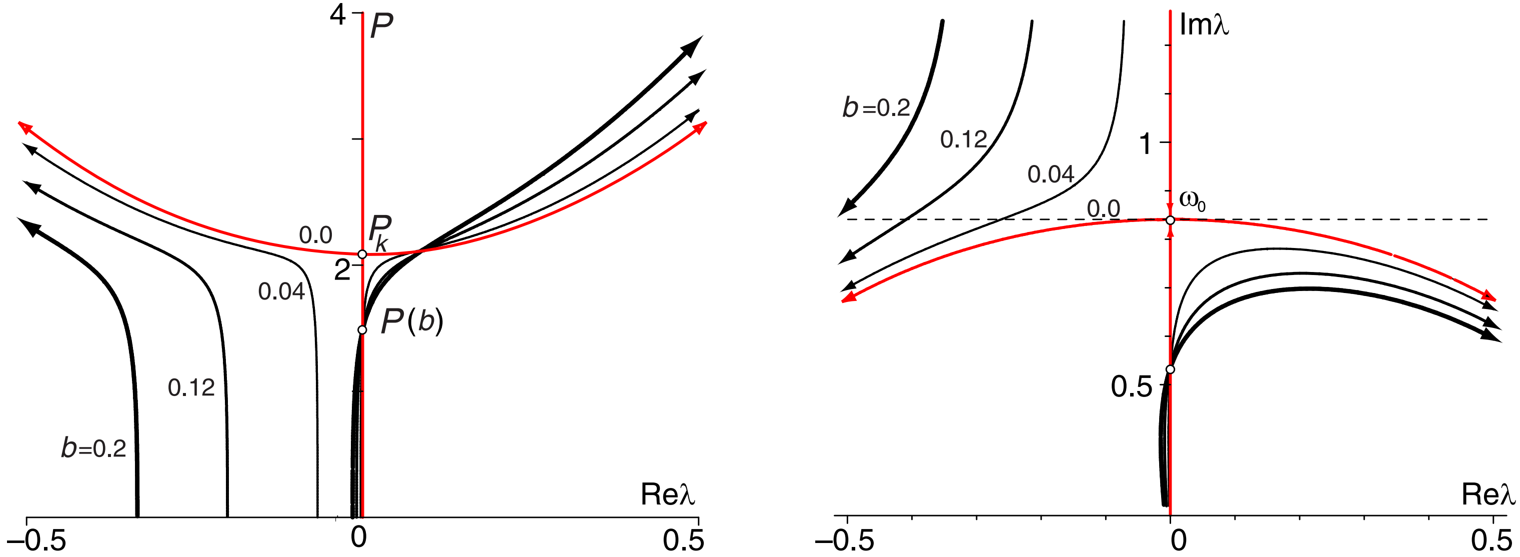
Dissipation-induced instability

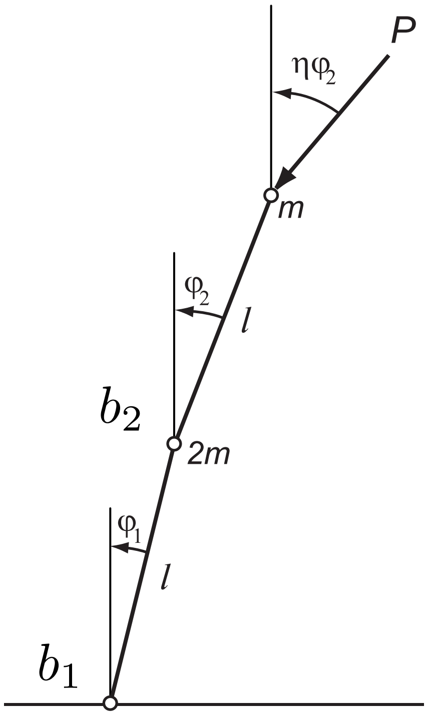


Pure imaginary eigenvalues get positive real increments under a dissipative perturbation

1966 Herrmann & Jong

Ziegler's pendulum with the partially follower force

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\ddot{\mathbf{x}} + \begin{pmatrix} b_1 + b_2 & -b_2 \\ -b_2 & b_2 \end{pmatrix}\dot{\mathbf{x}} + \begin{pmatrix} 2-p & \eta p - 1 \\ -1 & 1-(1-\eta)p \end{pmatrix}\mathbf{x} = 0$$



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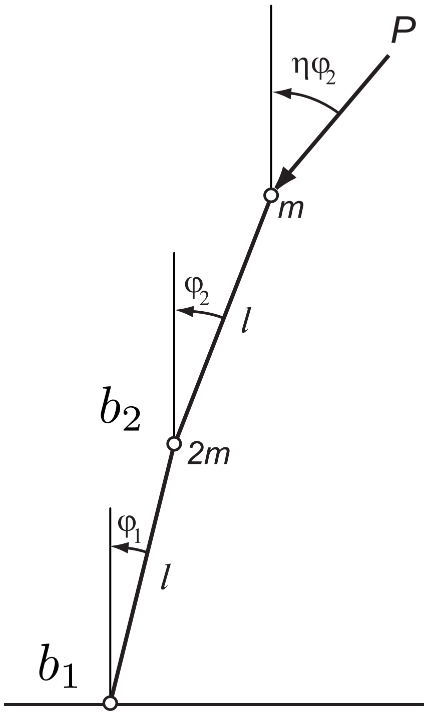
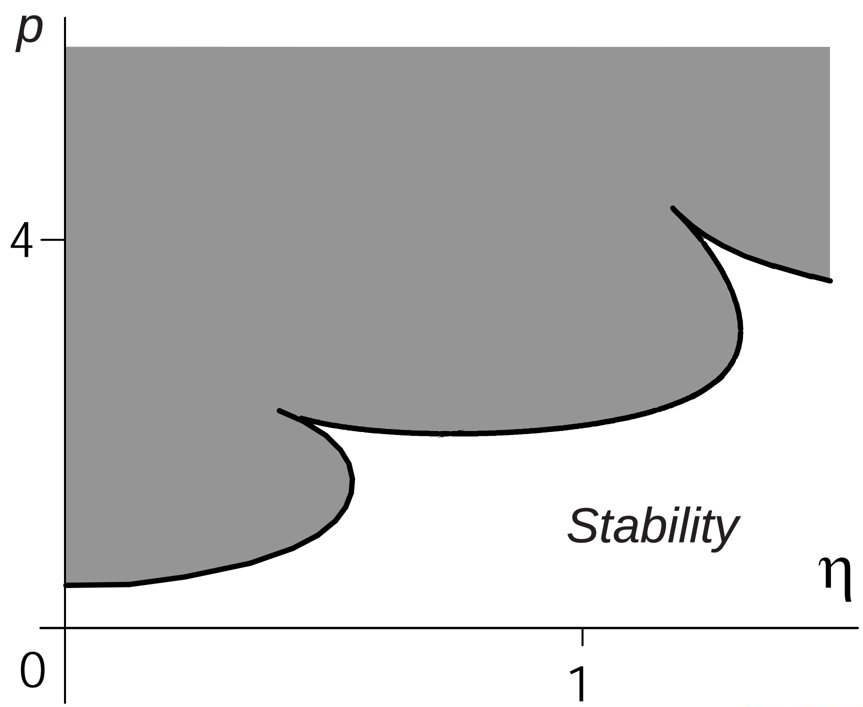
Undamped instability domain ($b_1=0$, $b_2=0$)





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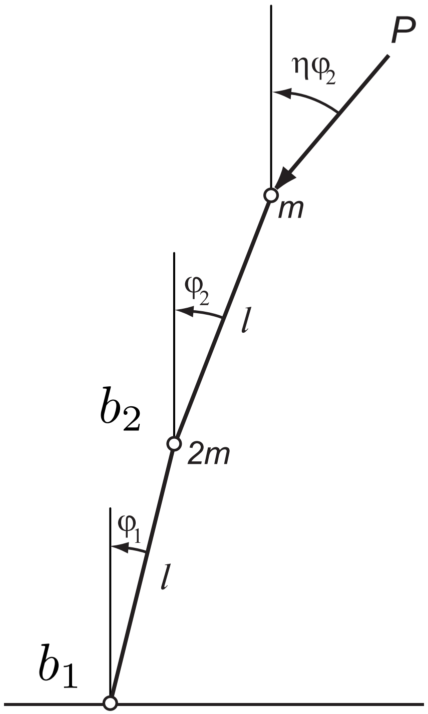
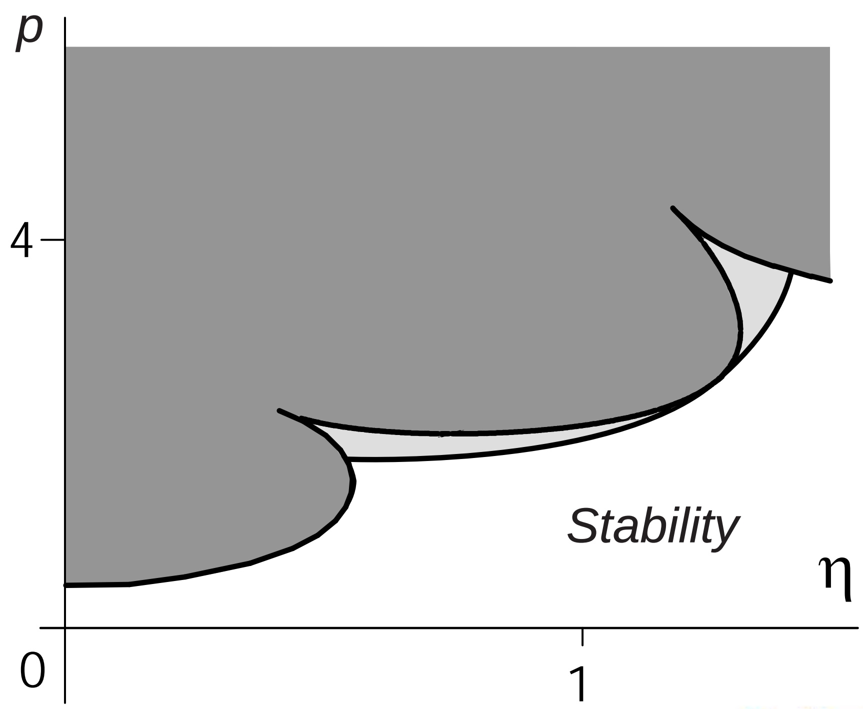
Undamped instability domain ($b_1=0$, $b_2=0$)
is not in the limit
of vanishing damping ($b_2=0.3b_1$, $b_1 \to 0$)



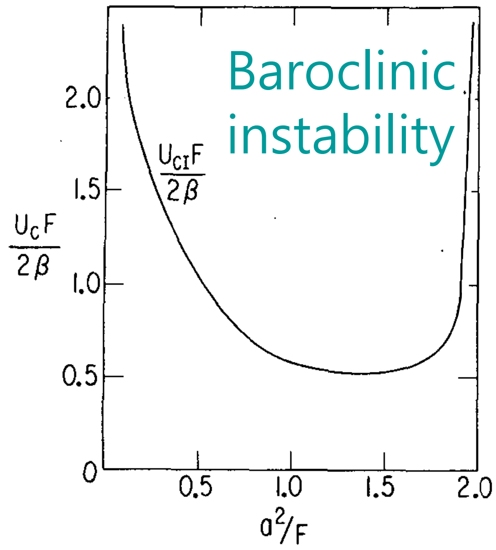


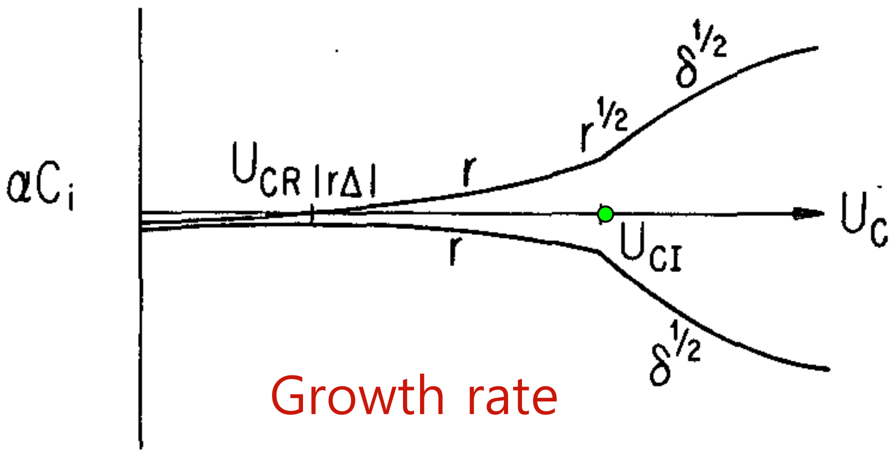
1961 Holopainen, 1977 Romea

Ekman layer dissipation enhances the baroclinic instability

Inviscid instability ($r = 0$)

$$U_{cI} = \frac{2\beta F}{a^2\sqrt{4F^2 - a^4}}$$





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Ekman layer dissipation enhances the baroclinic instability

Vanishing viscosity ($r \to 0$)

$$U_{cR} = \frac{2\beta F}{a(a^2+F)\sqrt{2F-a^2}}$$

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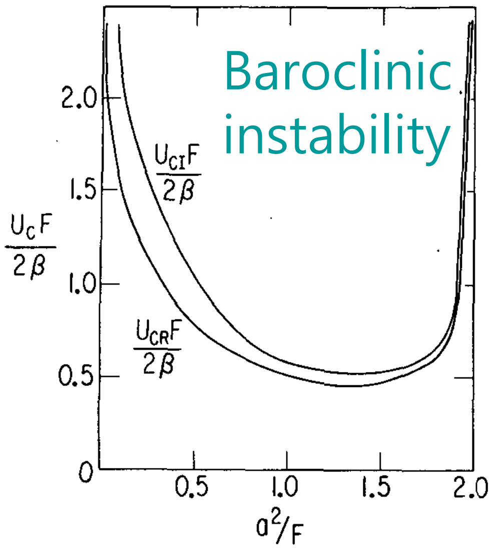

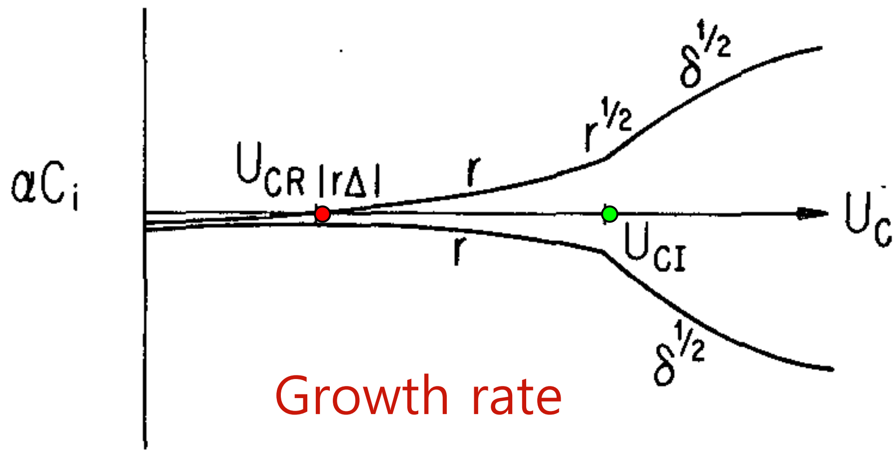

1961 Bolotin

highlights the structural instability

- “Suppose that a region of stability has been found based on two assumptions, the first ignoring damping and the second taking it into account. In the first case all the characteristic exponents were found to lie on the imaginary axis, and in the second case they were all in the left half-plane of the complex variable.

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- For a system in equilibrium under the action of potential forces the addition of dissipative forces with complete dissipation ensures asymptotic stability of the undisturbed equilibrium (Kelvin-Tait).

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- Does the addition of dissipative forces stabilize the undisturbed equilibrium?
- For a system in equilibrium under the action of potential forces the addition of dissipative forces with complete dissipation ensures asymptotic stability of the undisturbed equilibrium (Kelvin-Tait).
- In the case of non-conservative systems the addition of dissipative forces can in certain cases have a destabilizing effect."

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highlights the structural instability

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- **The greatest theoretical interest** is evidently centered in the unique effect of damping in the presence of non-potential forces, and in particular, in the differences in the results for systems with slight damping which then becomes zero and systems in which damping is absent from the start.

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- **The greatest theoretical interest** is evidently centered in the unique effect of damping in the presence of non-potential forces, and in particular, in the differences in the results for systems with slight damping which then becomes zero and systems in which damping is absent from the start.

- These interesting aspects require further study for obtaining further, more definite, results.”

1956 Bottema

resolves the Ziegler's paradox

A linear non-conservative system with 2 d.o.f.

$$\ddot{\mathbf{x}} + (\mathbf{D} + \mathbf{G})\dot{\mathbf{x}} + (\mathbf{K} + \mathbf{N})\mathbf{x} = 0$$

Forces:

Dissipative, $\mathbf{D} = \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix}$ Potential, $\mathbf{K} = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}$

Gyroscopic, $\mathbf{G} = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}$ Circulatory, $\mathbf{N} = \begin{pmatrix} 0 & \nu \\ -\nu & 0 \end{pmatrix}$

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Characteristic polynomial:

$$\mathbf{x} = e^{\mu t}\mathbf{u}, \quad q(\mu) = \mu^4 + q_1\mu^3 + q_2\mu^2 + q_3\mu + q_4$$

$$q_1 = \mathrm{tr}\mathbf{D}, \quad q_3 = \mathrm{tr}\mathbf{K}\mathrm{tr}\mathbf{D} - \mathrm{tr}\mathbf{K}\mathbf{D} + 2\Omega\nu$$

$$q_2 = \mathrm{tr}\mathbf{K} + \det\mathbf{D} + \Omega^2, \quad q_4 = \det\mathbf{K} + \nu^2$$

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$$q(\mu) = \mu^4 + q_1\mu^3 + q_2\mu^2 + q_3\mu + q_4$$

Hurwitz condition:

$$q_i > 0, \quad q_2 > \frac{q_1^2 q_4 + q_3^2}{q_1 q_3}$$

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$$q(\mu) = \mu^4 + q_1\mu^3 + q_2\mu^2 + q_3\mu + q_4$$

$$\mu = c\lambda, \quad c = \sqrt[4]{q_4}, \quad a_i = \frac{q_i}{c^i}$$

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$$p(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + 1$$

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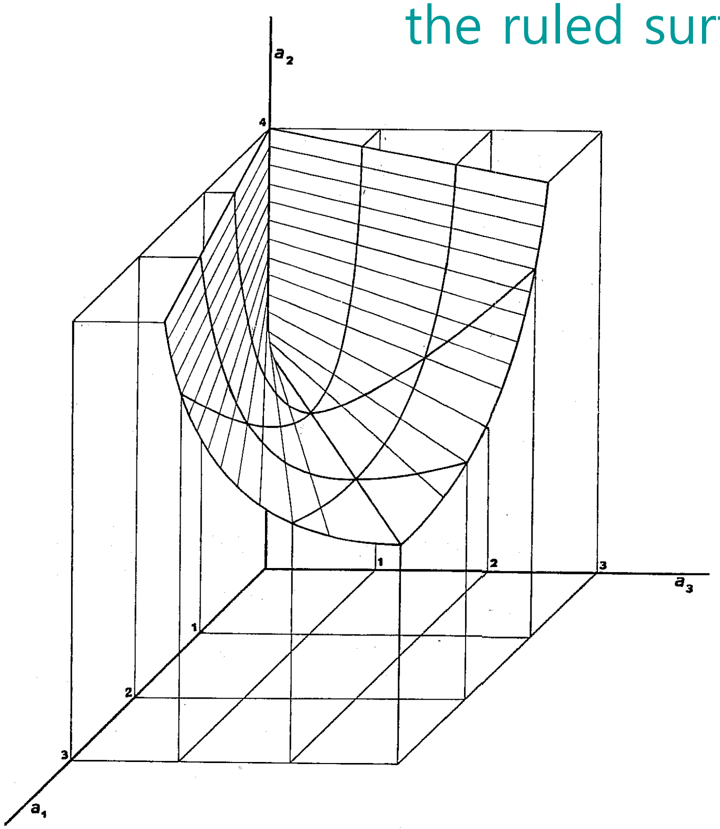
Asymptotic stability inside the ruled surface



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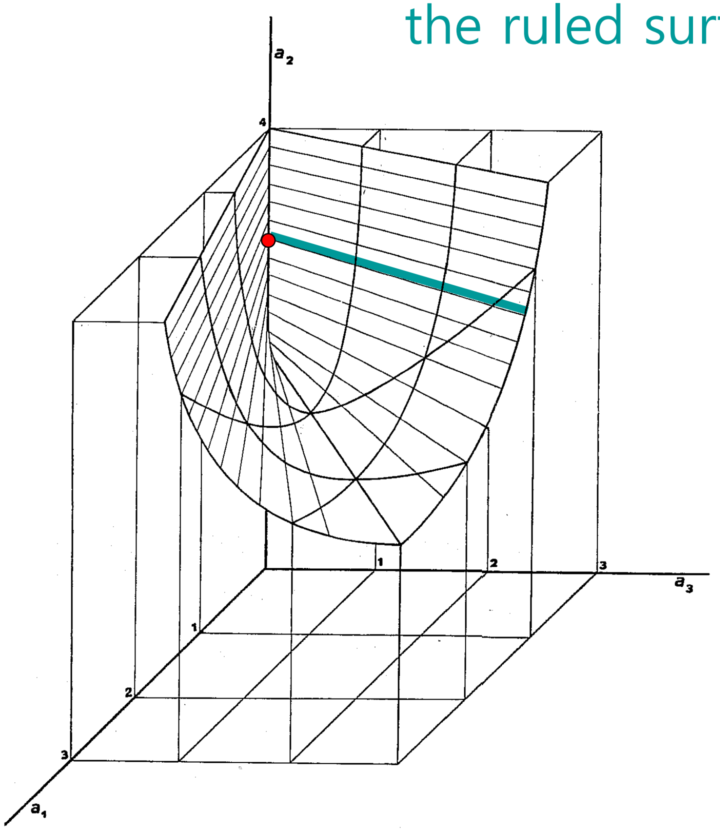

Hurwitz condition:

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Generators:

$$a_3 = r a_1, \quad a_2 = r + \frac{1}{r}$$

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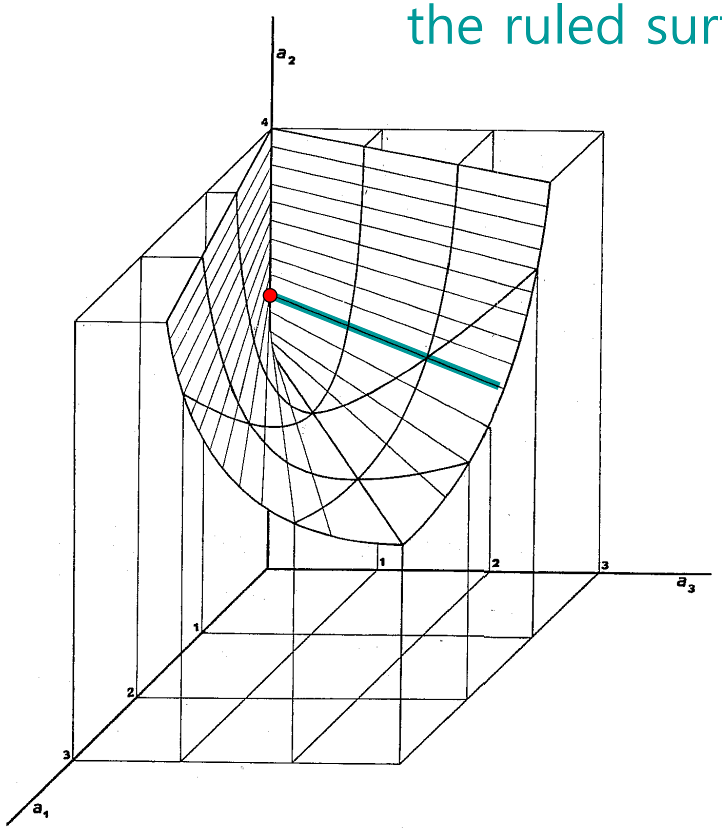

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$$r \in (0, \infty)$$

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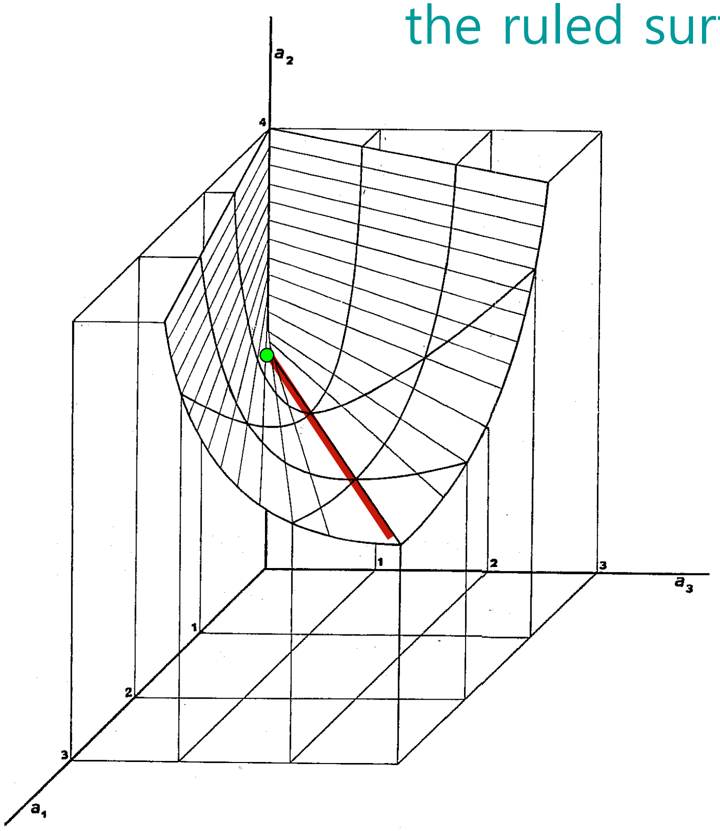

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Generators:

$$a_3 = r a_1, \quad a_2 = r + \frac{1}{r}$$

$$r \in (0, \infty)$$

Minimum: $a_2 = 2$

$$r = 1, \quad i.e. \quad a_3 = a_1$$

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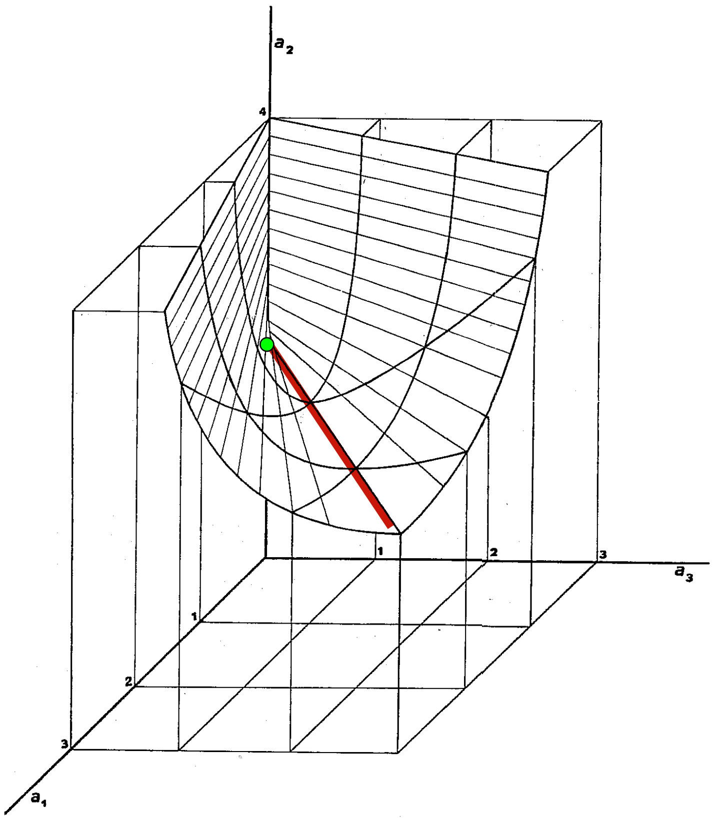
resolves the Ziegler's paradox

$$a_2 = \frac{a_1^2 + a_3^2}{a_1 a_3} > 2, \quad a_3 \neq a_1$$

$$a_2 = 2, \quad a_3 = a_1$$



„Here is the discontinuity we mentioned above. It plays a part in questions regarding the stability of equilibrium.

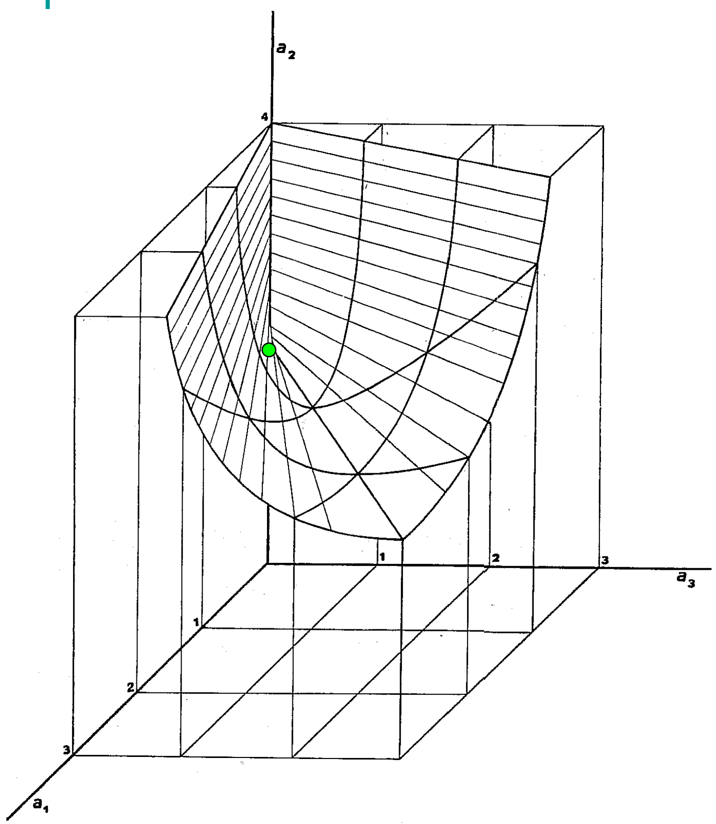
The coefficients a_1 and a_3 depend on the linear damping forces and it is well known that the stability condition may change in a discontinuous way if a very small damping vanishes at all.

The phenomenon may be illustrated by a geometrical diagram." Bottema, 1956

1971 Arnold

Whitney umbrella singularity on the stability boundary

Double pure imaginary eigenvalue at the singular point



two-parameter families.[1] *These singularities can be listed to within diffeomorphism as follows:*

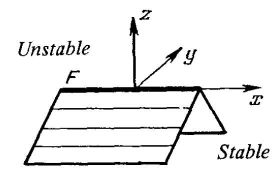


Fig. 4.14.

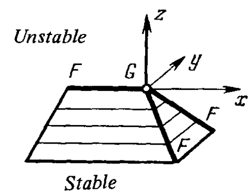


Fig. 4.15.

Two faces meeting along a ridge (F_i; Fig. 4.14): $z + |y| = 0$.

Three faces meeting at a corner ($G_{3,4,5}$; Fig. 4.15): $z + \max(x, |y|) = 0$.

Cuspidal point on a ridge (G_2; Fig. 4.17): $z + |\mathrm{Re}\sqrt{(x + iy)}| = 0$. (*This surface in* $\mathbf{R}^3$ *is diffeomorphic to that given by the equations* $XY^2 = Z^2$, *where* $Y \geqslant 0$.)

Node on a ridge (G_1; Fig. 4.16): $z + \lambda(x, y) = 0$, *where* λ *is the greatest real part of the roots of the equation* $\lambda^3 = x\lambda + y$. (*This surface in* $\mathbf{R}^3$ *is diffeomorphic to that given by* $X^2Y^2 = Z^2$, $X \geqslant 0$, $Y \geqslant 0$.)

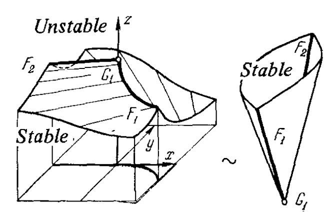


Fig. 4.16.

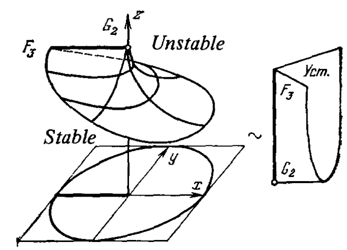


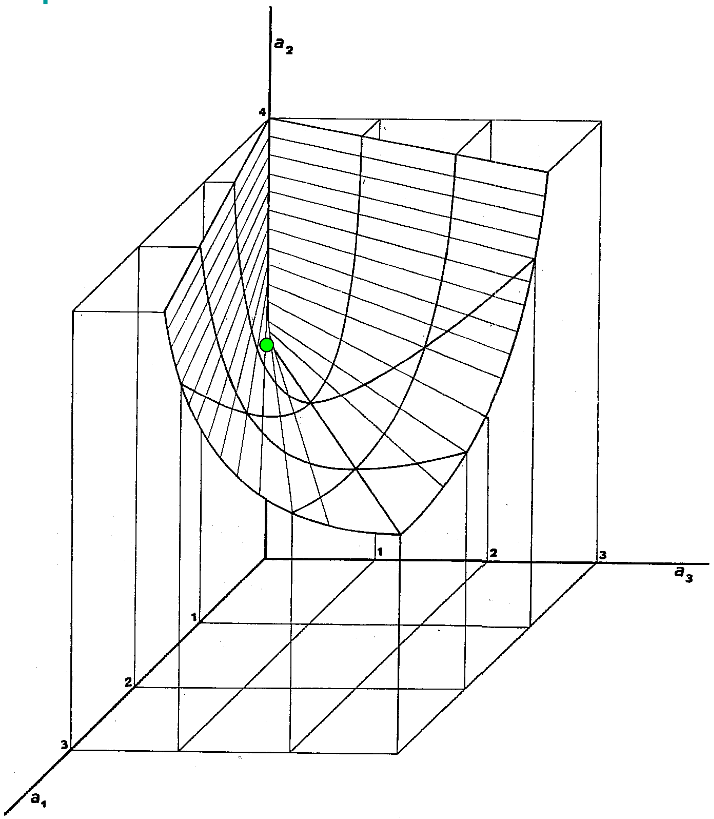
Fig. 4.17.

The acute angles of the stability boundary always point into the domain of instability.

1971 Arnold

Whitney umbrella singularity on the stability boundary

Double pure imaginary eigenvalue at the singular point: λ=i



$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + 1$$

Companion matrix:

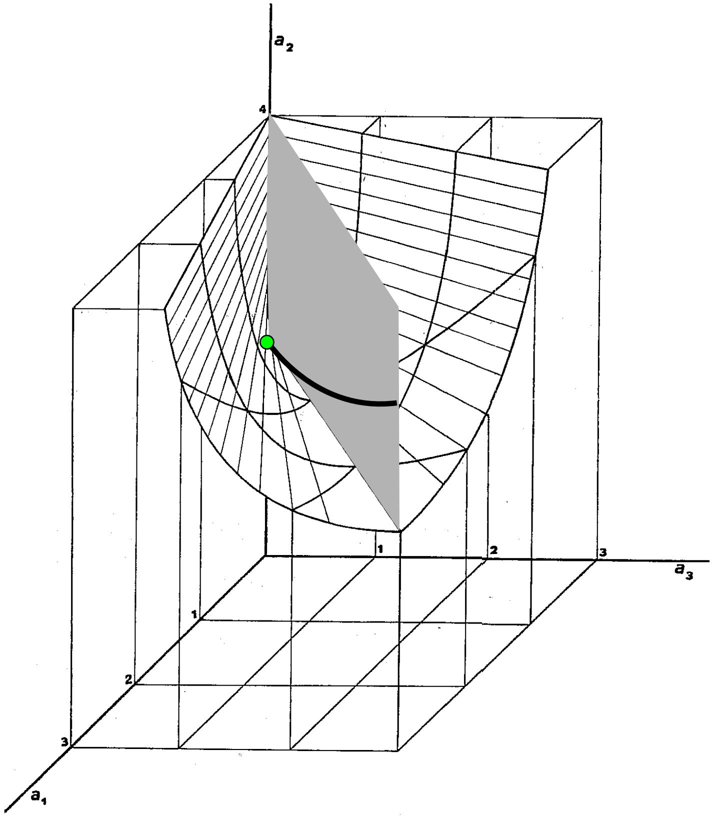
$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -a_3 & -a_2 & -a_1 \end{pmatrix}$$

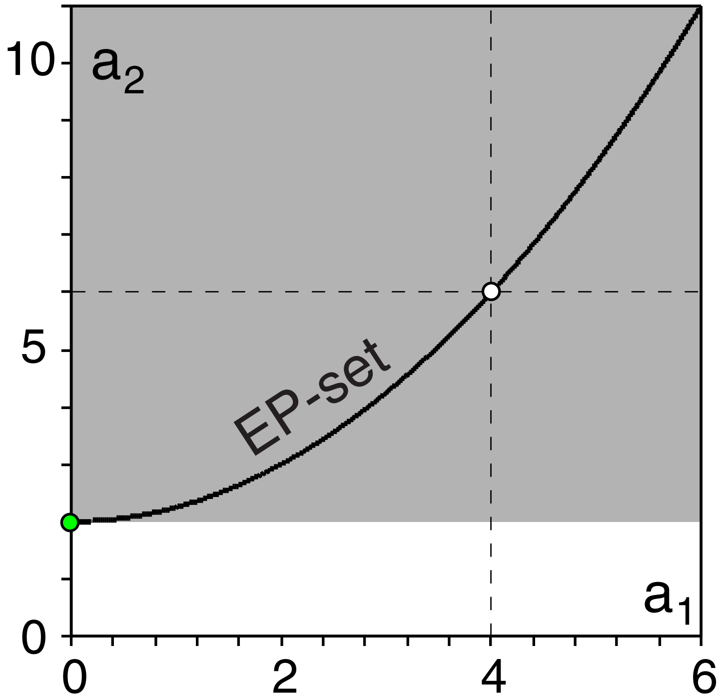
Jordan form:

$$\mathbf{J}_2 = \begin{pmatrix} i & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

Tangent cone: $\{(a_1, a_3, a_2): \quad a_1 = a_3, \quad a_1 > 0, \quad a_2 > 0\}$

EP-set: $\left\{(a_1, a_3, a_2): \quad a_1 = a_3, \quad a_2 = 2 + \dfrac{a_1^2}{4}\right\}$





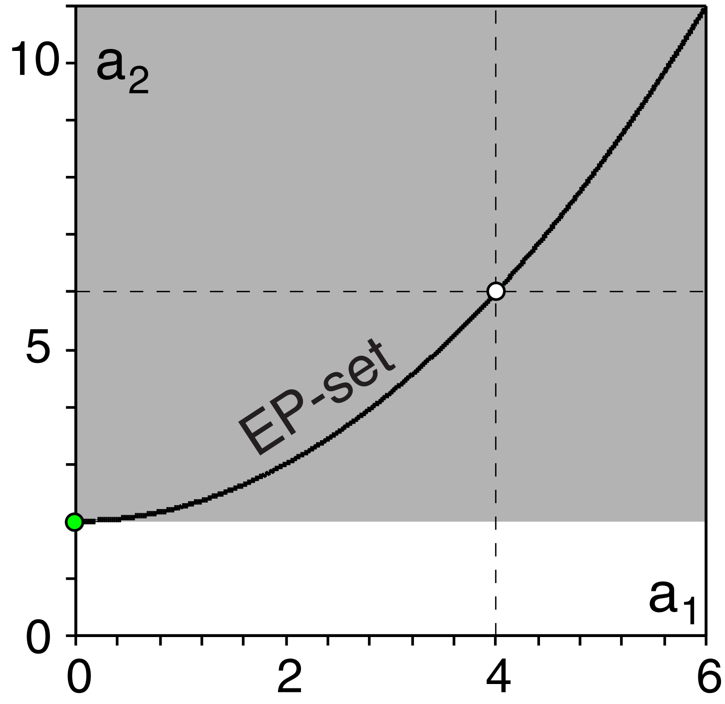
Tangent cone: $\{(a_1, a_3, a_2): \quad a_1 = a_3, \quad a_1 > 0, \quad a_2 > 0\}$

EP-set: $\left\{(a_1, a_3, a_2): \quad a_1 = a_3, \quad a_2 = 2 + \frac{a_1^2}{4}\right\}$

Multiple eigenvalues at the EP-set:

$$\lambda_1 = \lambda_2 = -\frac{a_1}{4} - \frac{1}{4}\sqrt{a_1^2 - 16}$$

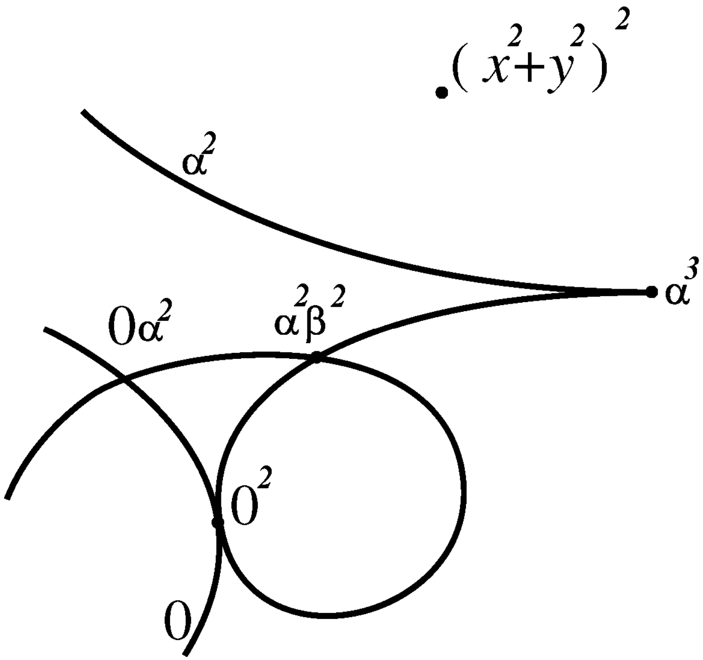
$$\lambda_3 = \lambda_4 = -\frac{a_1}{4} + \frac{1}{4}\sqrt{a_1^2 - 16}$$

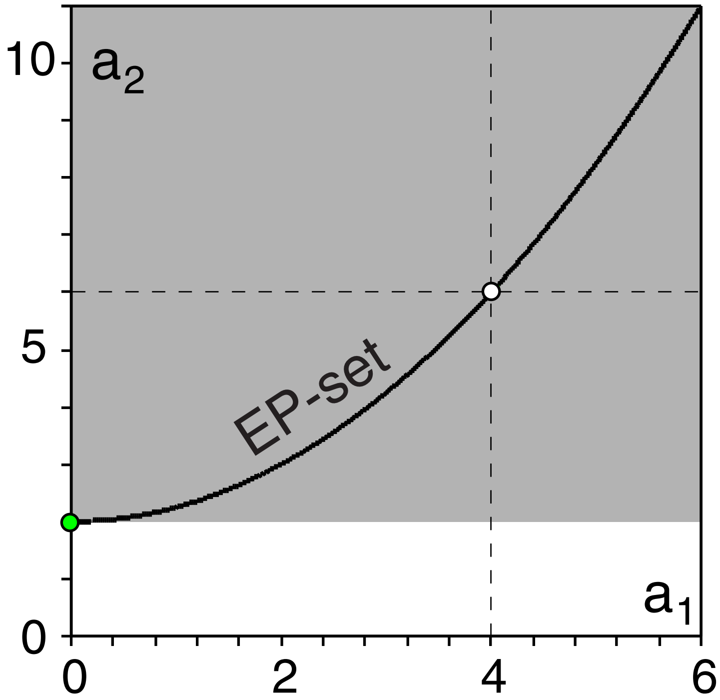


1972 Galin

bifurcation diagrams of families of real matrices

Double eigenvalues x+iy with the Jordan block:
codim=2



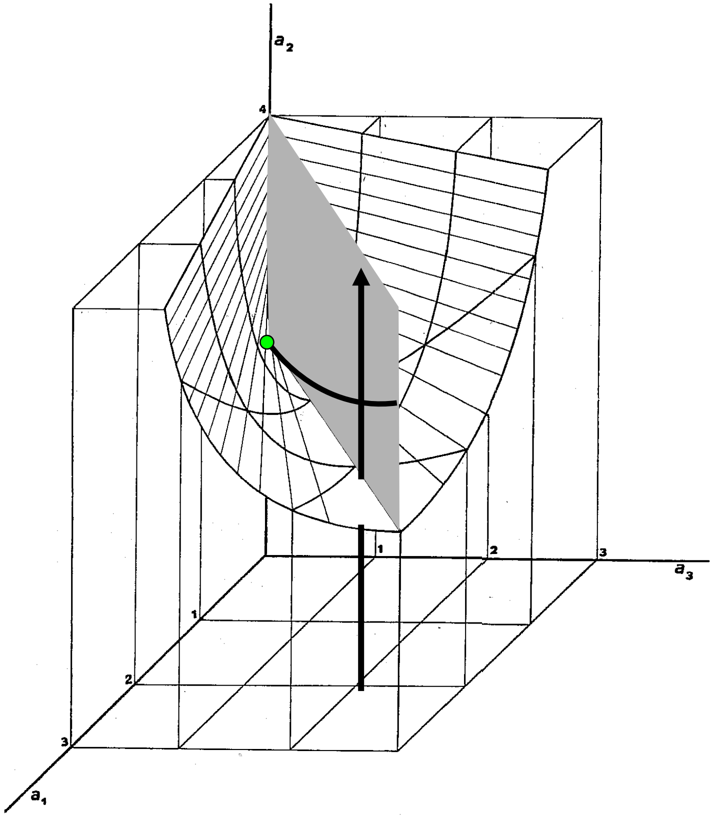


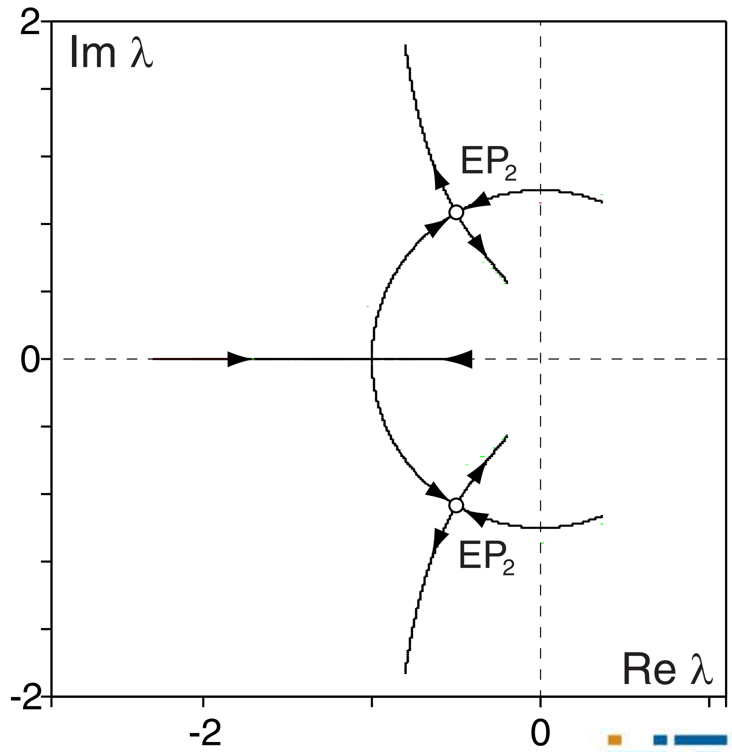
Movement of eigenvalues

Parameters change within the tangent cone

$$a_1 = a_3 = 2, \quad 0 \le a_2 \le 6$$

Collisions on the unit circle at exceptional points (EP_2), $\mathrm{Re}\lambda < 0$



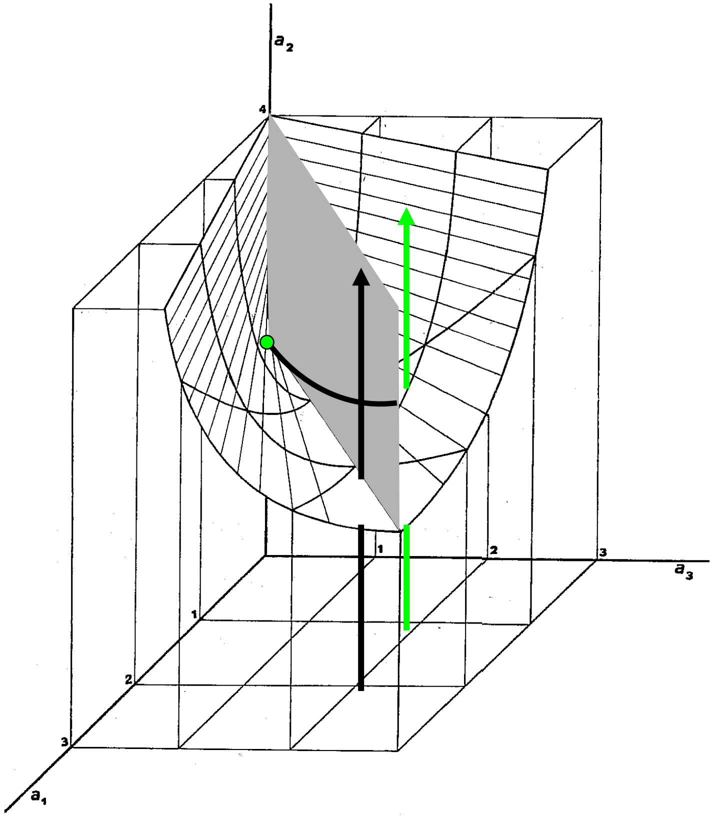


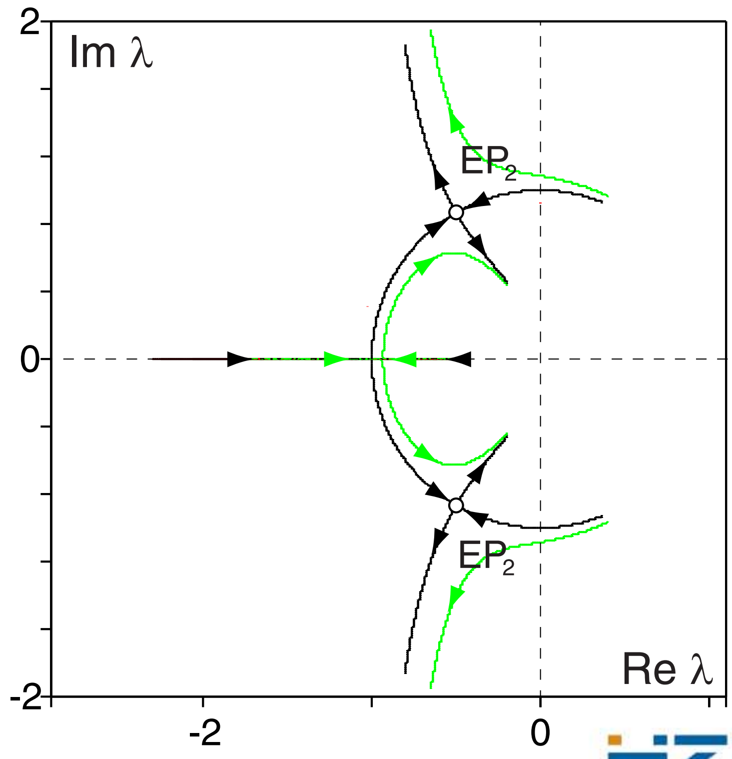
Imperfect merging of modes

Parameters change right to the tangent cone

$$a_1 = 1.7, \quad a_3 = 2, \quad 0 \le a_2 \le 6$$

Avoided crossings near EP_2, $\mathrm{Re}\lambda < 0$



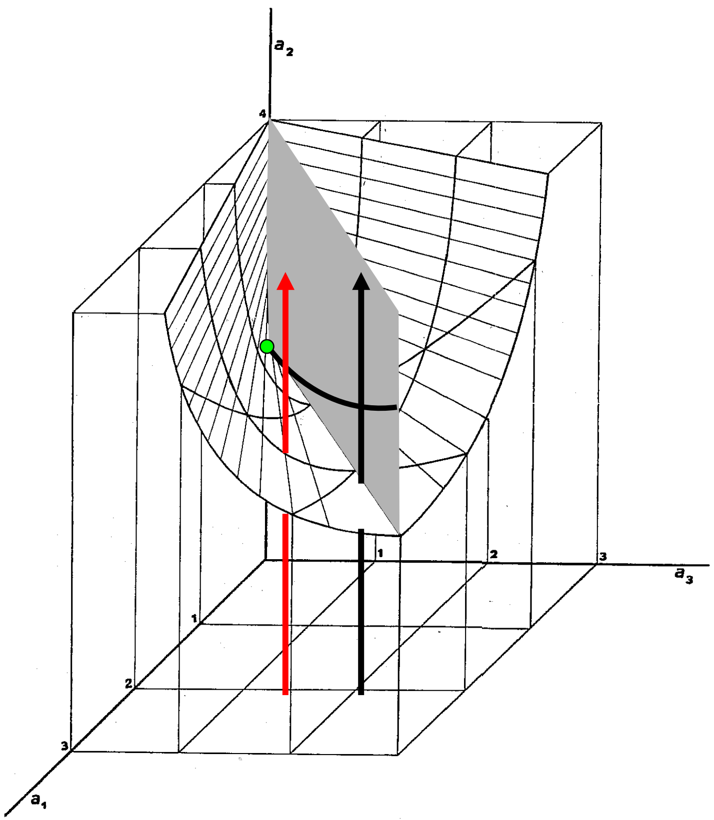


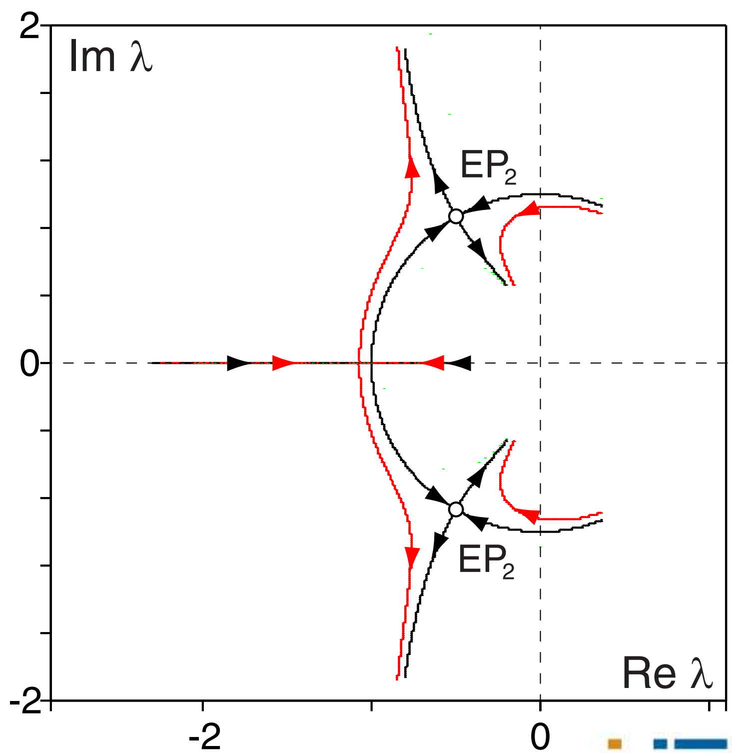
Exchange of instability between branches

Parameters change left to the tangent cone

$$a_1 = 2, \quad a_3 = 1.7, \quad 0 \le a_2 \le 6$$

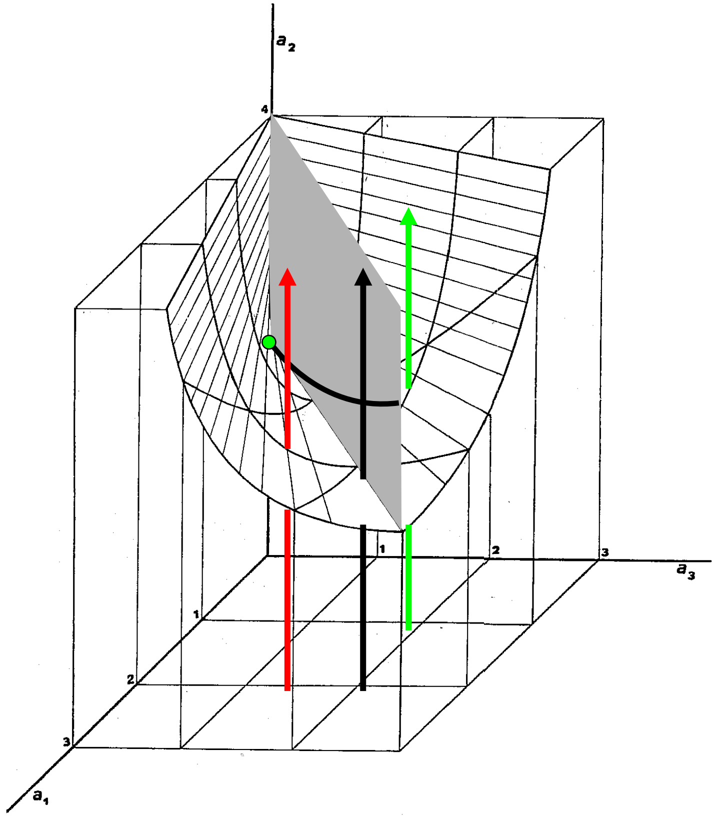
Avoided crossings near EP_2, $\mathrm{Re}\lambda < 0$

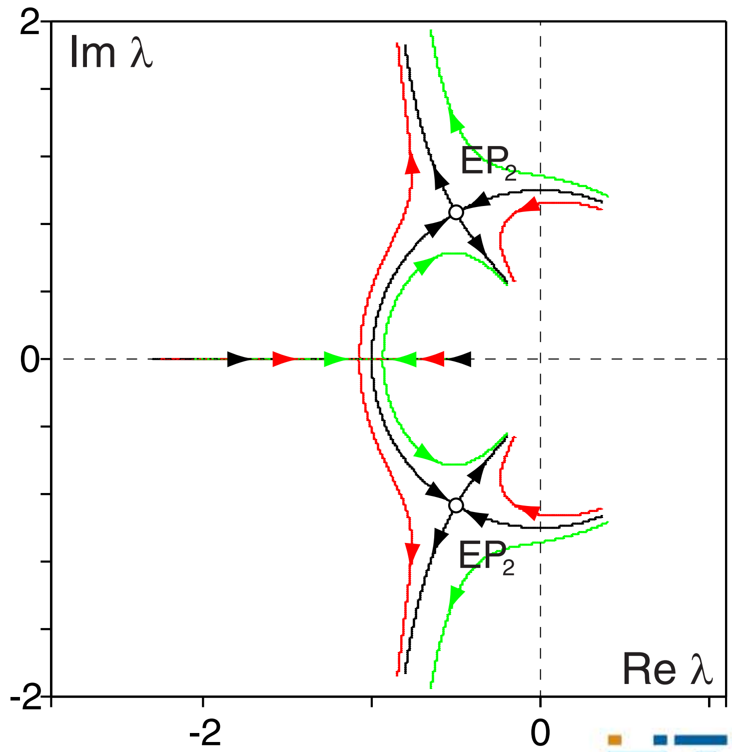




Selective role of the tangent cone

It determines which mode is destabilized by dissipation
because
the set of multiple complex eigenvalues (EP-set) is within it

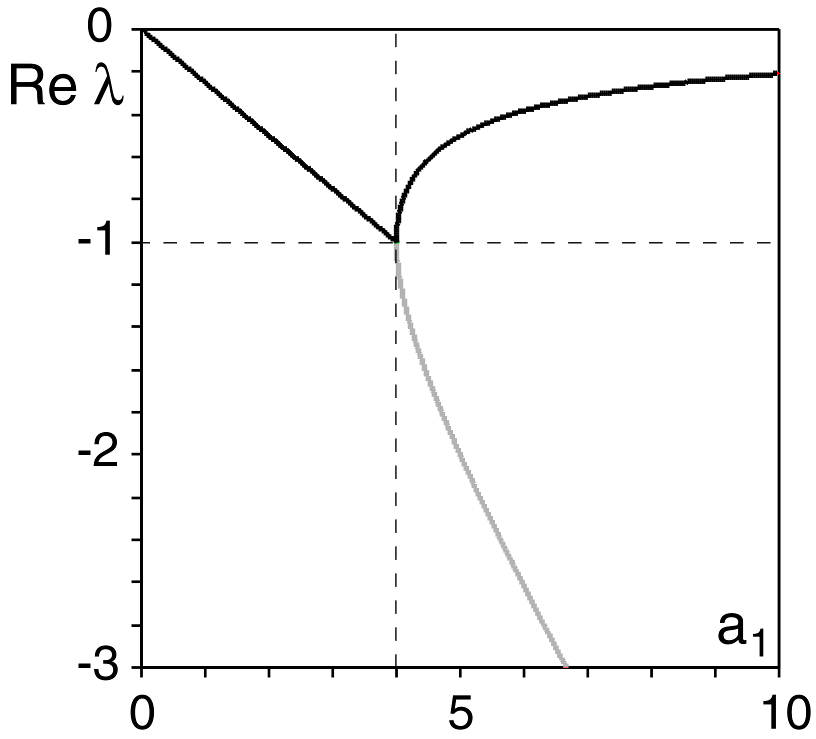


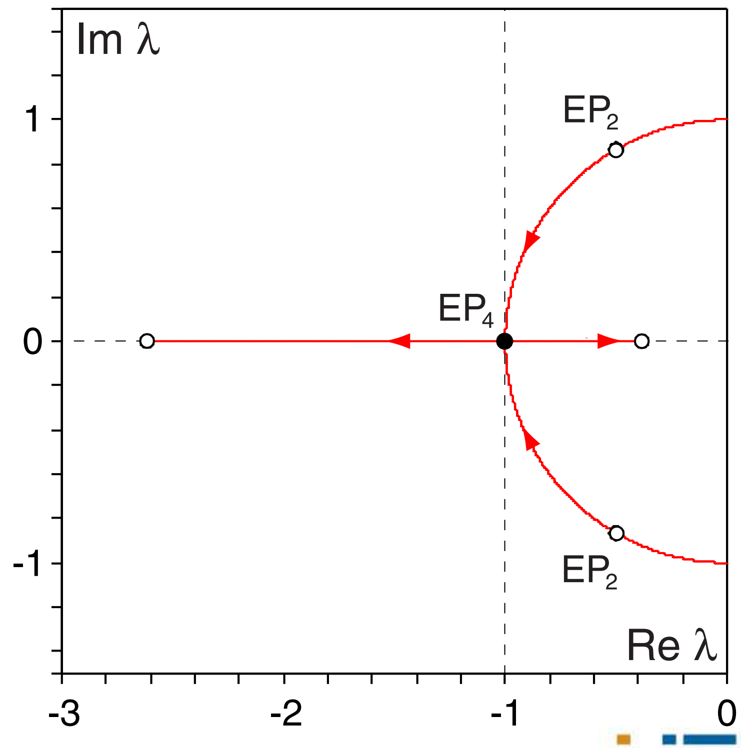


HZDR

Spectral abscissa minimization

$$\alpha(\mathbf{A}) = \max_k \mathrm{Re}\lambda_k(\mathbf{A}) \qquad \alpha(\mathbf{A}) \rightarrow \min_{a_1, a_2, a_3}$$

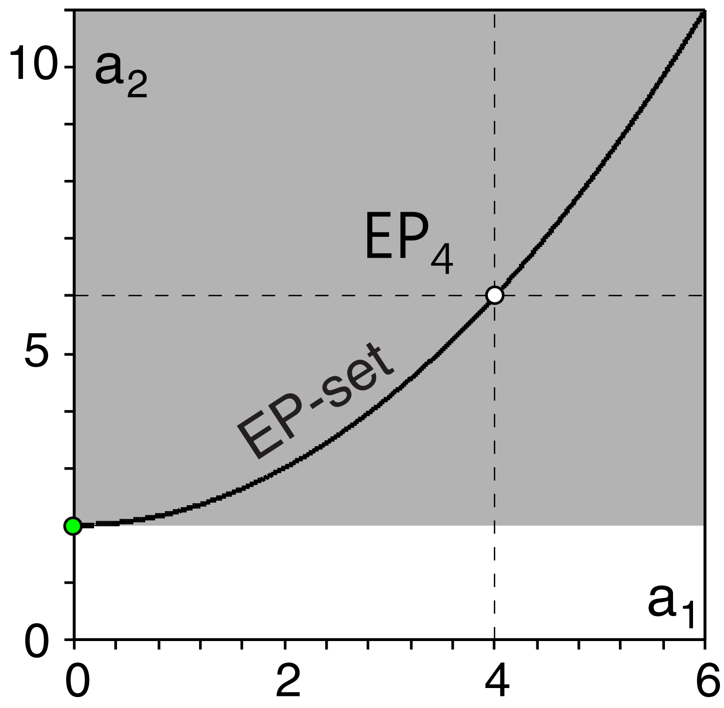


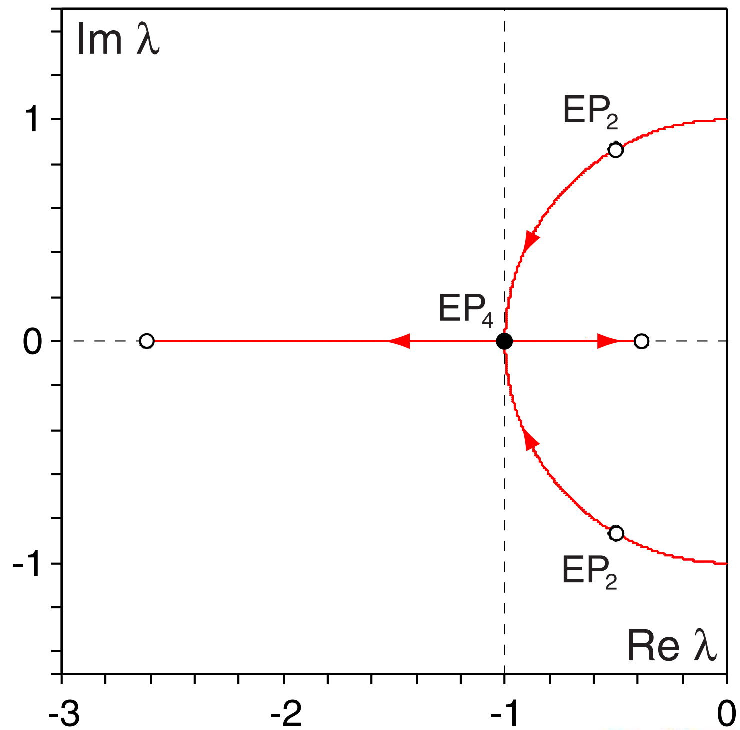


Spectral abscissa minimization

$$\alpha(\mathbf{A}) = \max_k \mathrm{Re}\lambda_k(\mathbf{A}) \qquad \alpha(\mathbf{A}) \to \min_{a_1,a_2,a_3}$$

$$\min_{a_1,a_2,a_3} \alpha(\mathbf{A}) = -1$$ The minimizer is at the EP-set





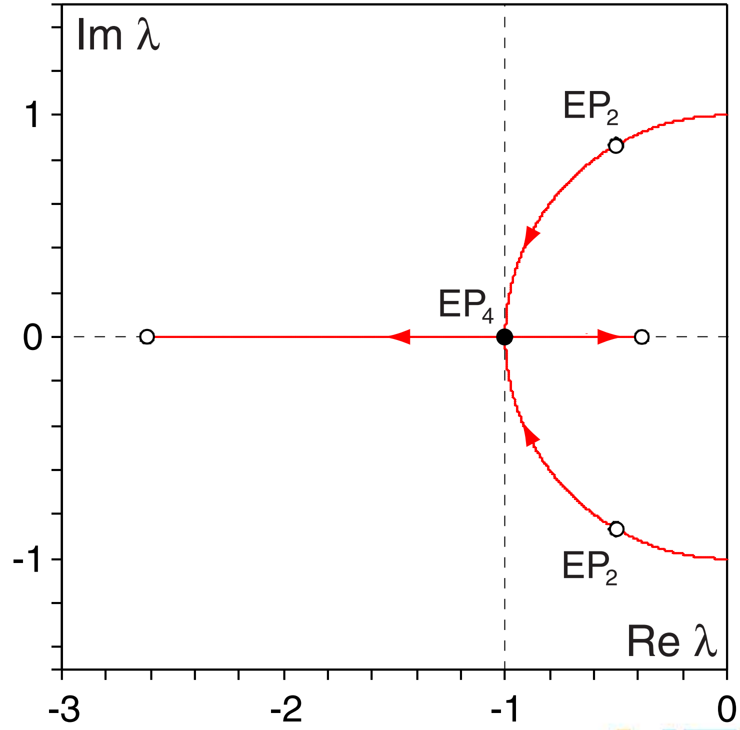
2001 Burke, Lewis, Overton

non-derogatory matrices are minimizers of the spectral abscissa

$$\min_{a_1,a_2,a_3} \alpha(\mathbf{A}) = -1$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{pmatrix}$$

$$\mathbf{J}_4(-1) = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



Non-conservative gyroscopic system

$$\ddot{\mathbf{z}} + (\delta \mathbf{D} + \Omega \mathbf{J})\dot{\mathbf{z}} + (\mathbf{K} + \nu \mathbf{J})\mathbf{z} = 0,$$

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{D} = \mathbf{D}^T, \quad \mathbf{K} = \mathbf{K}^T, \qquad \kappa_1, \kappa_2$$ eigenvalues of $\mathbf{K}$

δ=0, ν=0, eigenvalues: $\lambda = i\omega_\pm(\Omega)$

$$\omega_\pm(\Omega) = \sqrt{\omega_0^2 + \frac{\Omega_2}{2}\left(\sqrt{\Omega^2 - \Omega_2^2} \pm \sqrt{\Omega^2 - \Omega_1^2}\right)\sqrt{\frac{\Omega^2}{\Omega_2^2} - 1}}$$

$$\omega_0 = \frac{1}{2}\sqrt{\Omega_2^2 - \Omega_1^2}$$

$$0 \le \sqrt{-\kappa_2} - \sqrt{-\kappa_1} =: \Omega_1 \le \Omega_2 := \sqrt{-\kappa_1} + \sqrt{-\kappa_2}$$

Gyroscopic stabilization

$$\mathbf{K} = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}$$

δ=0, ν=0: When Ω increases, complex eigenvalues $\lambda = i\omega_{\pm}(\Omega)$

move along the circle in the complex plane

$$(\mathrm{Re}\lambda)^2 + (\mathrm{Im}\lambda)^2 = \omega_0^2$$

After the Krein collision at $\Omega=\Omega_2$,
pure imaginary eigenvalues diverge along the imaginary axis

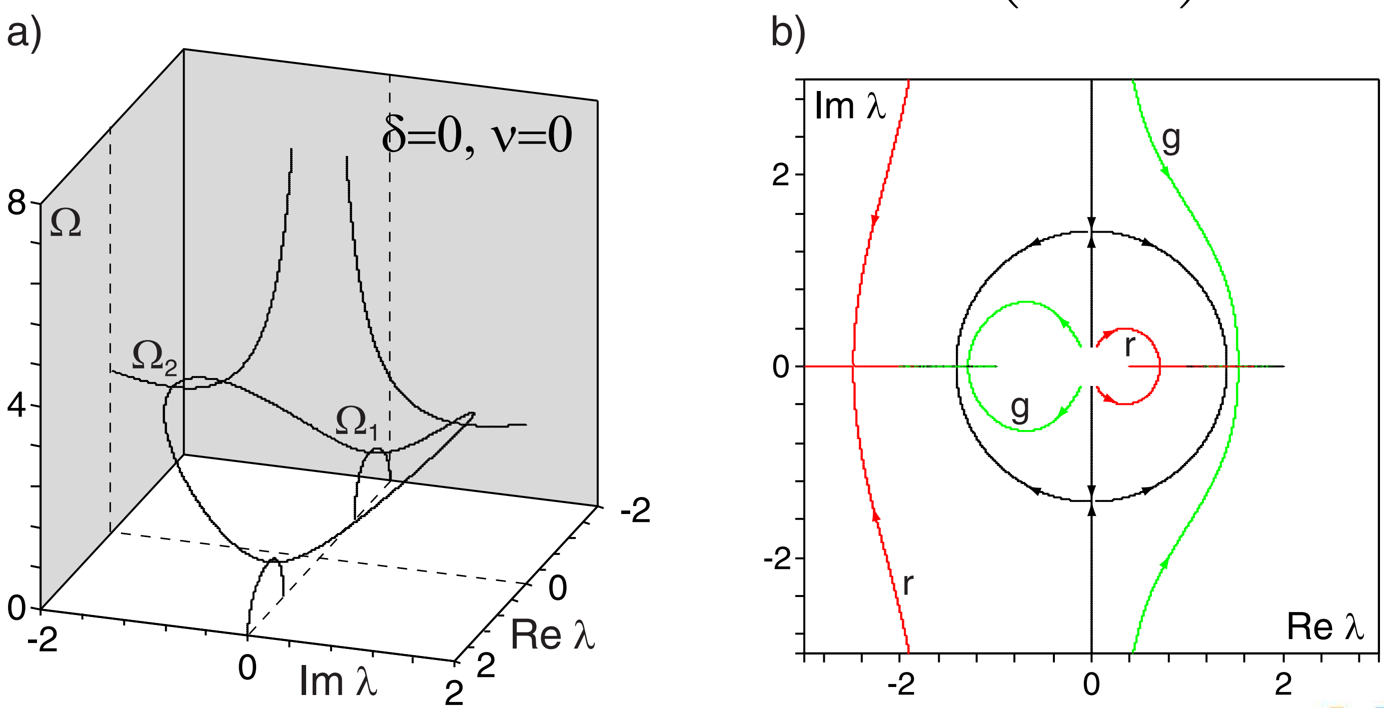
$$\omega_+(\Omega) > \omega_-(\Omega) > 0$$

Krein signature is positive for $i\omega_+(\Omega)$, negative for $i\omega_-(\Omega)$

Full dissipation (δ=1, ν=0) destabilizes eigenvalues with negative Krein signature (red curves)

Circulatory forces (δ=0, ν=1) destabilize eigenvalues with positive Krein signature (green curves)

$$\mathbf{D} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} > 0$$

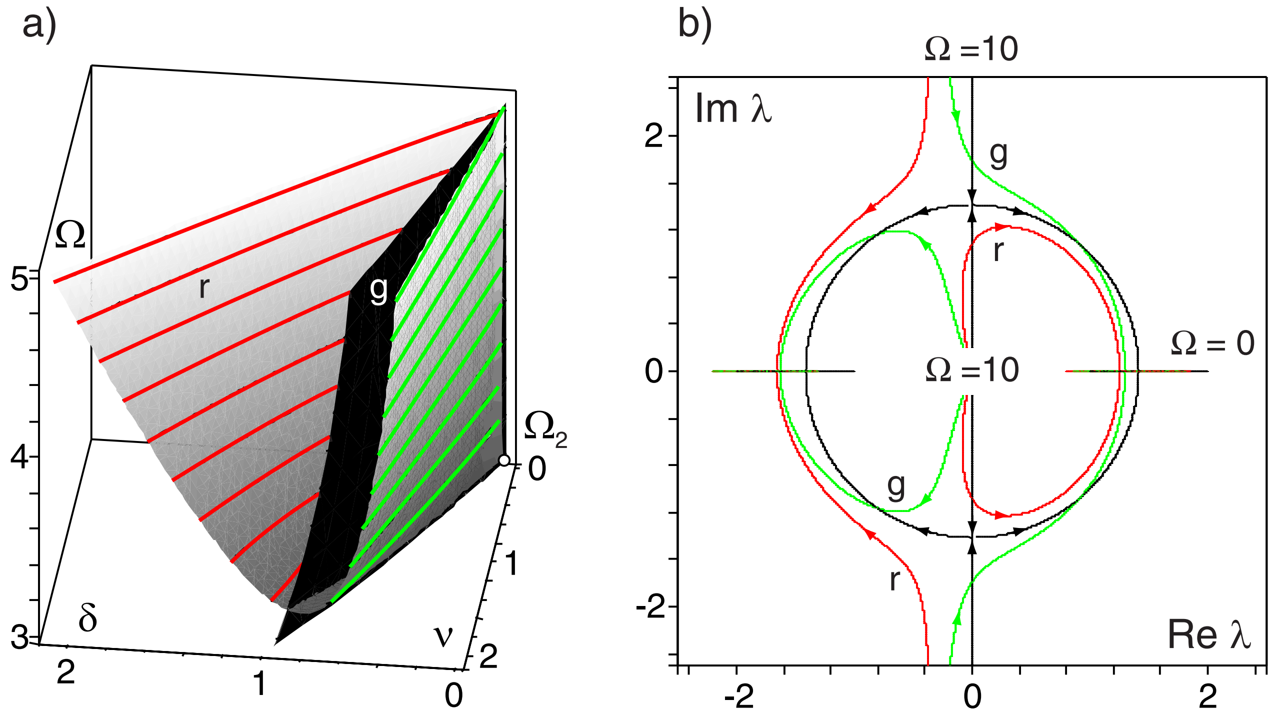


Gyroscopic stabilization

in the presence of damping and non-conservative positional forces

$$\kappa_1 = -1, \quad \kappa_2 = -4, \quad \mathrm{tr}\mathbf{D} = 3, \quad \mathrm{tr}\mathbf{KD} = -6, \quad \det \mathbf{D} = 1$$

$\delta = 0.3, \nu = 0.6$ red eigencurves $\delta = 0.3, \nu = 0.9$ green eigencurves



Switching surface

has a tangent cone as its linear approximation at the singular point

$$\nu = \delta\Omega \frac{\delta^2 \mathrm{tr}\mathbf{D} \det \mathbf{D} + 4\Omega_2\gamma_* + \mathrm{tr}\mathbf{D}(\Omega^2 - \Omega_2^2)}{\delta^2(\mathrm{tr}\mathbf{D})^2 + 4\Omega^2}$$

Tangent cone:

$$\{\nu = \gamma_*\delta, \quad \Omega > \Omega_2, \quad \nu > 0, \quad \delta > 0\} \qquad \gamma_* = \frac{\mathrm{tr}\mathbf{KD} + \left(\Omega_2^2 - \omega_0^2\right)\mathrm{tr}\mathbf{D}}{2\Omega_2}$$

Movement of eigenvalues (approximation near singularity):

$$(\mathrm{Im}\lambda - \omega_0 - \mathrm{Re}\lambda - a/2)^2 - (\mathrm{Im}\lambda - \omega_0 + \mathrm{Re}\lambda + a/2)^2 = 2d$$

$$a = \frac{\mathrm{tr}\mathbf{D}}{2}\delta, \quad d = \frac{\Omega_2}{2\omega_0}(\gamma_*\delta - \nu)$$

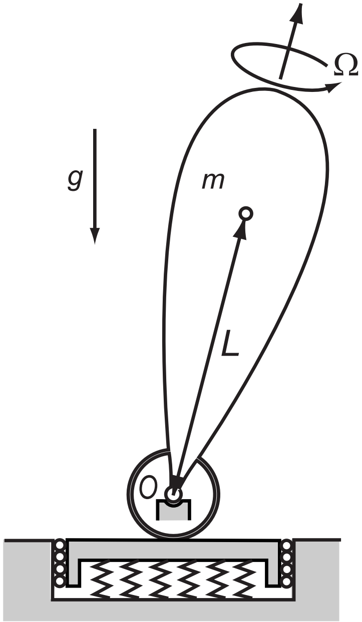
HZDR

1995 Crandall

gyropendulum with stationary and rotating damping

$$\ddot{\mathbf{z}} + \begin{pmatrix} \sigma + \rho & \eta\Omega \\ -\eta\Omega & \sigma + \rho \end{pmatrix} \dot{\mathbf{z}} + \begin{pmatrix} -\alpha^2 & \rho\Omega \\ -\rho\Omega & -\alpha^2 \end{pmatrix} \mathbf{z} = 0$$

$$\eta = \frac{I_a}{I_d}, \quad \sigma = \frac{b_s}{I_d}, \quad \rho = \frac{b_r}{I_d}, \quad \alpha^2 = \frac{mgL}{I_d}$$



Drag force,
stationary damping (b_s)

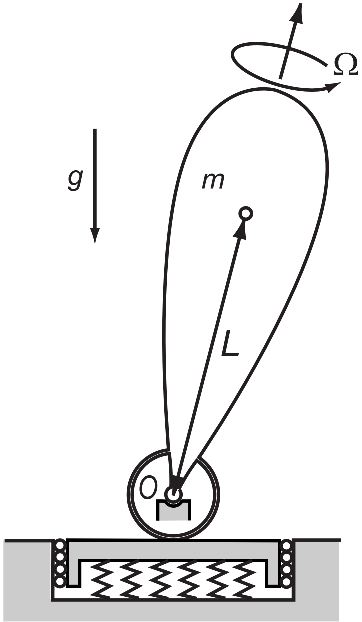
„Viscous friction force",
rotating damping (b_r)

1995 Crandall

gyropendulum with stationary and rotating damping

$$\ddot{\mathbf{z}} + \begin{pmatrix} \sigma+\rho & \eta\Omega \\ -\eta\Omega & \sigma+\rho \end{pmatrix}\dot{\mathbf{z}} + \begin{pmatrix} -\alpha^2 & \rho\Omega \\ -\rho\Omega & -\alpha^2 \end{pmatrix}\mathbf{z} = 0$$

$$\eta = \frac{I_a}{I_d}, \quad \sigma = \frac{b_s}{I_d}, \quad \rho = \frac{b_r}{I_d}, \quad \alpha^2 = \frac{mgL}{I_d}$$



Gyroscopic stabilization ($\sigma,\rho=0$):

$$\Omega > \Omega_0^+ = \frac{2\alpha}{\eta}$$

Asymptotic stability ($\sigma,\rho\neq 0$):

$$\Omega^2 > {\Omega_0^+}^2 + \frac{1}{\rho}\frac{\alpha^2}{\eta^2}\frac{(\sigma\eta+\rho(\eta-2))^2}{\sigma\eta+\rho(\eta-1)} \geq {\Omega_0^+}^2$$

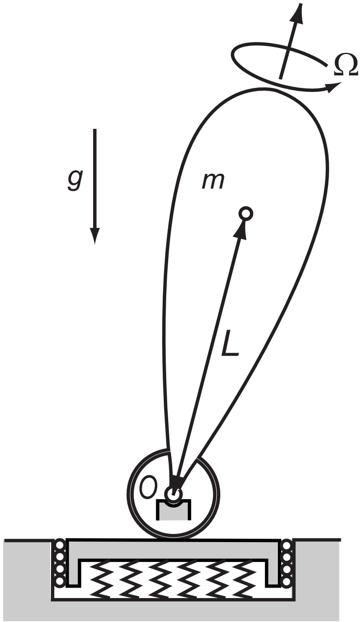
$$\sigma+\rho>0$$

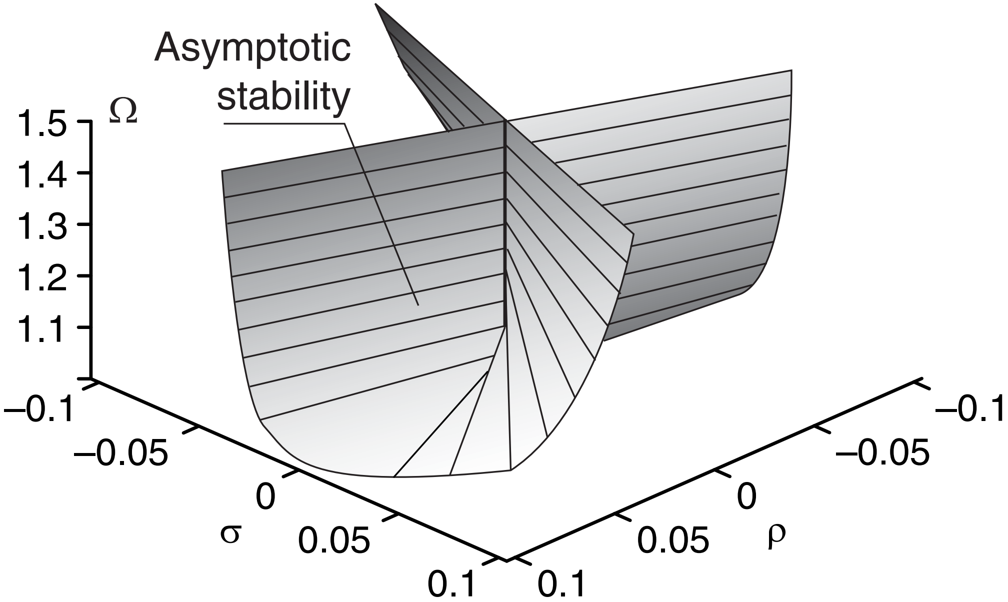
1995 Crandall

gyropendulum with stationary and rotating damping

$$\ddot{\mathbf{z}} + \begin{pmatrix} \sigma+\rho & \eta\Omega \\ -\eta\Omega & \sigma+\rho \end{pmatrix}\dot{\mathbf{z}} + \begin{pmatrix} -\alpha^2 & \rho\Omega \\ -\rho\Omega & -\alpha^2 \end{pmatrix}\mathbf{z} = 0$$

$$\eta = \frac{I_a}{I_d}, \quad \sigma = \frac{b_s}{I_d}, \quad \rho = \frac{b_r}{I_d}, \quad \alpha^2 = \frac{mgL}{I_d}$$





2008 Samantaray et al.

Fast/slow precession destabilization of the Crandall gyropendulum

Fast: $\frac{\sigma}{\rho} < \frac{2-\eta}{\eta}$ Slow: $\frac{\sigma}{\rho} > \frac{2-\eta}{\eta}$

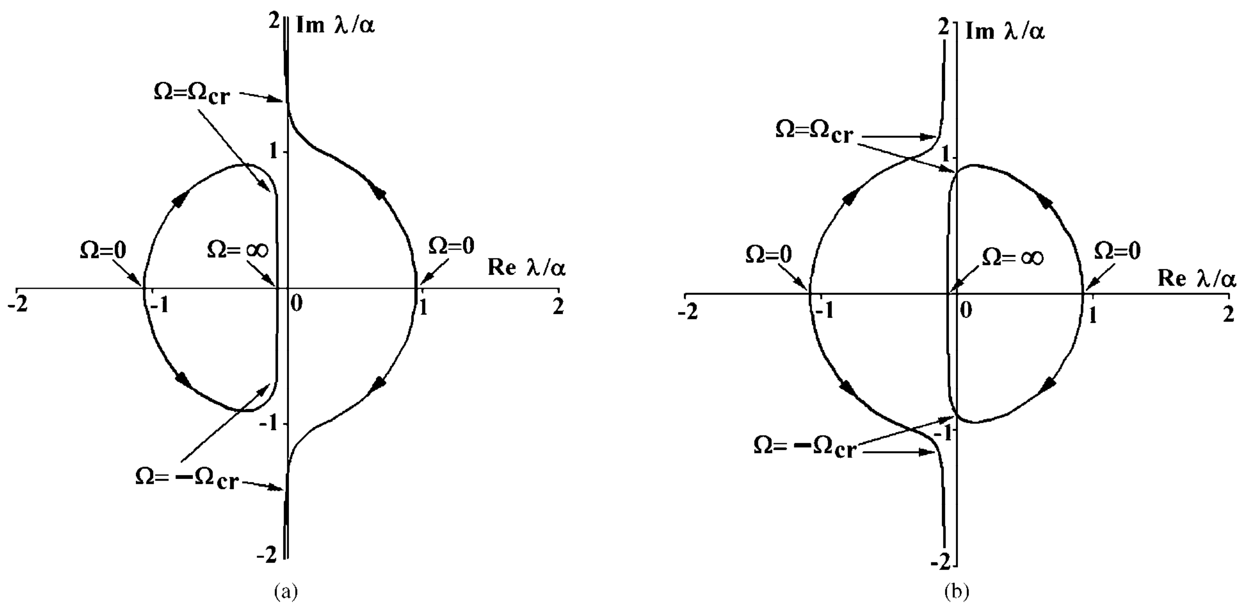


Fig. 2. Variation of the eigenvalues (scaled) of the gyropendulum. (a) Case $\sigma/\rho < (2-\eta)/\eta$ with parameter values $\eta = 1.5$, $\alpha = 10\ \mathrm{s}^{-1}$, $\sigma = 0$ and $\rho = 1\ \mathrm{s}^{-1}$. (b) Case $\sigma/\rho > (2-\eta)/\eta$ with parameter values $\eta = 1.5$, $\alpha = 10\ \mathrm{s}^{-1}$, $\sigma = 0.5\ \mathrm{s}^{-1}$ and $\rho = 1\ \mathrm{s}^{-1}$.

2008 Samantaray et al.

Fast/slow precession destabilization in the Crandall gyropendulum

Fast: $\frac{\sigma}{\rho} < \frac{2-\eta}{\eta}$

Slow: $\frac{\sigma}{\rho} > \frac{2-\eta}{\eta}$

Whitney umbrella:

$$\Omega^2 = {\Omega_0^+}^2 + \frac{1}{\rho}\frac{\alpha^2}{\eta^2}\frac{(\sigma\eta + \rho(\eta-2))^2}{\sigma\eta + \rho(\eta-1)}$$

Tangent cone: $\frac{\sigma}{\rho} = \frac{2-\eta}{\eta}, \quad \Omega > \Omega_0^+, \quad ,\rho > 0$

Energy balance at the tangent cone: the work done by damping equals to the work of circulatory forces

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