Dark Solitons and Vortices in Bose-Einstein Condensates

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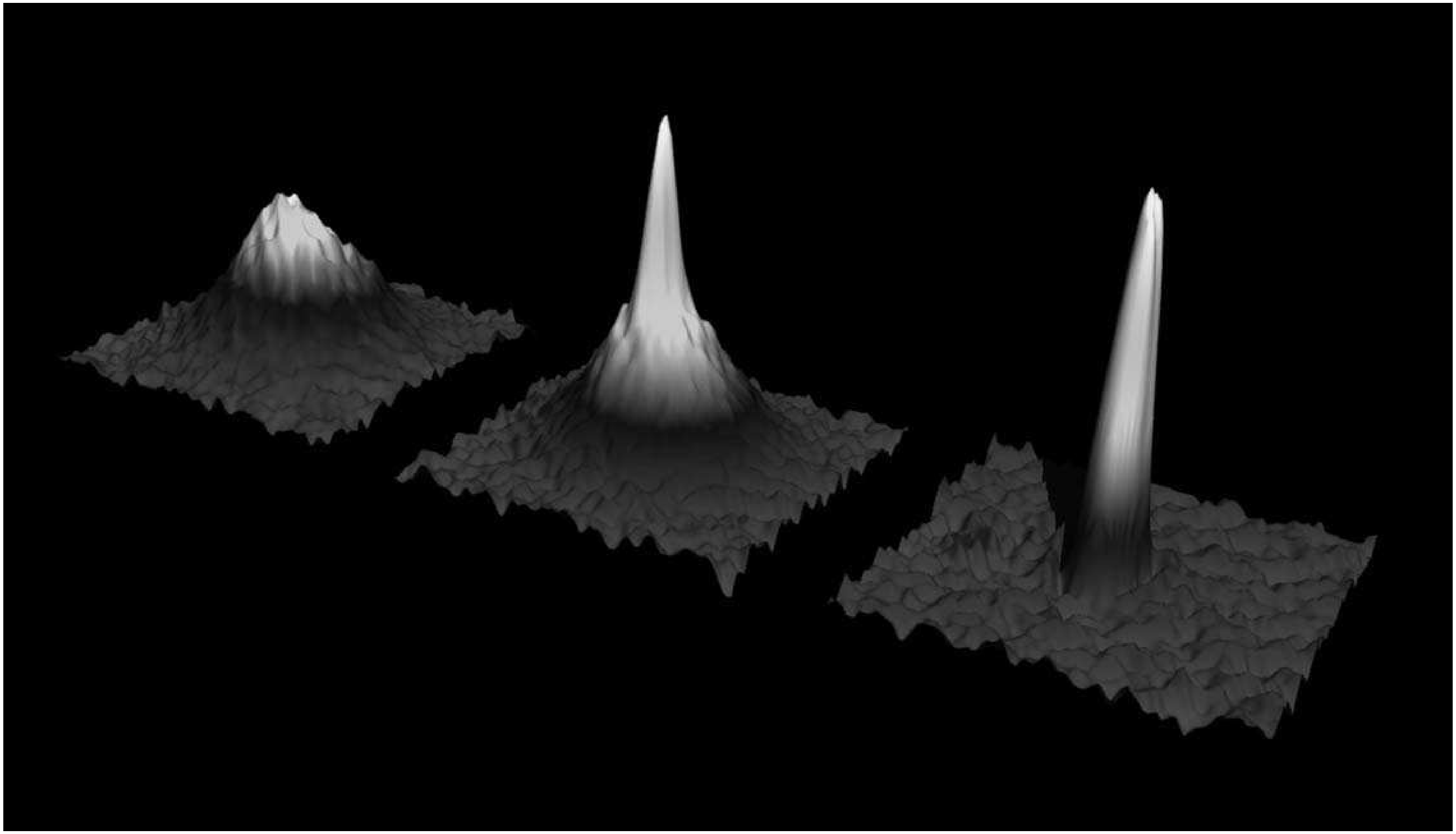
References

- 1d GPE:
  - Contemporary Mathematics **473**, 159 (2008)
  - Z. Agnew. Math. Phys. **59**, 559 (2008)
  - Phys. Rev. Lett. **104**, 244302 (2010)
  - Nonlinearity **23**, 1753 (2010)
  - Phys. Rev. A **81**, 053618 (2010)
- Experimental Results:
  - Phys. Rev. Lett. **101**, 130401 (2008)
  - Phys. Rev. A **81**, 063604 (2010)
  - Phys. Rev. A **84**, 011605 (2011)
- Generalizations to Vortices:
  - J. Phys. B **43**, 155303 (2010)
  - Phys. Rev. A **82**, 013646 (2010)
  - EPL **93**, 20008 (2011)
  - arXiv:1207.0386

Brief Introduction to BECs

- 1924: S. Bose and A. Einstein realize that Bose statistics predicts a Maximum Atom Number in the Excited States: a Quantum Phase Transition.
- 1995: E. Cornell, C. Wieman and W. Ketterle realize BEC in a dilute gas of ^{87}Rb and ^{23}Na: 2001 Nobel Prize.
- Today:
  - ~ 35 Experimental Groups have achieved BEC (in 100-10^8 atoms of Rb, Li, Na, H).
  - O(10^3) Theoretical and O(10^2) Experimental papers ! Check out: http://amo.phy.gasou.edu/bec.html/bibliography.html



Mean-Field Models of BEC: why do we care ?

BEC

- Many Body Hamiltonian

$$\hat{H} = \int d\mathbf{r}\hat{\Psi}^\dagger \left[-\frac{\hbar^2}{2m}\Delta + V_{\text{ext}}(\mathbf{r})\right]\hat{\Psi} + \frac{1}{2}\int d\mathbf{r}d\mathbf{r}'\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}^\dagger(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\hat{\Psi}(\mathbf{r}')\hat{\Psi}(\mathbf{r}) \tag{1}$$

- Bogoliubov Decomposition:

$$\hat{\Psi} = \Phi(\mathbf{r},t) + \hat{\Psi}'(\mathbf{r},t) \tag{2}$$

- Φ is now a regular wavefunction (the expectation value of the field operator). Its equation:

$$i\hbar\frac{\partial\Phi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Phi + V_{\text{ext}}(\mathbf{r})\Phi + g|\Phi|^2\Phi \tag{3}$$

- for dilute, cold, binary collision gas.

- But: This is 3D NLS with a Potential: GP !

Low Dimensional Reductions

- 1d Magnetic Trap and/or Optical Lattice

$$V(x) = \frac{1}{2}\Omega^2 x^2 + V_0 \sin^2(kx + \theta) \tag{4}$$

- 2d Magnetic Trap and/or Optical Lattice

$$V(x,y) = \frac{1}{2}\left(\Omega_x^2 x^2 + \Omega_y^2 y^2\right) + V_0\left(\sin^2(kx+\theta) + \sin^2(ky+\theta)\right) \tag{5}$$

- Discrete Models: Reduction through Wannier functions (WF)

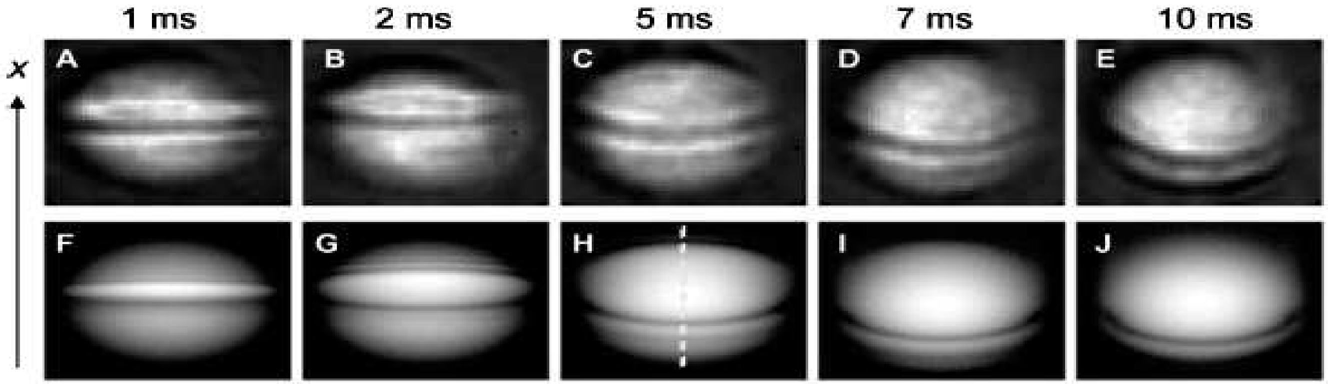
$$\psi(x,t) = \sum_{n\alpha} c_{n,\alpha}(t) w_{n,\alpha}(x) \tag{6}$$

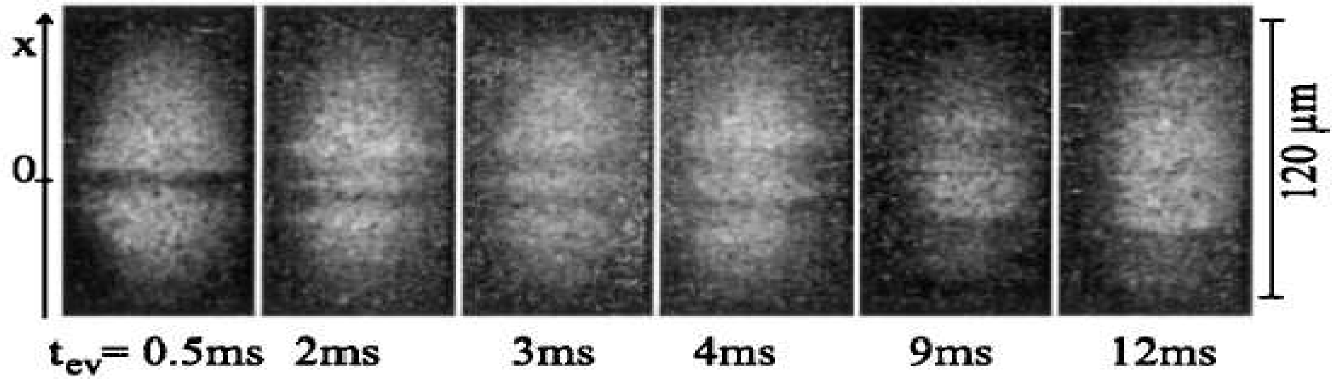
where the WF w_n of band α are expressed in terms of the Bloch functions $\phi_{k,\alpha}$ as:

$$w_\alpha(x - nL) = \sqrt{\frac{L}{2\pi}} \int_{-\pi/L}^{\pi/L} \varphi_{k,\alpha}(x) e^{-inkL} dk. \tag{7}$$

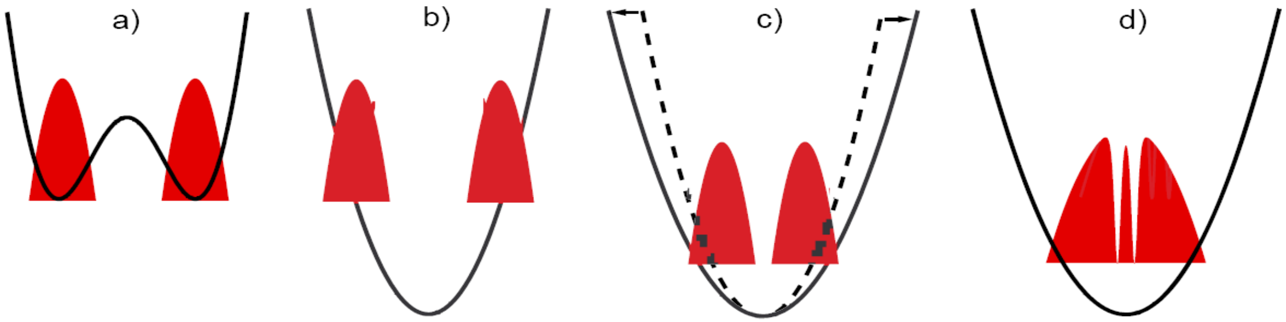
An Interesting Aspect: Dark Soliton Dynamics

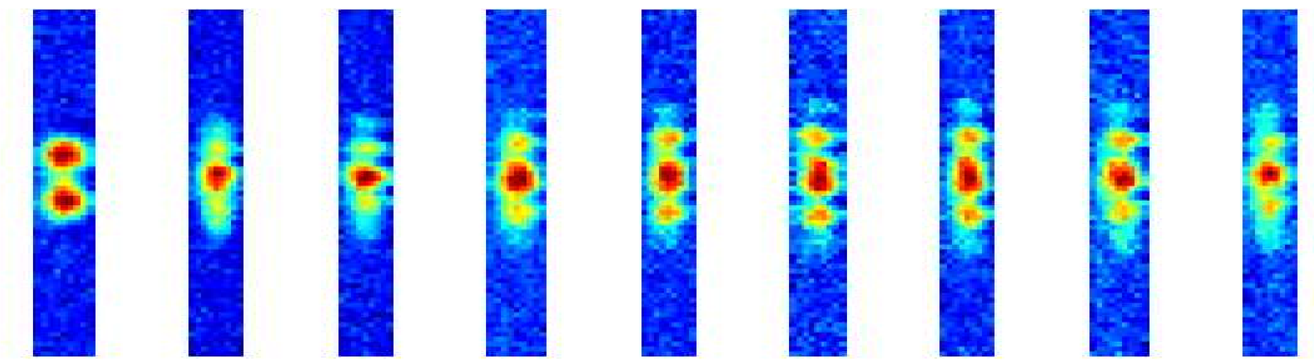
Early Experiments



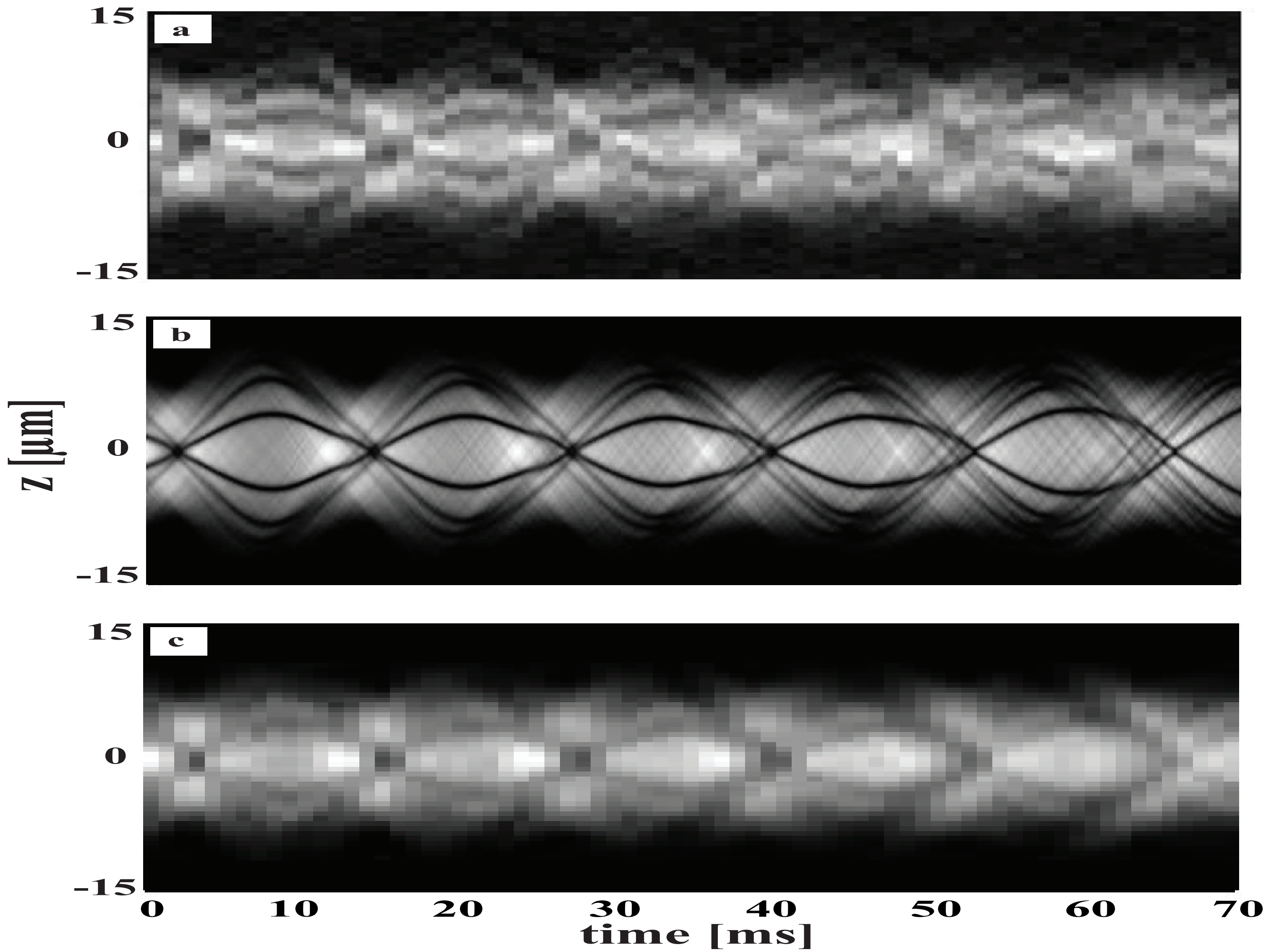


Our Specific Motivation: Dark Soliton Experiments in Heidelberg

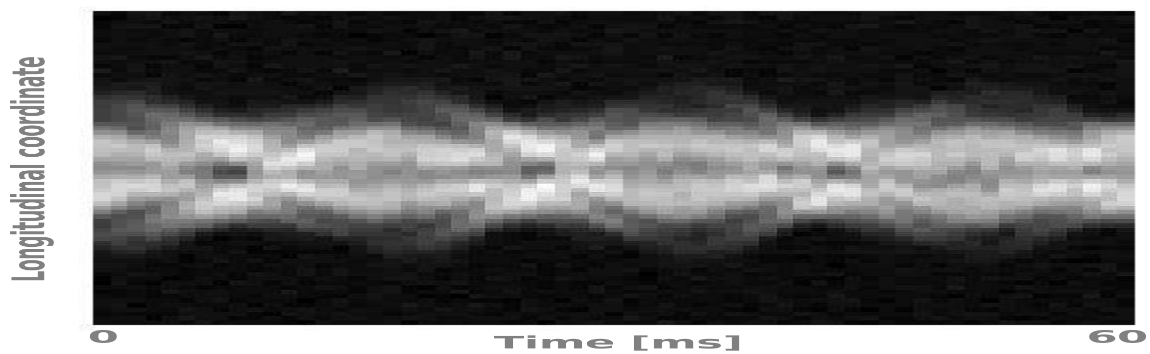


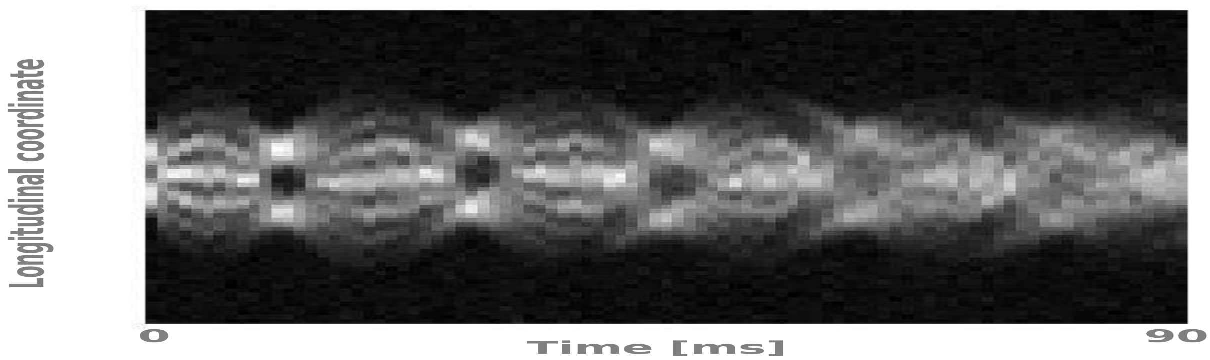


Detailed Dynamics and Computational Comparison



3-, 4-, N-soliton States





1st Line of Attack: 1d GPE

- Model reads

$$iu_t + \frac{1}{2}u_{xx} - |u|^2 u = \frac{1}{2}\omega^2 x^2 u \tag{8}$$

- Model assumes strong anisotropy ($\omega \ll 1$), and $\mu \ll \hbar\omega_\perp$, so that $\phi_0(r) \propto \exp(-r^2/2a_r^2)$.
- Consider Linear Limit $u(x,t) = \exp(-i\mu t)\sum_n A_n\phi_n(x)$ to obtain Bifurcation Function

$$F_n = (\mu - n)A_n - \sum_{n_1,n_2,n_3=0}^{\infty} K_{n,n_1,n_2,n_3} A_{n_1}\bar{A}_{n_2}A_{n_3}, \qquad \forall n = 0,1,2,... \tag{9}$$

- This leads to Near-Linear Solutions

$$\|\mathbf{A} - \varepsilon \mathbf{e}_1\|_{l^2_{1/2}} \leq C_1\varepsilon^3, \qquad \left|\mu - 1 - \sigma\varepsilon^2 K_{1,1,1,1}\right| \leq C_2\varepsilon^4, \tag{10}$$

where $K_{n,n_1,n_2,n_3} = (\phi_n, \phi_{n_1}\phi_{n_2}\phi_{n_3})$.
- Spectral Stability of These Solutions can be studied via:

$$\mathbf{a}(t) = e^{-i\mu t}\left[\mathbf{A} + (\mathbf{B} - \mathbf{C})\,e^{i\Omega t} + \left(\bar{\mathbf{B}} + \bar{\mathbf{C}}\right)e^{-i\bar{\Omega} t} + \mathrm{O}(\|\mathbf{B}\|^2 + \|\mathbf{C}\|^2)\right], \tag{11}$$

- This leads to Eigenvalue Problem:

$$L_+\mathbf{B} = \Omega\mathbf{C}, \qquad L_-\mathbf{C} = \Omega\mathbf{B}, \tag{12}$$

- In the above expression:

$$\left\{ \begin{array}{lcl} (L_+\mathbf{B})_n & = & (n-\mu)B_n + 3\sum_{m=0}^{\infty} V_{n,m}B_m, \\ (L_-\mathbf{C})_n & = & (n-\mu)C_n + \sum_{m=0}^{\infty} V_{n,m}C_m, \end{array} \right. \qquad \forall n = 0,1,2,3,..., \tag{13}$$

where $V_{n,m} = \sum_{n_2,n_3=0}^{\infty} K_{n,m,n_2,n_3} A_{n_2} A_{n_3}$.

- Using Perturbation Theory, we obtain:

$$\left| \Omega_m - m + \varepsilon^2 \sigma \left(K_{1,1,1,1} - 2K_{m+1,1,1,m+1} \right) \right| \leq C_2 \varepsilon^4, \tag{14}$$

$$\left| \Omega_1 - 1 + \frac{\varepsilon^2 \sigma}{8\sqrt{2\pi}} \right| \leq C_1 \varepsilon^4 \tag{15}$$

- Another Limit known is the so-called Thomas-Fermi Limit

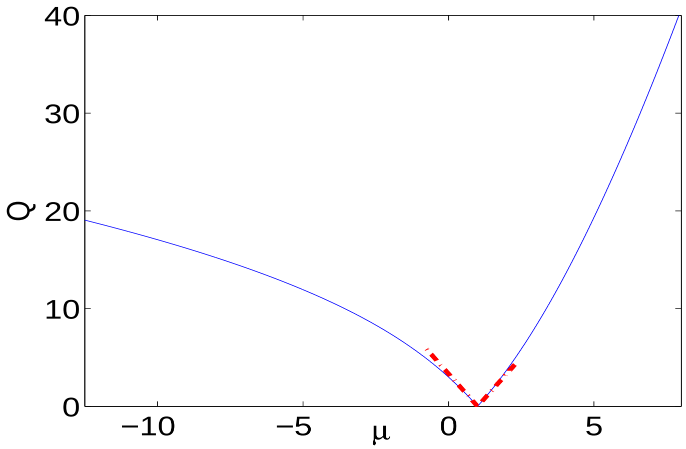
$$\sigma = 1: \quad \Omega_0 = 1, \;\; \lim_{\mu\to\infty} \Omega_1 = \frac{1}{\sqrt{2}}, \;\; \lim_{\mu\to\infty} \Omega_m = \frac{\sqrt{m(m+1)}}{\sqrt{2}}, \qquad m \geq 2 \tag{16}$$

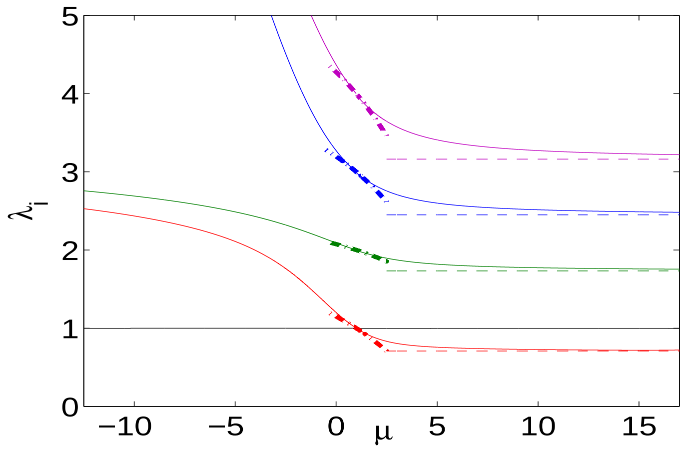
- Dipolar Oscillation Frequency $\Omega_0 = 1$ is fixed due to Transformation

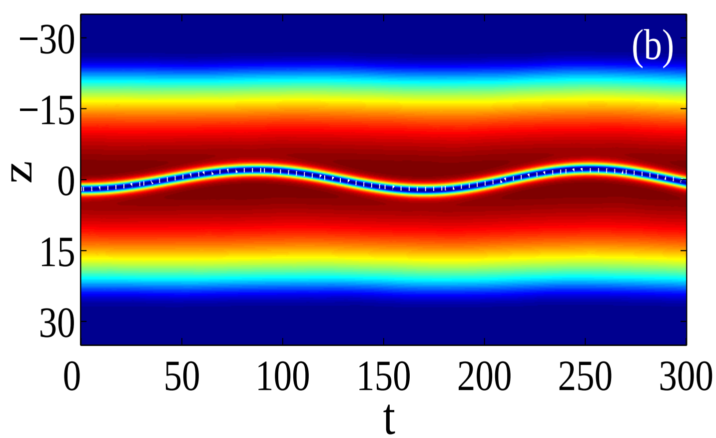
$$u(x,t) = e^{ip(t)x - \frac{i}{2}p(t)s(t) - \frac{i}{2}t - i\mu t - i\theta_0} \phi(x - s(t)), \tag{17}$$

where $\dot{s} = p$, $\dot{p} = -s$

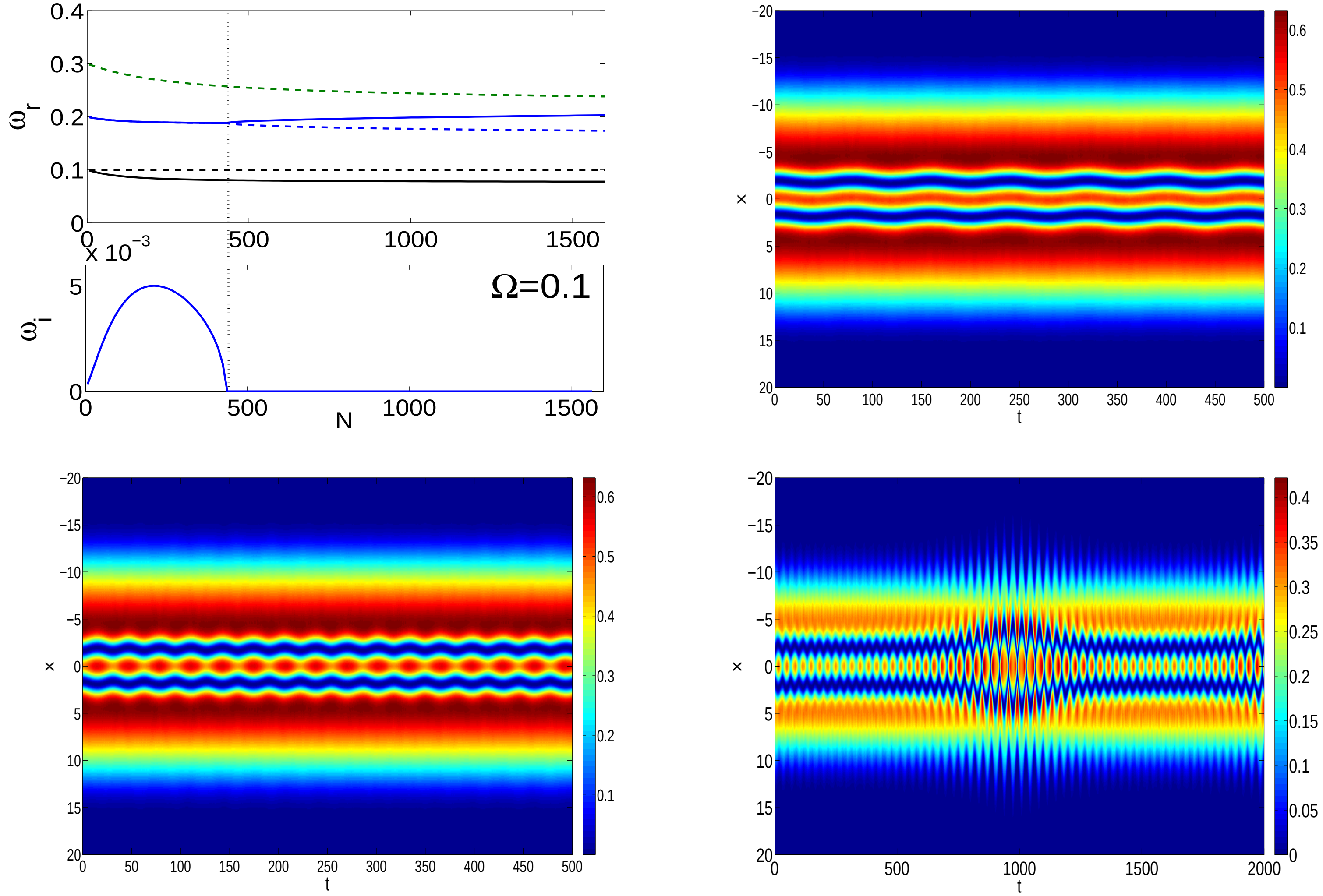
Numerical Findings in 1d Case



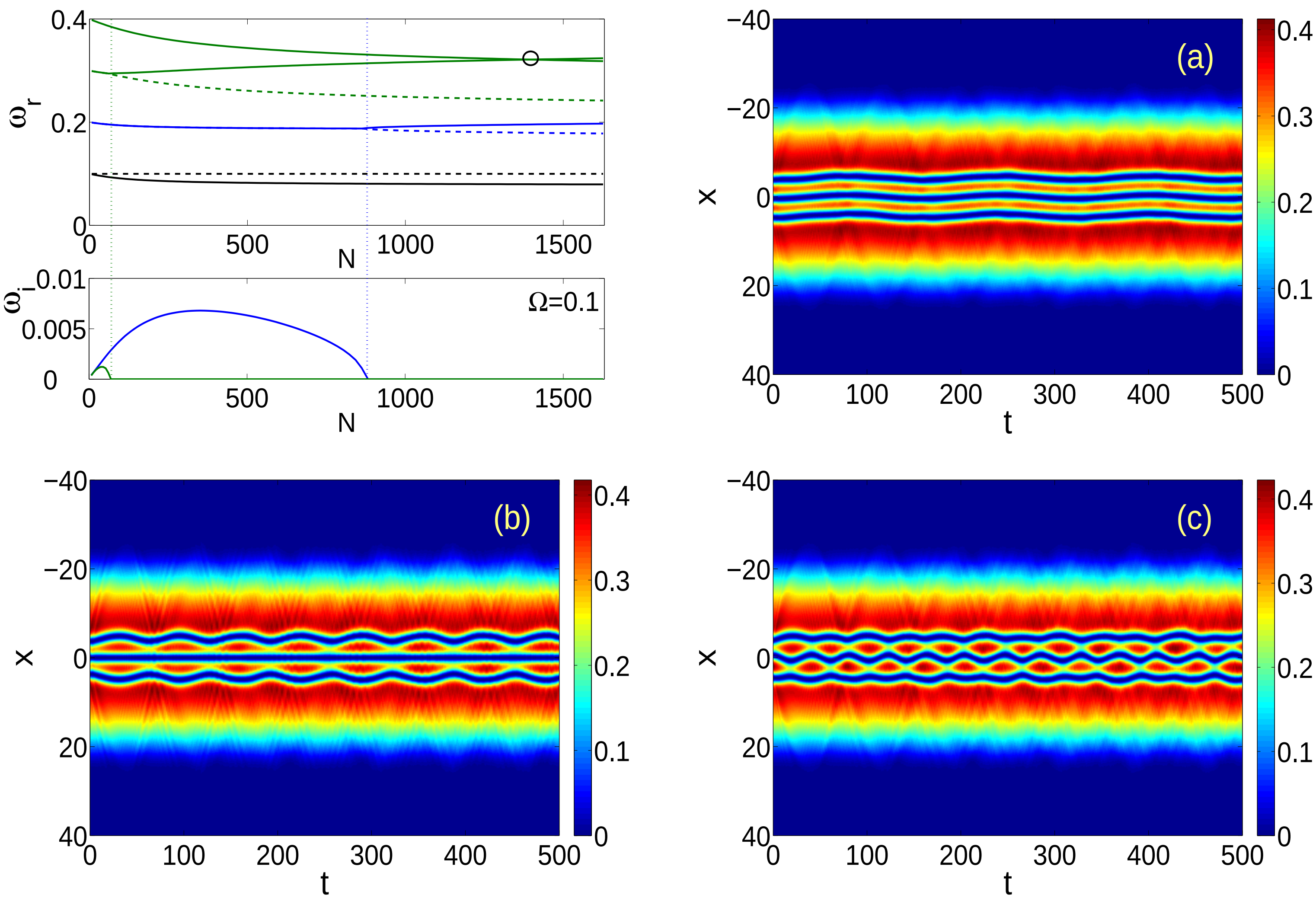




2-soliton Statics/Stability in the Crossover Regime



3-soliton Statics/Stability in the Crossover Regime



2nd Line of Attack: Developing Soliton Dynamics

- Integrable 1d NLS has 2-soliton solution (Akhmediev et al.)

$$u(x,t) = \frac{(2a_3 - 4a_1)\cosh(\frac{\mu t}{2}) - 2\sqrt{a_1 a_3}\cosh(2px) + i\sinh(\frac{\mu t}{2})}{2\sqrt{a_3}\cosh(\frac{\mu t}{2}) + 2\sqrt{a_1}\cosh(2px)} e^{i a_3 t} \tag{18}$$

where $\mu = 4\sqrt{a_1(a_3 - a_1)}$ and $p = \sqrt{a_3 - a_1}$.

- By computing $\partial |u|^2 / \partial x = 0$, we find soliton trajectories

$$\cosh(2px_0) = \sqrt{\frac{a_3}{a_1}}\cosh(\frac{\mu t}{2}) - 2\sqrt{\frac{a_1}{a_3}}\frac{1}{\cosh(\frac{\mu t}{2})} \tag{19}$$

- Interestingly, this yields insight on soliton interaction with point of closest approach

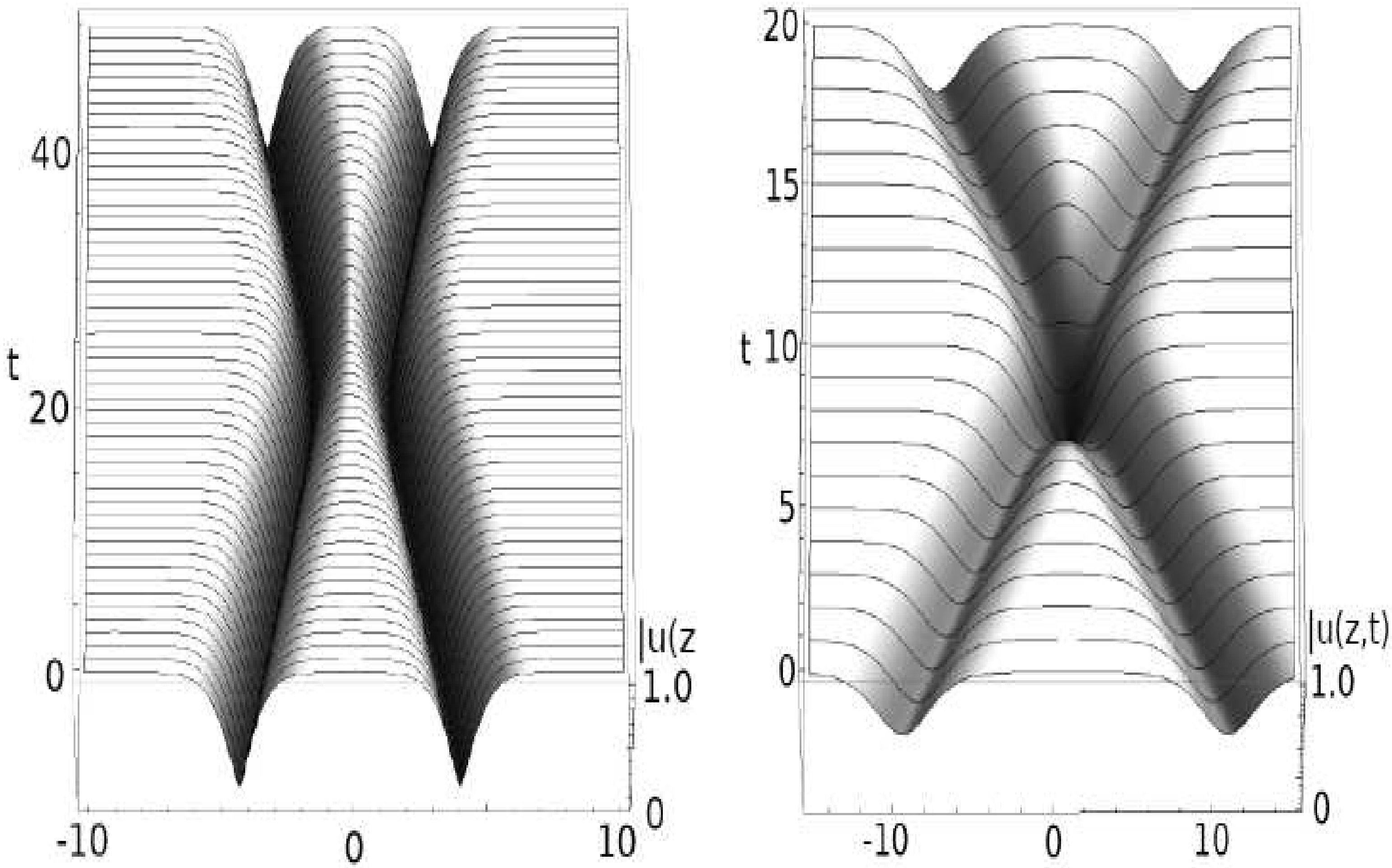
$$x_0 = \frac{1}{2\sqrt{a_3 - a_1}}\cosh^{-1}\left(\sqrt{\frac{a_3}{a_1}} - 2\sqrt{\frac{a_1}{a_3}}\right) \tag{20}$$

with $x_0 = 0$ for $a_1/a_3 = 1/4$ ($v = 0.5$)

- More importantly, for well-separated solitons (notice asymptotic limits)

$$\frac{dx_0}{dt} = \frac{\mu\sqrt{\frac{a_3}{a_1}}\sinh(\frac{\mu t}{2})}{2p\sqrt{\frac{a_3}{a_1}\cosh^2(\frac{\mu t}{2}) - 1}} \tag{21}$$

Integrable Dynamics



Developing Soliton Dynamics (With and Without Trap)

- Soliton Equation of Motion

$$\frac{d^2x_0}{dt^2} = 8p^3\frac{\cosh(2px_0)}{\sinh^3(2px_0)} \tag{22}$$

- Thus, obtain Inter-Soliton Interaction Potential (cf. Krolikowski-Kivshar)

$$V = \frac{1}{2}\frac{p^2}{\sinh^2(2px_0)} \tag{23}$$

- Importantly, notice that, in principle, V is dependent on $p = \sqrt{1 - \dot{x_0}^2}$.
- This expression can be generalized to Multi-Soliton Dynamics as:

$$V_{int} = \sum_{i \neq j}^{n} \frac{\mu p_{ij}^2}{2\sinh^2[\sqrt{\mu} p_{ij}(x_i - x_j)]}. \tag{24}$$

- The Single Soliton Trapping Potential can be added:

$$V_{full} = V_{int} + \frac{1}{2}\omega_{eff}^2 x^2 \tag{25}$$

An Alternative View of this "Superposition"

4-th Line of Attack: Variational Approach for Dark Solitons

- For simplicity, consider the GPE model:

$$iv_\tau = -\frac{1}{2}v_{\xi\xi} + \frac{1}{2}\xi^2 v + |v|^2 v - \mu v, \tag{26}$$

- under the Change of Variables:

$$v(\xi,t) = \mu^{1/2}u(x,t), \quad \xi = (2\mu)^{1/2}x, \quad \tau = 2t, \tag{27}$$

leads to (with $\epsilon = (2\mu)^{-1}$ used as a Small Parameter)

$$i\epsilon u_t + \epsilon^2 u_{xx} + (1 - x^2 - |u|^2)u = 0, \tag{28}$$

- Then using information established about the ground state:

$$\eta_0(x) := \lim_{\epsilon\to 0}\eta_\epsilon(x) = \begin{cases} (1-x^2)^{1/2}, & \text{for } |x| < 1, \\ 0, & \text{for } |x| > 1. \end{cases} \tag{29}$$

we can use the ansatz $u(x,t) = \eta_\epsilon(x)v(x,t)$ within the Lagrangian $L(v) = K(v) + \Lambda(v)$, where

$$\Lambda(v) = \epsilon^2 \int_{\mathbb{R}} \eta_\epsilon^2(x)|v_x|^2 dx + \frac{1}{2}\int_{\mathbb{R}} \eta_\epsilon^4(x)(1-|v|^2)^2 dx. \tag{30}$$

$$K(v) = \frac{i}{2}\epsilon \int_{\mathbb{R}} \eta_\epsilon^2(x)(v\bar{v}_t - \bar{v}v_t)dx, \tag{31}$$

Single Trapped Dark Soliton

- For a Single Dark Soliton (choosing $A = \sqrt{1-b^2}$)

$$v_1(x,t) = A(t)\ \tanh\big(\epsilon^{-1}B(t)(x-a(t))\big) + ib(t), \tag{32}$$

the effective Lagrangian becomes

$$L_1 := \lim_{\epsilon\to 0}\frac{L(v_1)}{2\epsilon} = -\frac{\dot b}{\sqrt{1-b^2}}(a - \frac{1}{3}a^3) + b\sqrt{1-b^2}(1-a^2)\dot a$$
$$+\frac{2}{3}(1-a^2)(1-b^2)B + \frac{1}{3B}(1-a^2)^2(1-b^2)^2.$$

This leads to:

$$B = \frac{\sqrt{1-a^2}\sqrt{1-b^2}}{\sqrt{2}}.$$

$$\dot a = \sqrt{2}\sqrt{1-a^2}b, \quad \dot b = -\frac{\sqrt{2}a(1-b^2)}{\sqrt{1-a^2}},$$

which, in turn, leads to:

$$\ddot a + 2a = 0.$$

2 Trapped Dark Solitons

- Now: 2-Soliton State (use $a_1 = -a_2 = -a$ and $b_1 = -b_2 = -b$):

$$v_2(x,t) = \left[A_1(t)\ \tanh\left(\epsilon^{-1}B_1(t)(x-a_1(t))\right) + ib_1(t)\right] \times \left[A_2(t)\ \tanh\left(\epsilon^{-1}B_2(t)(x-a_2(t))\right) + ib_2(t)\right], \quad (33)$$

yields the corresponding Lagrangian. After simplifications, one can write

$$\Lambda_2 := \frac{\Lambda(v_2)}{2\epsilon} = \Lambda_+ + \Lambda_- + \Lambda_{\text{overlap}}, \quad \Lambda_\pm = \frac{4(1-a^2)^{3/2}(1-b^2)^{3/2}}{3\sqrt{2}} + \mathcal{O}(\epsilon^{1/3}).$$

$$\Lambda_{\text{overlap}} = -8\sqrt{2}(1-a^2)^{3/2}(1-b^2)^{5/2}\, e^{-4Ba\epsilon^{-1}}\left(1+\mathcal{O}(\epsilon^{1/3})\right).$$

- This, in turn, yields the nonlinear oscillator:

$$\ddot{a} + 2a = 8\sqrt{2}\epsilon^{-1}e^{-\frac{2\sqrt{2}a}{\epsilon}},$$

with equilibrium position:

$$a = \frac{\epsilon}{\sqrt{2}}\left(-\log(\epsilon) - \frac{1}{2}\log|\log(\epsilon)| + \frac{3}{2}\log(2) + o(1)\right) \quad \text{as} \quad \epsilon \to 0 \quad (34)$$

and oscillation frequency around it:

$$\omega_0^2(\epsilon) = 2 + \frac{32}{\epsilon^2}e^{-2\sqrt{2}a_0(\epsilon)\epsilon^{-1}} = 2 + \frac{4\sqrt{2}a_0(\epsilon)}{\epsilon} = -4\log(\epsilon) - 2\log|\log(\epsilon)| + 2 + 6\log(2) + o(1), \quad \text{as} \quad \epsilon \to 0. \quad (35)$$

m Trapped Dark Solitons

- Use the Ansatz:

$$v_m(x,t) = \prod_{j=1}^{m} \left(A_j(t)\ \tanh\left(\epsilon^{-1}B_j(t)(x-a_j(t))\right) + ib_j(t)\right). \tag{36}$$

- Obtain the Lagrangian

$$\L_m \sim -\sqrt{2}\sum_{j=1}^{m}\left(a_j^2+b_j^2\right) - 2\sum_{j=1}^{m} a_j\dot{b}_j - 8\sqrt{2}\sum_{j=1}^{m-1} e^{-\sqrt{2}(a_{j+1}-a_j)\epsilon^{-1}}.$$

- Derive the Equations of Motion

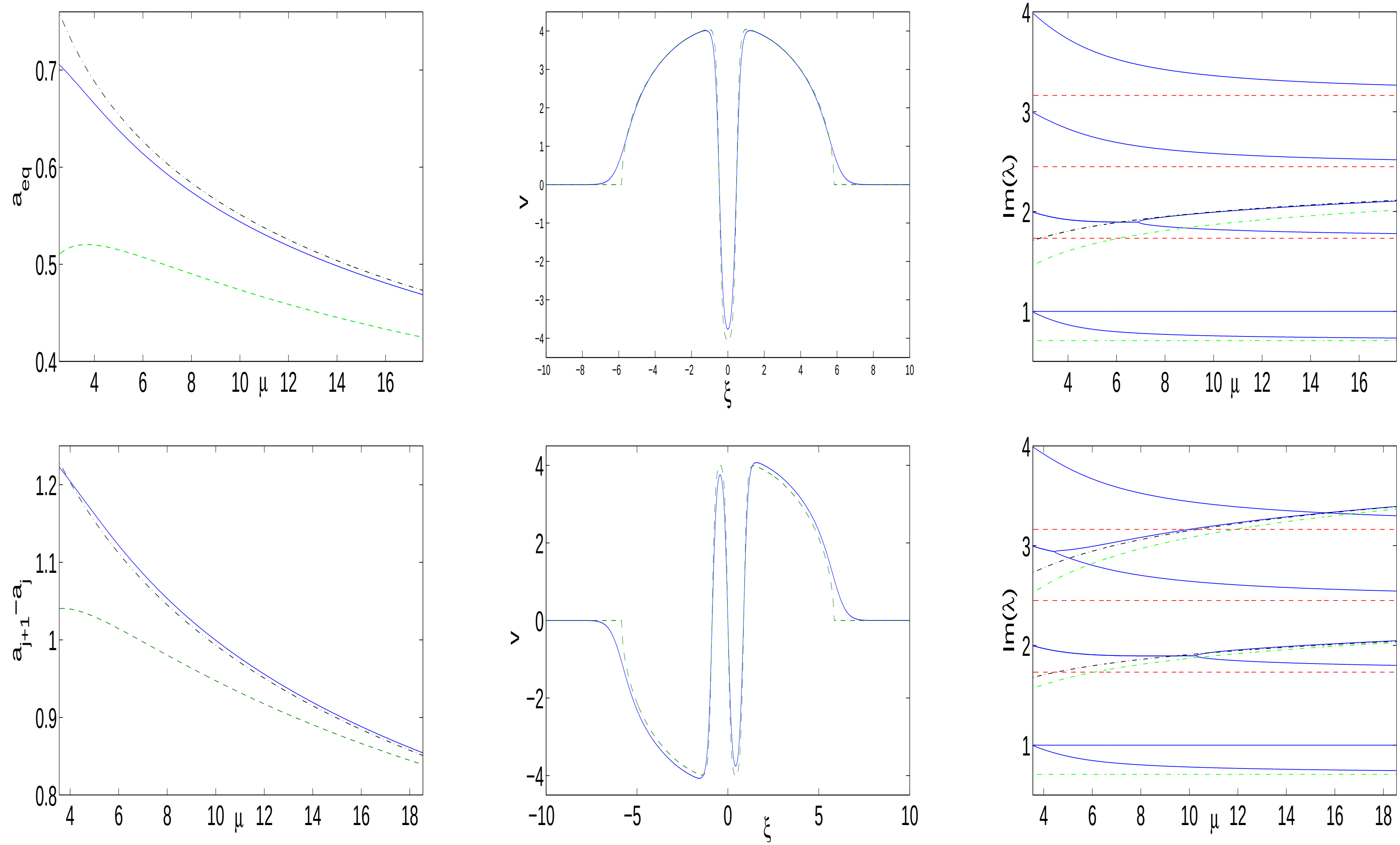
$$\dot{a}_j = \sqrt{2}b_j, \quad \dot{b}_j = -\sqrt{2}a_j - 8\epsilon^{-1}\left(e^{-\sqrt{2}(a_{j+1}-a_j)\epsilon^{-1}} - e^{-\sqrt{2}(a_j-a_{j-1})\epsilon^{-1}}\right). \tag{37}$$

- Define $x_j = \sqrt{2}(a_{j+1}-a_j)\epsilon^{-1}$, to find Equilibrium Distances and Oscillation Frequencies [With $\Omega^2 \in \left\{1,3,6,...,\frac{m(m-1)}{2}\right\}$].

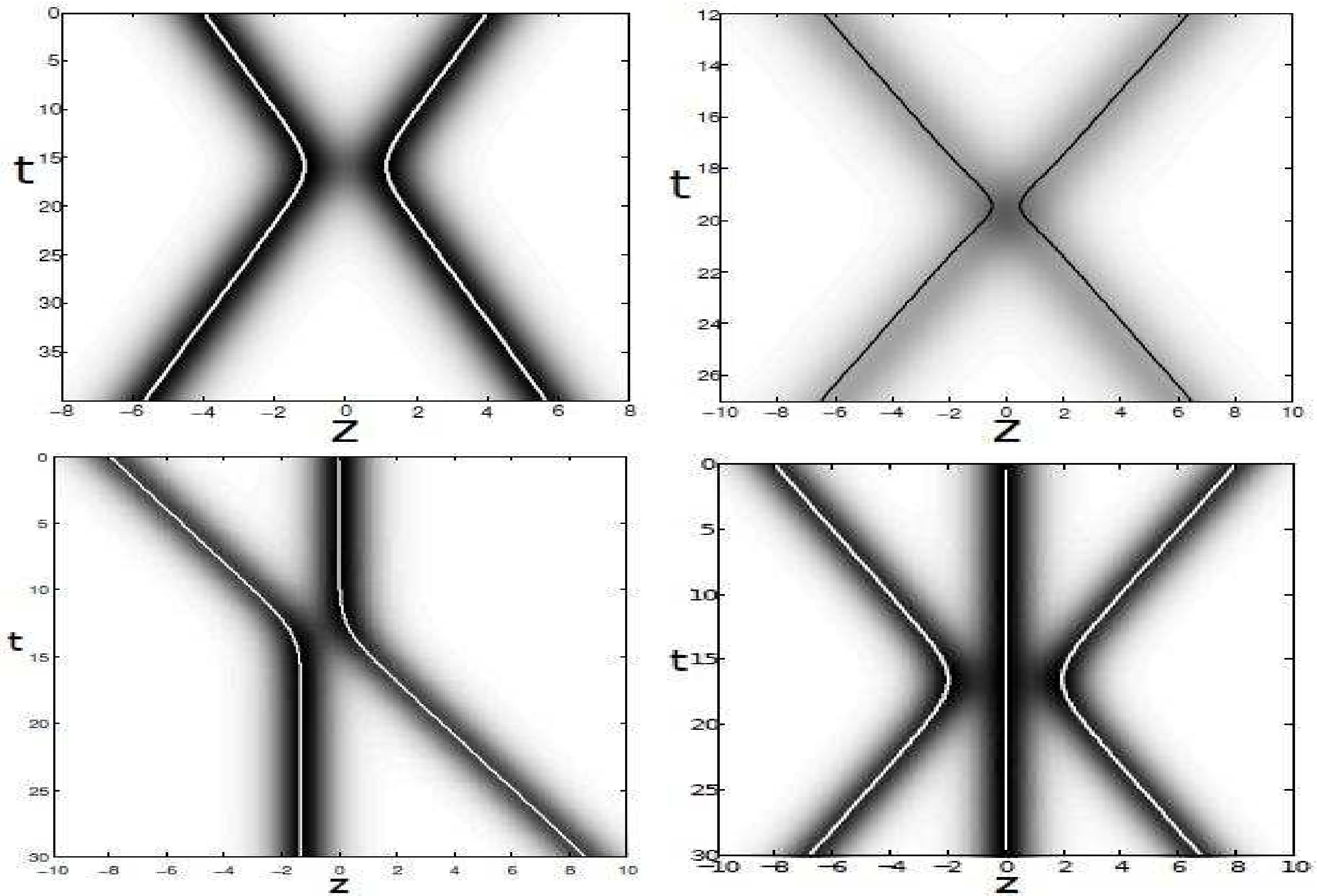
$$\mathbf{x} = -2\log(\epsilon)\mathbf{1} - \log|\log(\epsilon)|\mathbf{1} + 2\log(2)\mathbf{1} - \log(\mathbf{A}^{-1}\mathbf{1}) + o(1), \quad \text{as} \quad \epsilon \to 0, \tag{38}$$

$$\omega^2 = 2 + \left(-4\log(\epsilon) - 2\log|\log(\epsilon)| + 4\log(2)\right)\Omega^2 + \mathcal{O}(1). \tag{39}$$

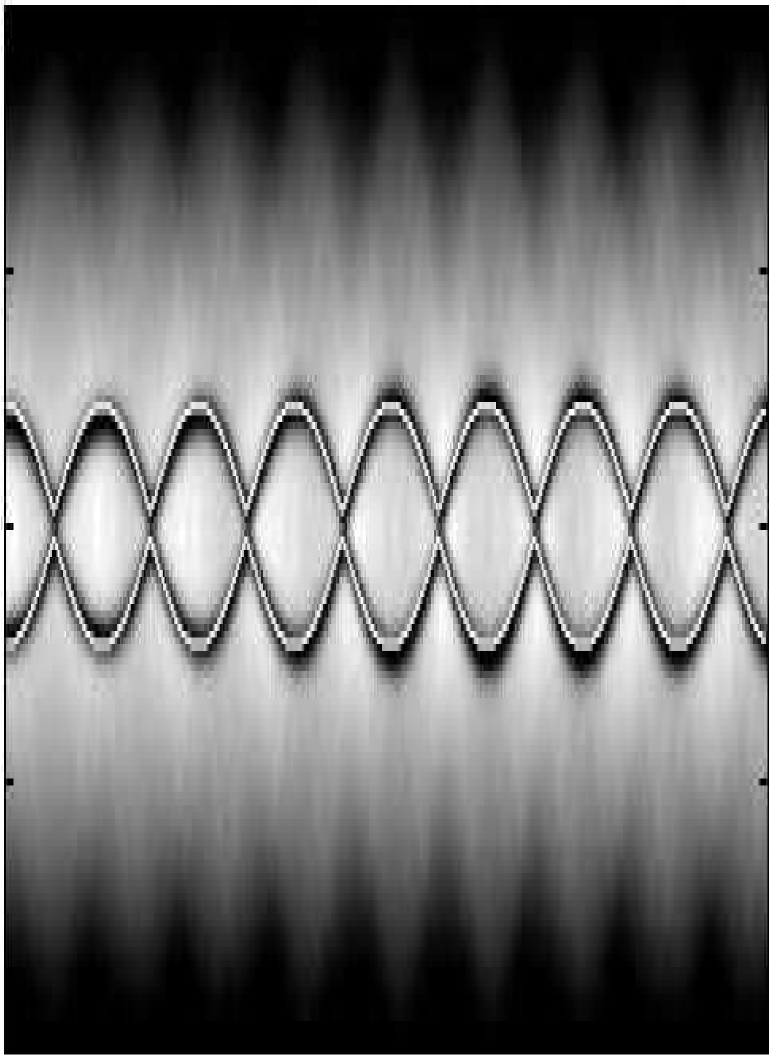
Testing the Prediction: Statics & Oscillation Frequencies

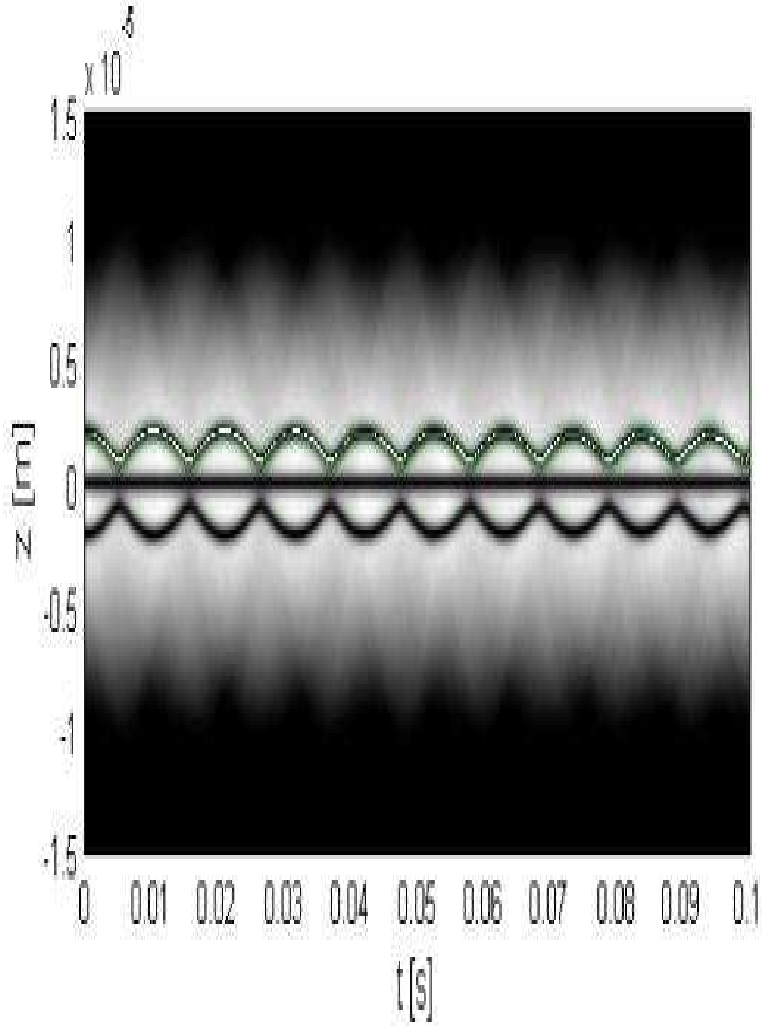


Testing the Prediction: Dynamics (Without Trapping)



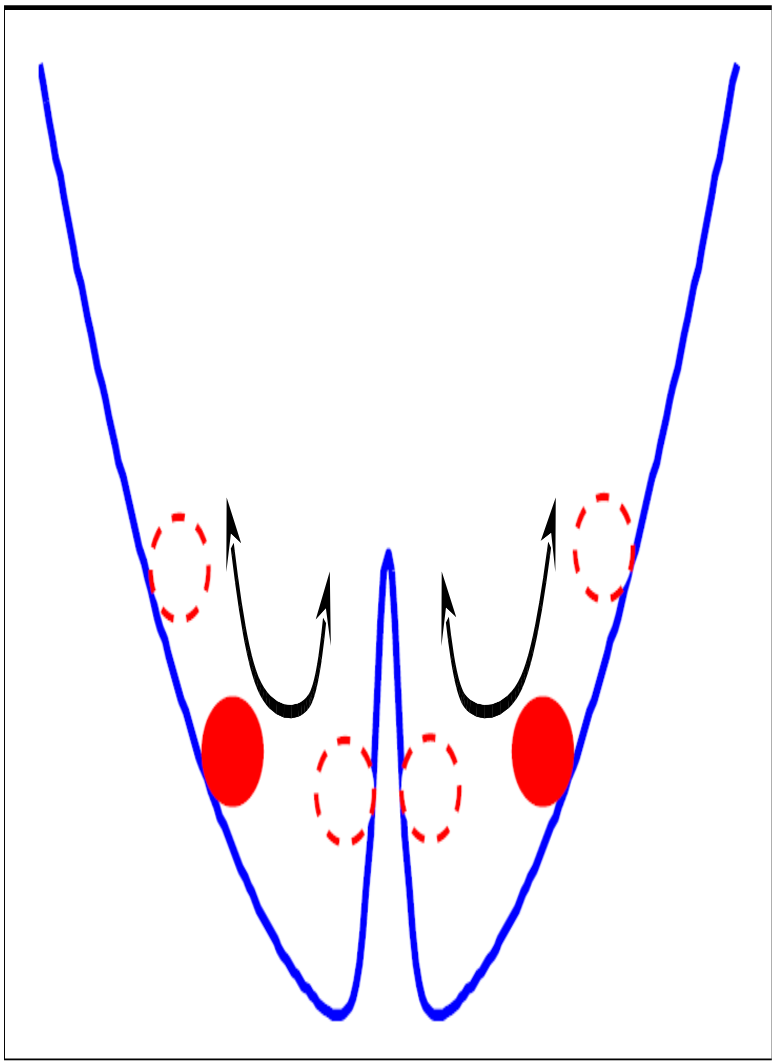
Testing the Prediction: Dynamics (With Trapping)

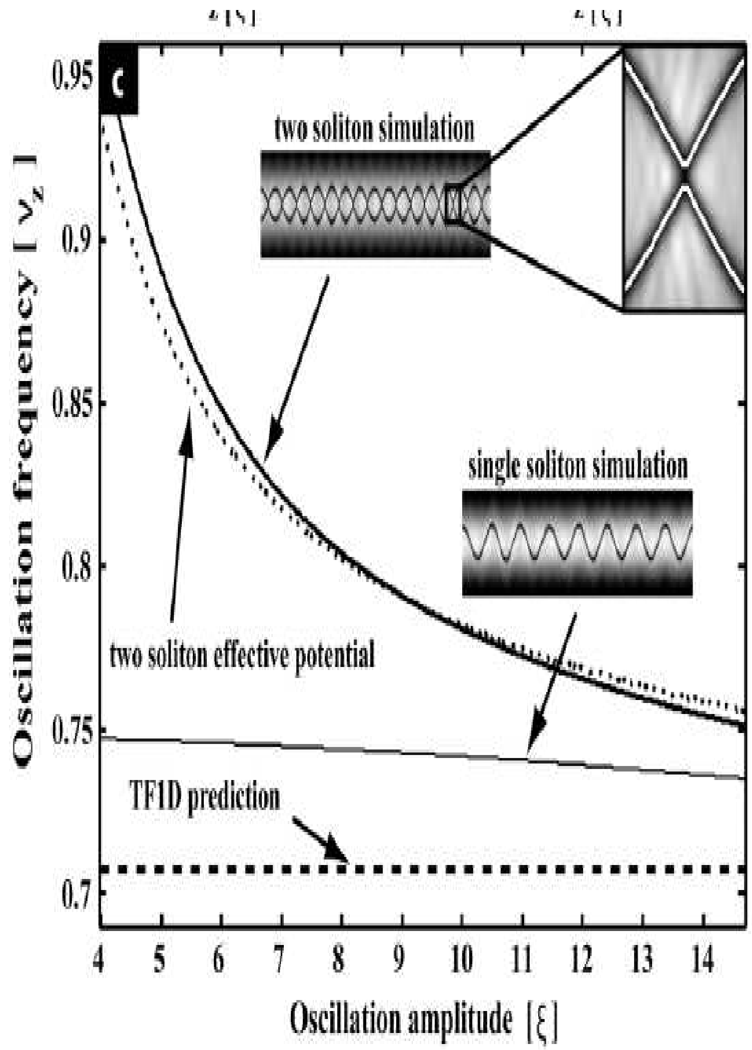




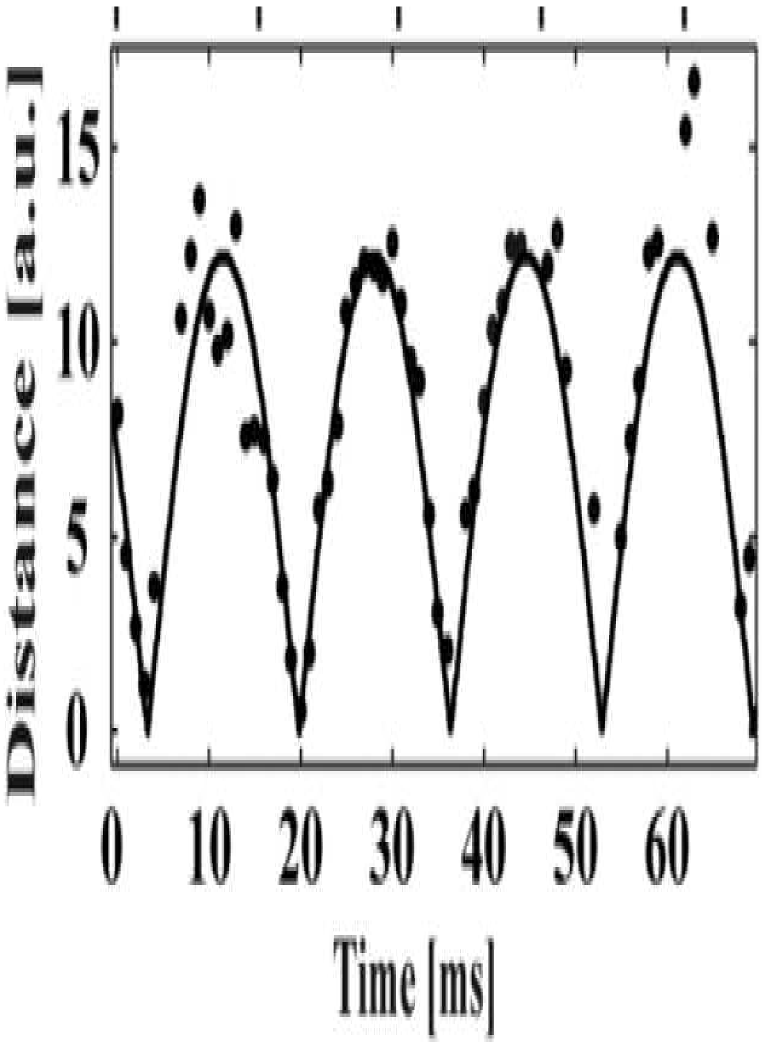
Now Collecting All the Chips !

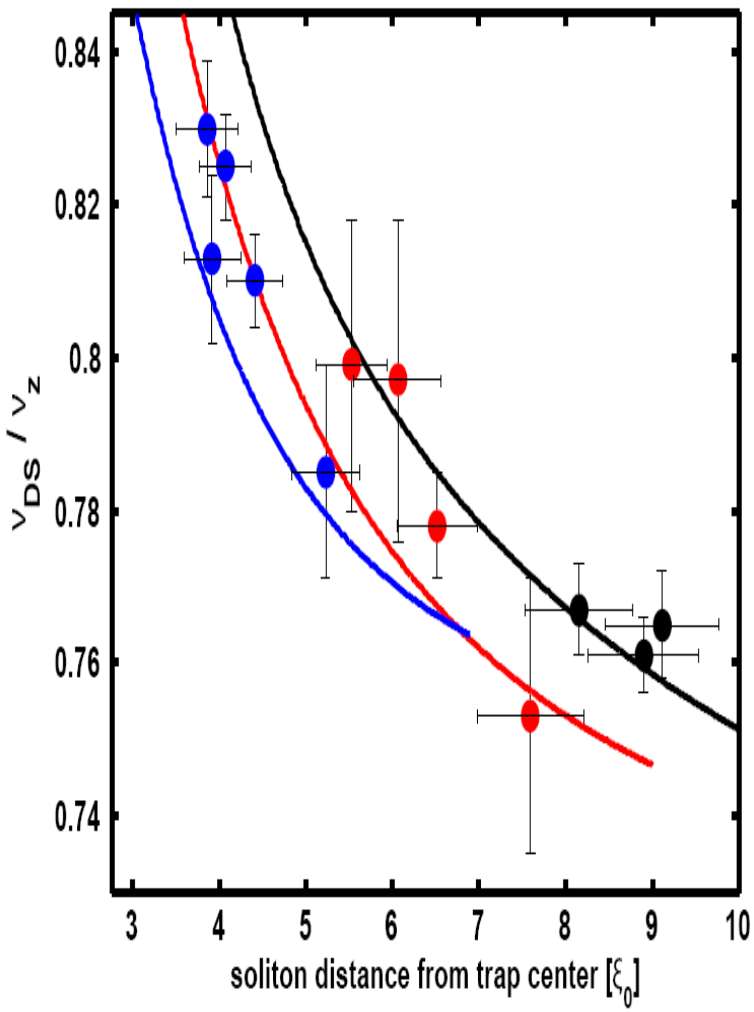
Effective Potential and its Oscillation Frequency Prediction





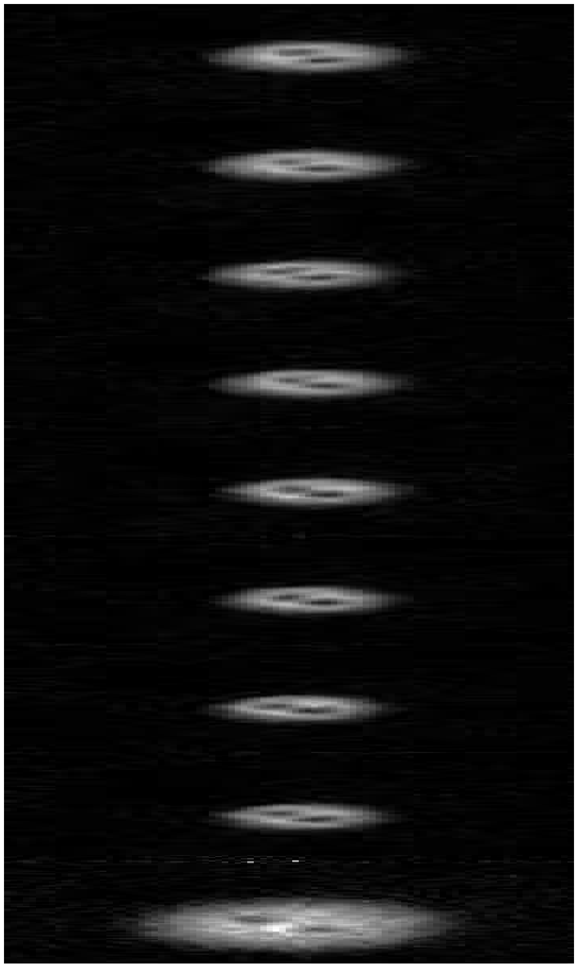
Comparison with Experiments

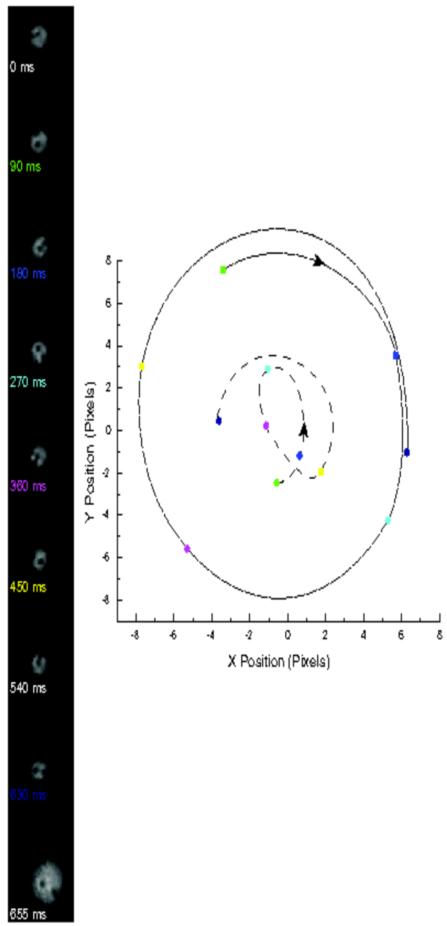


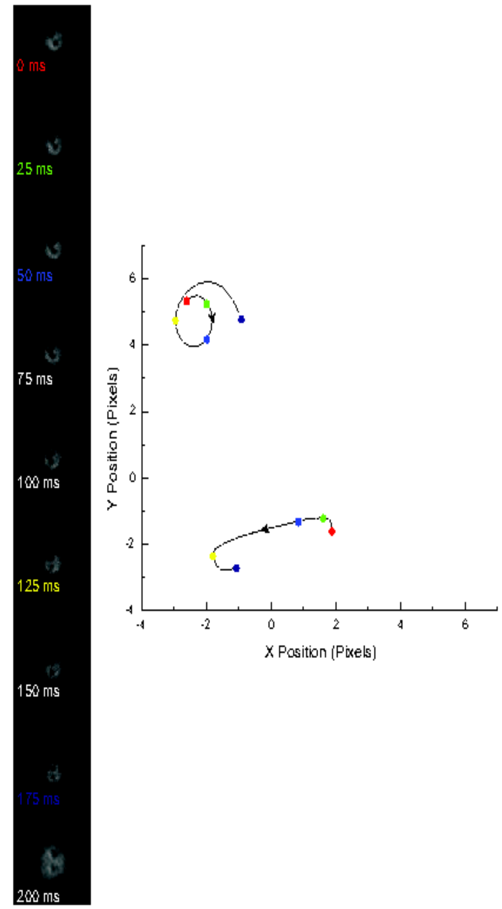


Generalizations, Part I: Higher Dimensions

1-Component, 2-dimensions: Vortex Dipoles in Amherst

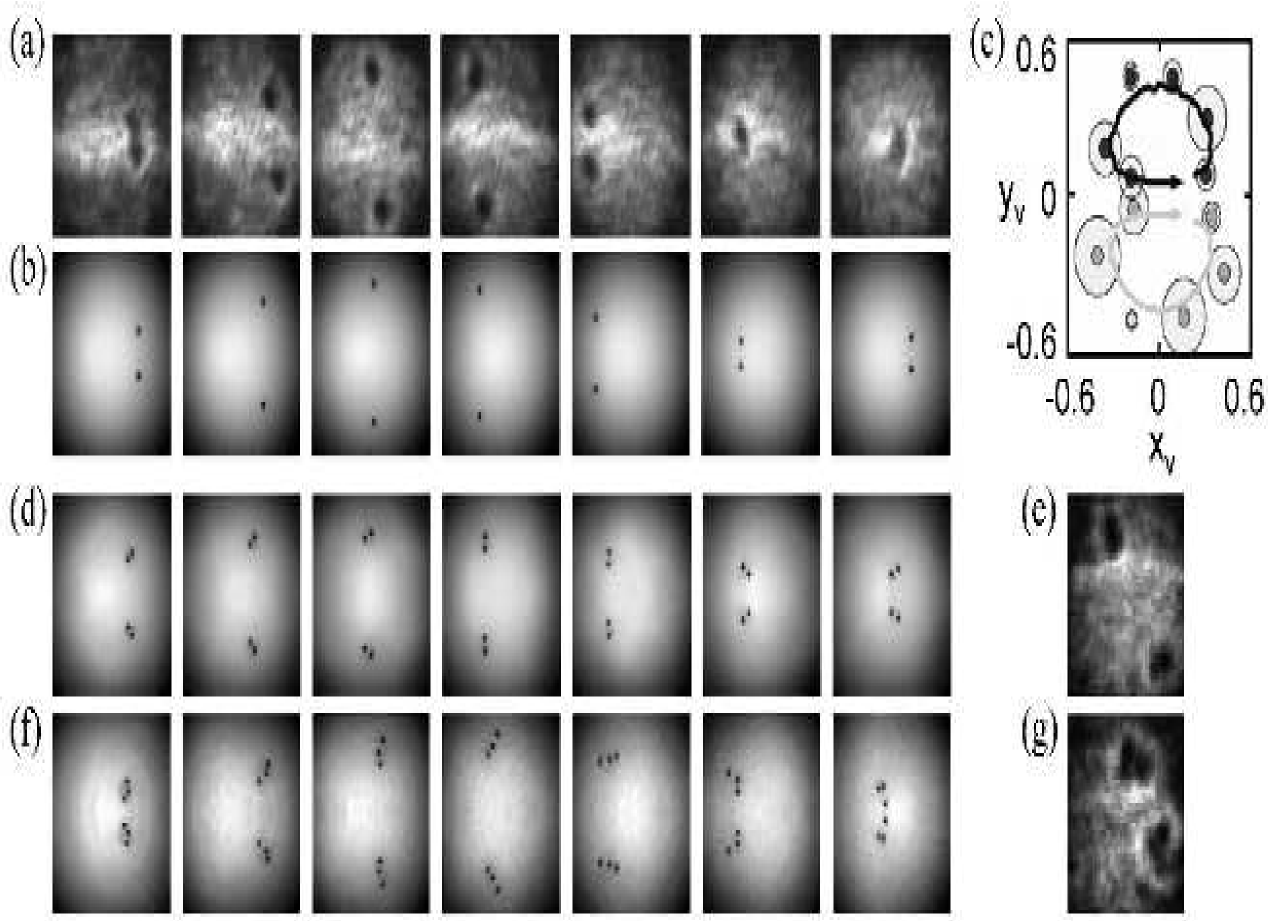






Generalizations, Part I: Higher Dimensions

1-Component, 2-dimensions: Vortex Dipoles in Tucson



Generalization in Higher Dimensions: First, the Single Vortex

- Vortices have Dynamics reminiscent of those of the Dark Solitons.
- They Bifurcate from the Linear Mode $\Psi(x,y) = \psi_0(x)\psi_1(y) + i\psi_1(x)\psi_0(y)$.
- They are dominated by an Oscillation Mode associated with their Precession Frequency. One can used Matched Asymptotics to obtain:

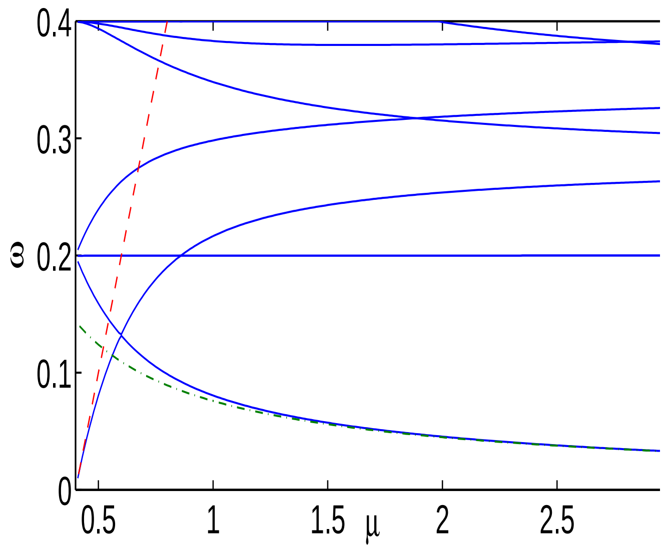
$$\dot{x}_v = \frac{\Omega^2}{2\mu} \log\left(A\frac{\mu}{\Omega}\right) y_v, \tag{40}$$

$$\dot{y}_v = -\frac{\Omega^2}{2\mu} \log\left(A\frac{\mu}{\Omega}\right) x_v, \tag{41}$$

- This yields the Precession Frequency:

$$\omega_{\text{prec}} = \frac{\Omega^2}{2\mu} \log(A\frac{\mu}{\Omega}) \tag{42}$$

- In the presence of Finite Temperature, this mode becomes Complex giving rise to Dynamical Instability.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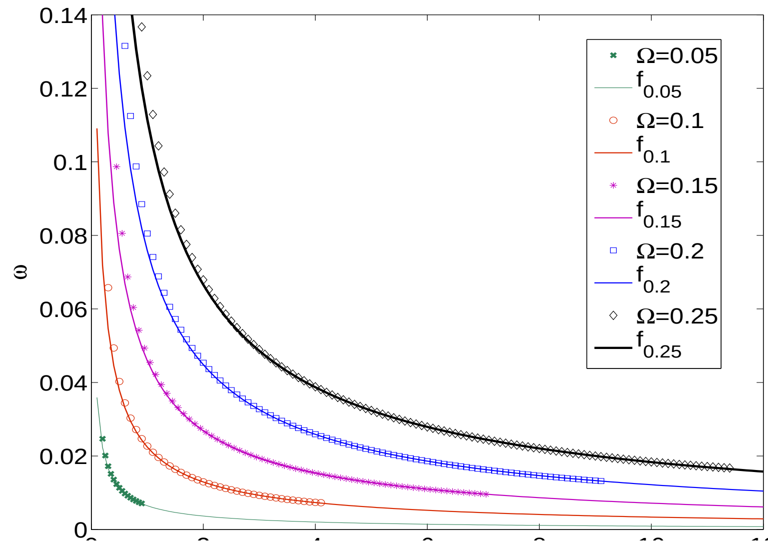
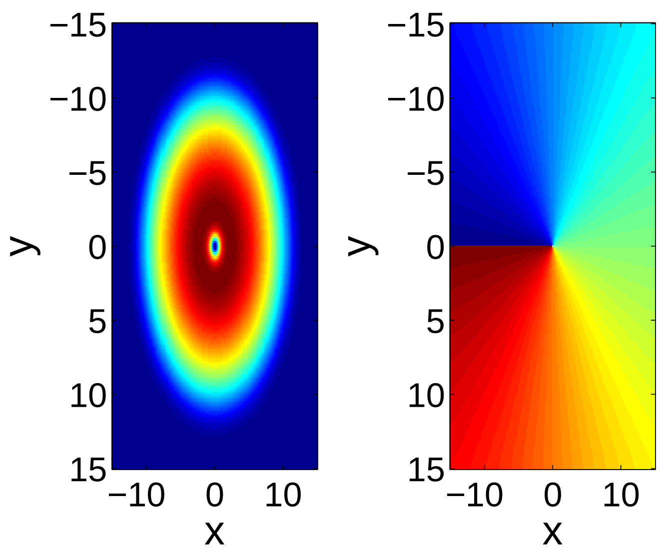
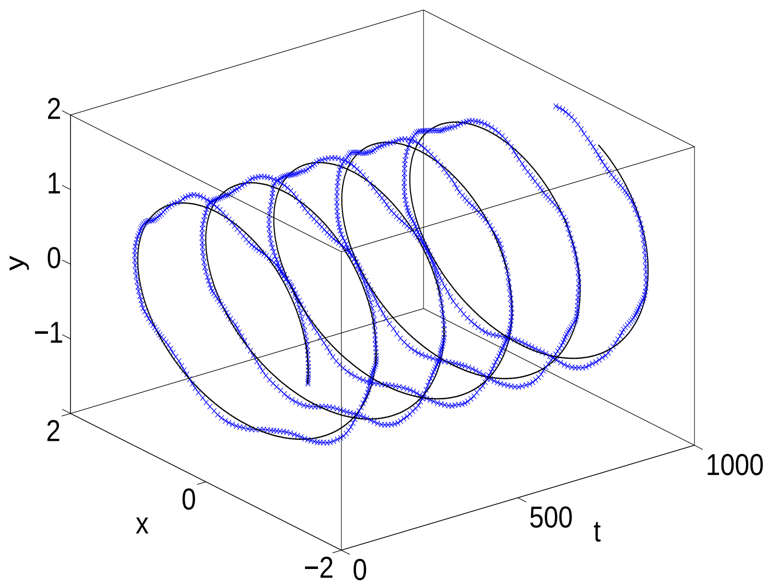
2
2.5


0.14
0.12
0.1
0.08
ω
0.06
0.04
0.02
0
Ω=0.05
f0.05
Ω=0.1
f0.1
Ω=0.15
f0.15
Ω=0.2
f0.2
Ω=0.25
f0.25


−15
−10
−5
y 0
5
10
15
−10 0 10
x
−15
−10
−5
y 0
5
10
15
−10 0 10
x


2
1
y 0
−1
2
0
x
−2
0
500
1000
t

Connection of Dark Solitons and Vortices: The Bifurcation Picture

- In 2d, Stable Dark Solitons bifurcate from the Linear Mode $\Psi_{10}(x,y) = \psi_1(x)\psi_0(y)$.
- Subsequently, they become Unstable due to a Cascade of Symmetry-Breaking Bifurcations.
- The Symmetry-Breaking stems from the mixing of $\Psi_{0m}(x,y) = \psi_0(x)\psi_m(y)$ with $\Psi_{10}(x,y) = \psi_1(x)\psi_0(y)$.
- Use Galerkin Truncation to study the Symmetry Breaking as a Bifurcation Problem:

$$u(x,z) = c_0(z)\Psi_{10} + c_1(z)\Psi_{0m}, \tag{43}$$

$$i\dot{c}_0 = (\mu + \omega_0)c_0 - a_{00}|c_0|^2 c_0 - a_{01}\left(2|c_1|^2 c_0 + c_0^\star c_1^2\right), \tag{44}$$

$$i\dot{c}_1 = (\mu + \omega_1)c_1 - a_{11}|c_1|^2 c_1 - a_{01}\left(2|c_0|^2 c_1 + c_1^\star c_0^2\right), \tag{45}$$

$$\dot{\rho}_0 = a_{01}\rho_1^2\rho_0 \sin(2\Delta\phi), \tag{46}$$

$$\dot{\Delta\phi} = -\Delta\omega + a_{11}\rho_1^2 - a_{00}\rho_0^2 + a_{01}\left(2 + \cos(2\Delta\phi)\right)\left(\rho_0^2 - \rho_1^2\right), \tag{47}$$

Supercritical Pitchfork Bifurcation & Ensuing Vortex Dynamics

- The Two-Mode Picture captures this $\pi/2$ relative phase bifurcation, yielding:

$$N_{\text{cr}} = \frac{\omega_1 - \omega_2}{B - A_0}, \quad \mu_{cr} = \omega_1 + A_0 N_{\text{cr}} \tag{48}$$

where $A_0 = \int \Psi_{10}^4 dxdy$, $B = \int \Psi_{10}^2 \Psi_{0m}^2 dxdy$. $\mu_{cr} = 10\Omega/3$ for $m = 2$ (vortex dipole), $86\Omega/19$ for $m = 3$ (vortex tripole), ...

- The resulting states are Multi-Vortex ones which can be described by a Particle Picture:

$$\dot{x}_m = -S_m \omega_{\text{pr}} y_m - B S_n \frac{y_m - y_n}{2\rho_{mn}^2} \tag{49}$$

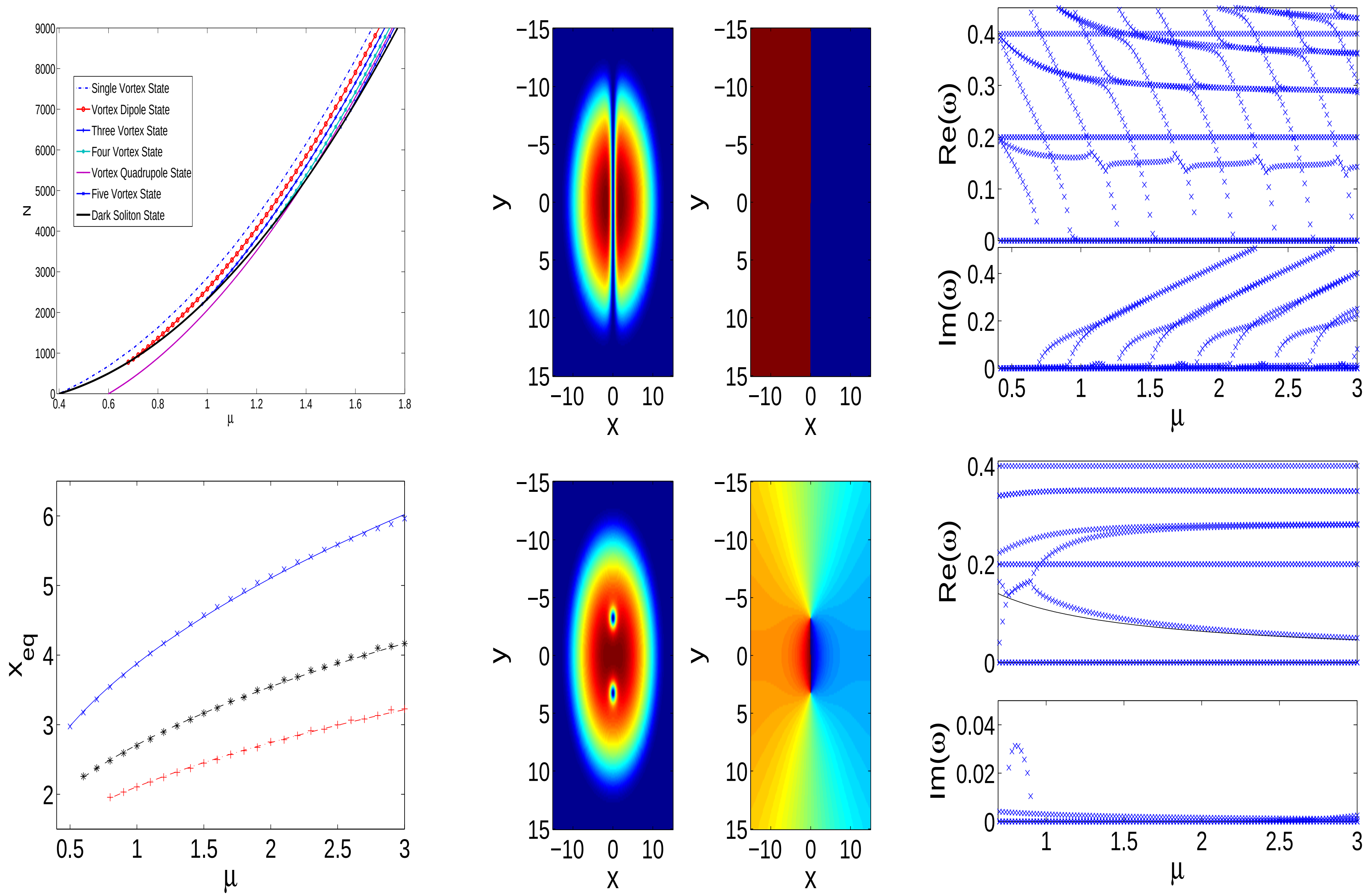
$$\dot{y}_m = S_m \omega_{\text{pr}} x_m + B S_n \frac{x_m - x_n}{2\rho_{mn}^2}, \tag{50}$$

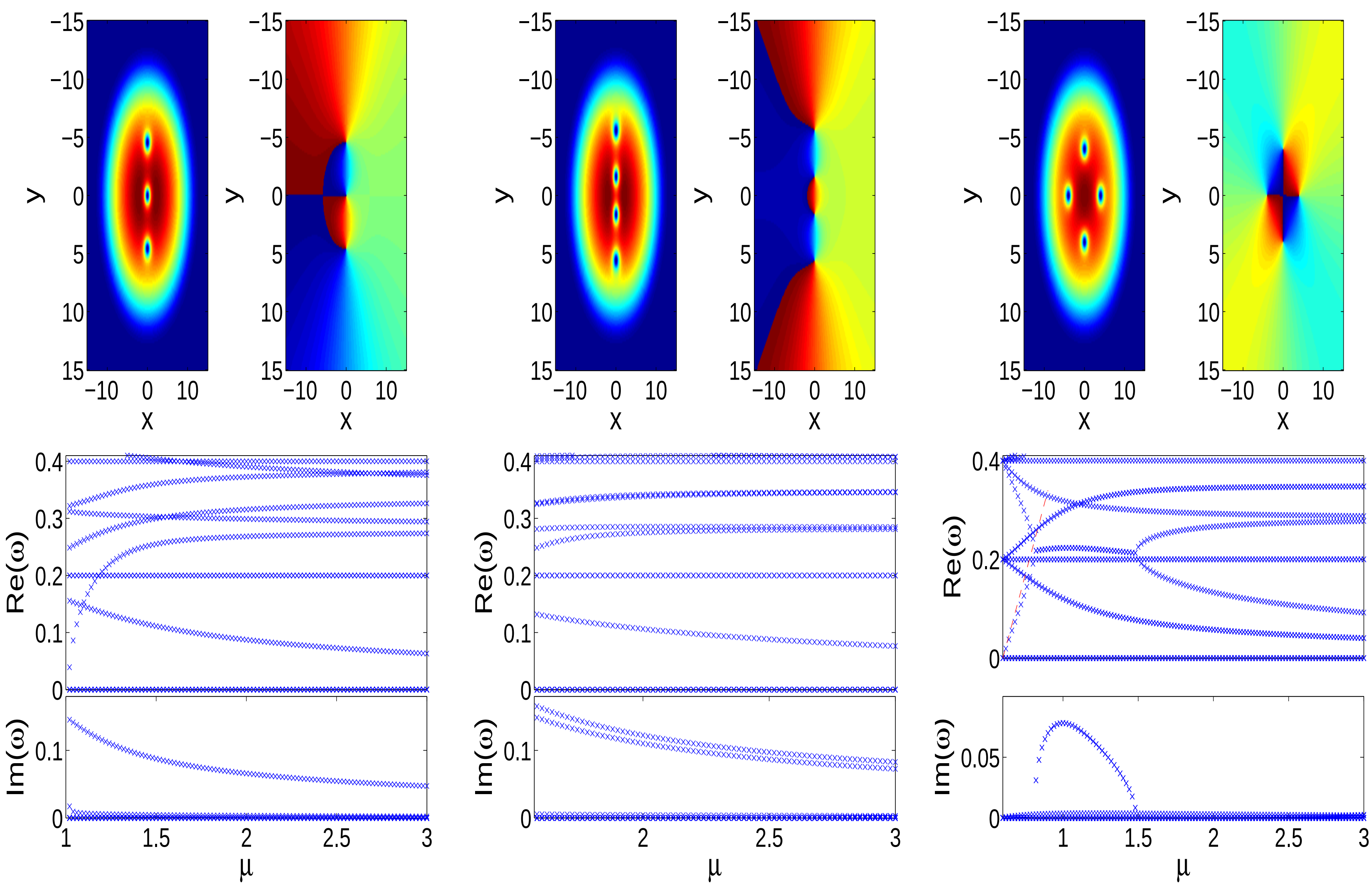
- This contains Precession and Interaction (cf. Oscillation and Interaction) and makes useful predictions: $y_{1,eq} = -y_{2,eq} = \sqrt{\frac{B}{4\omega_{\text{pr}}}}$ $\quad \omega_{\text{pr}}^{\text{VD}} = \pm\sqrt{2}\omega_{\text{pr}}$

- The Dynamical Stability of the Stripe is Inherited by the Bifurcation Byproducts: Dipole, Tripole, (Aligned) Quadrupole, Quintopole, etc. E.g. for Tripole

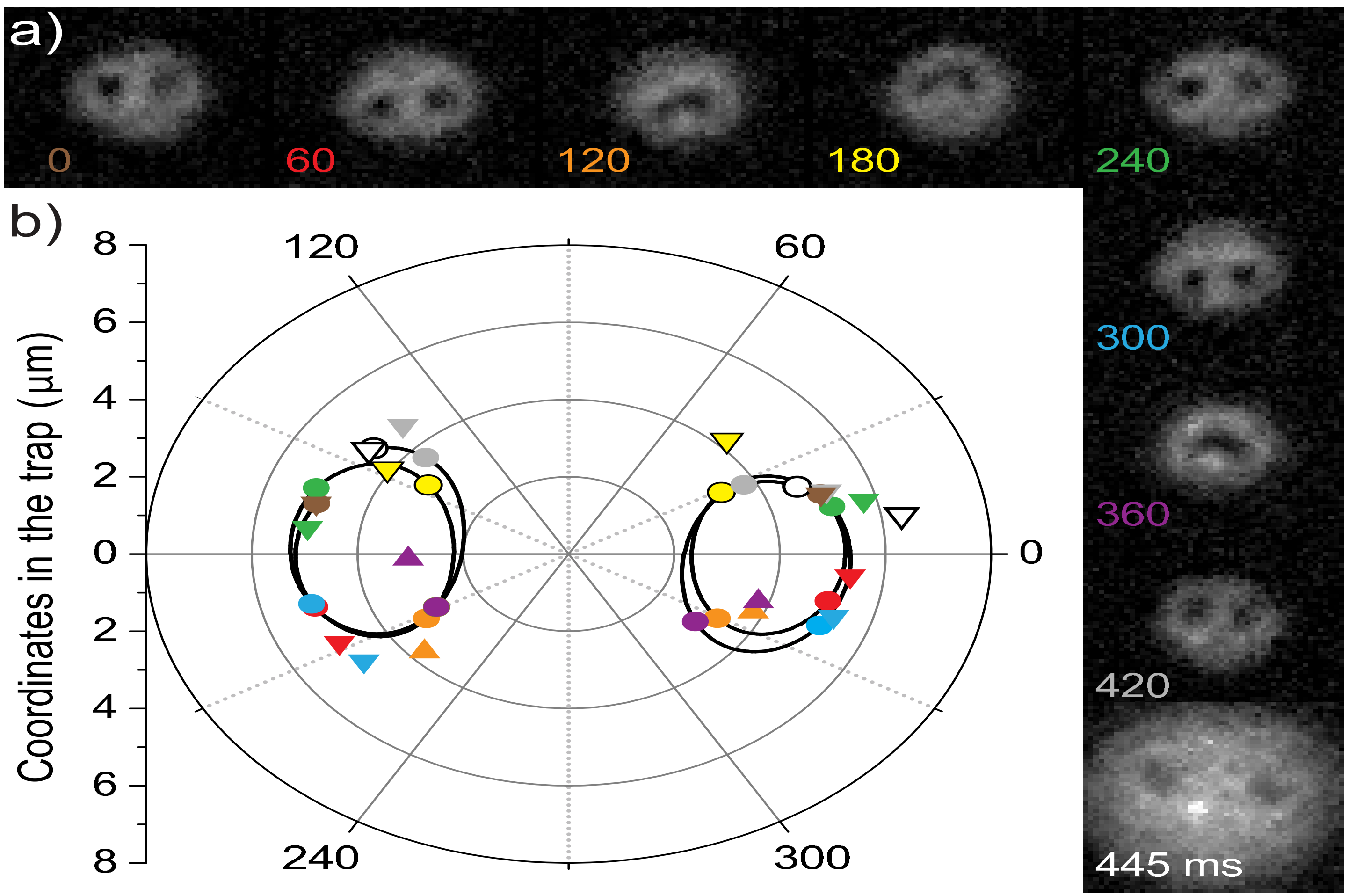
$$\omega_{\text{pr1}}^{\text{3v}} = \pm\sqrt{5}\omega_{\text{pr}}, \tag{51}$$

$$\omega_{\text{pr2}}^{\text{3v}} = \pm i\sqrt{7}\omega_{\text{pr}}. \tag{52}$$

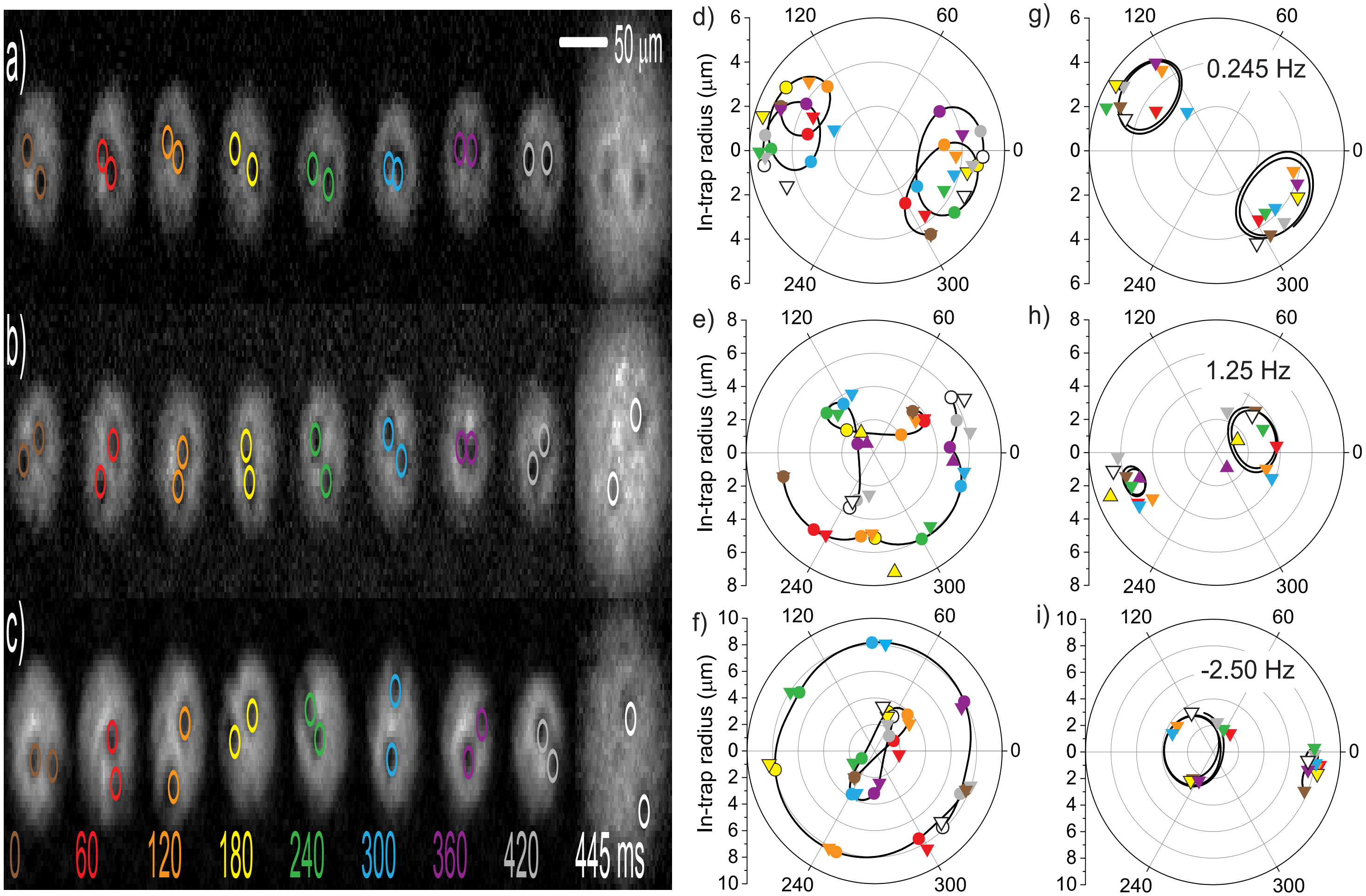

Single Vortex State
Vortex Dipole State
Three Vortex State
Four Vortex State
Vortex Quadrupole State
Five Vortex State
Dark Soliton State
N
μ
x_{eq}
x
y
Re(ω)
Im(ω)


y
x
Re(ω)
Im(ω)
μ

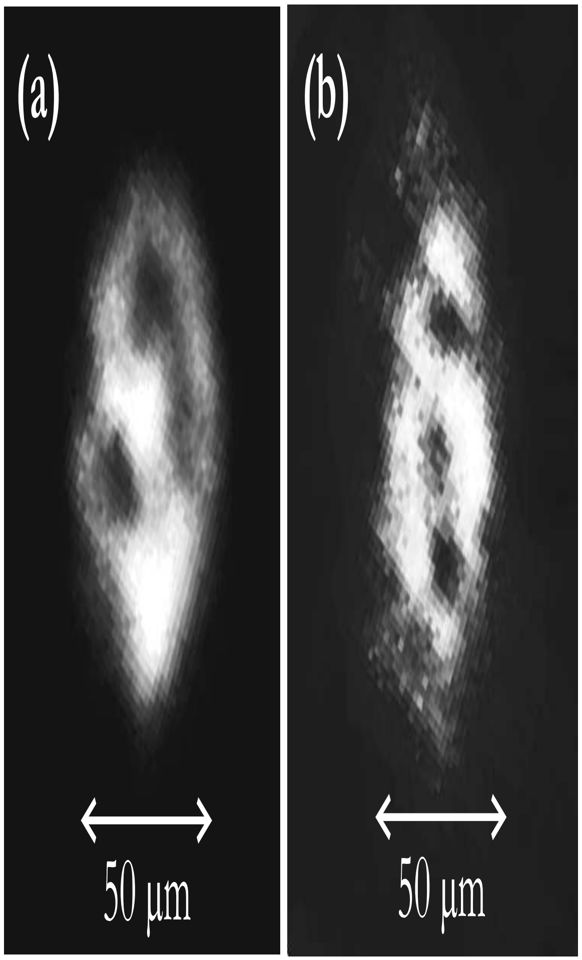
Experimental Verification Part I: Statics & Periodic Dynamics

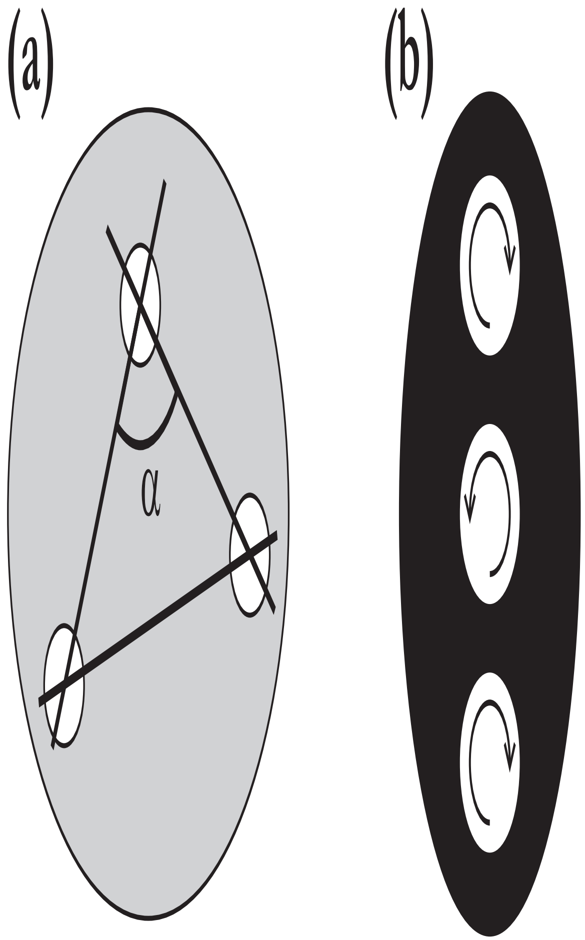


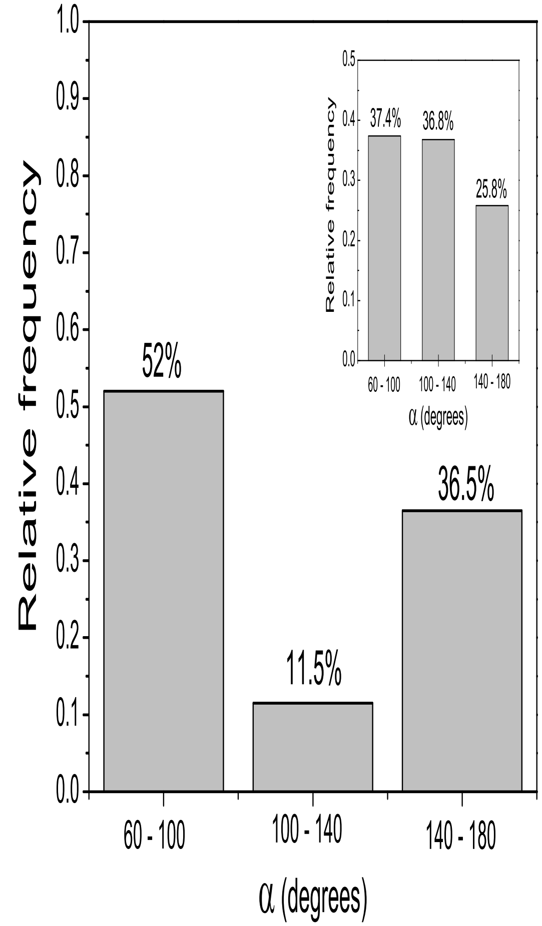
Experimental Verification Part II: General Quasi-Periodic Dynamics



Connections with Recent Work of V.S. Bagnato (PRA 82, 033616 (2010))

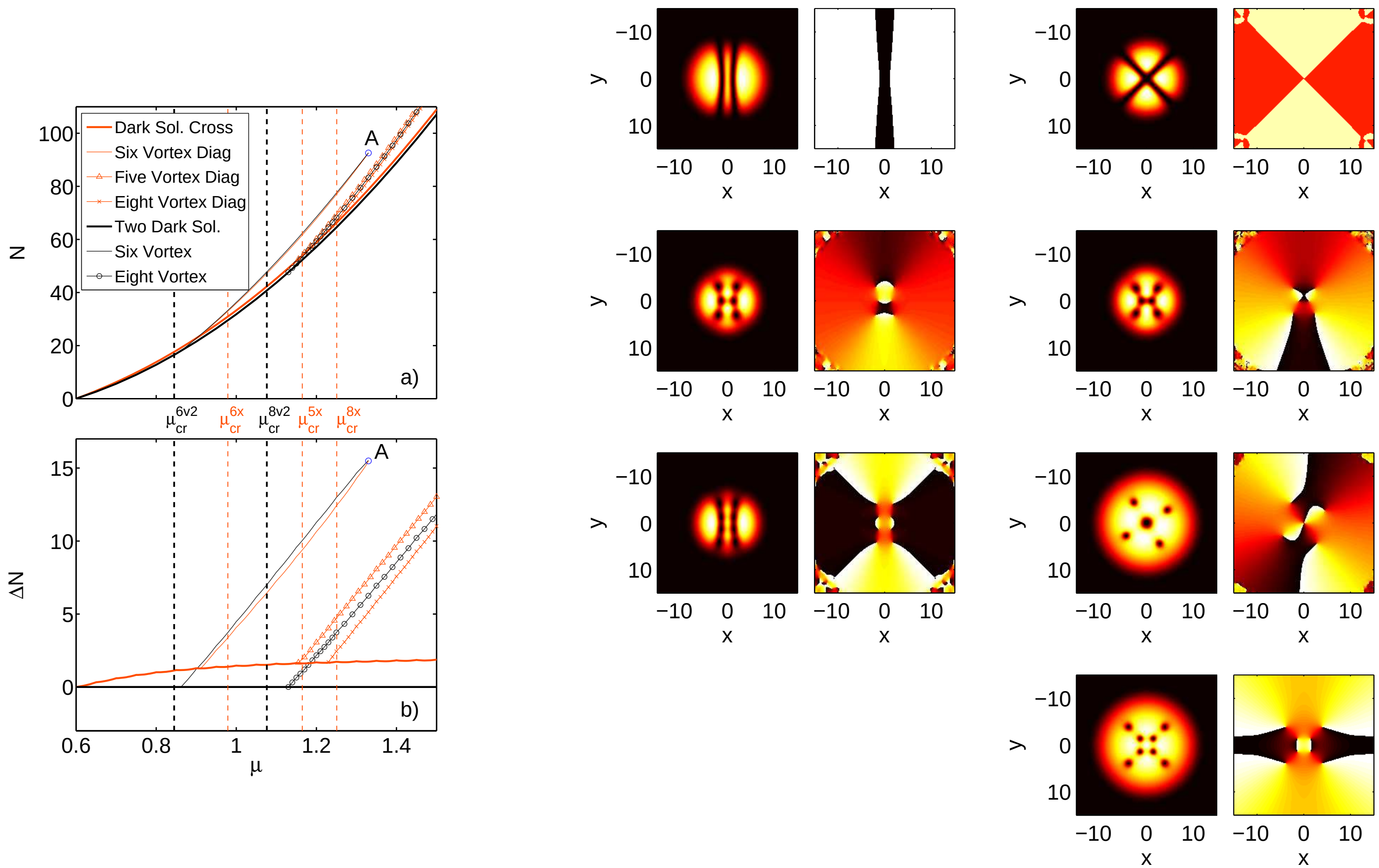






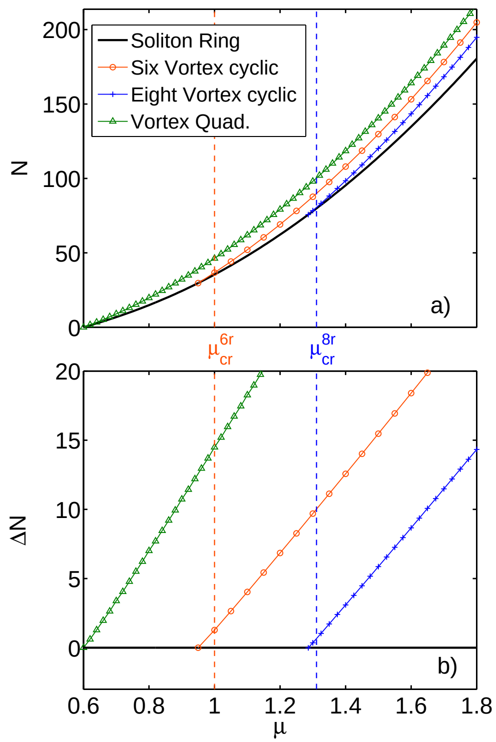
Twist I: Generalizing the Bifurcation Picture

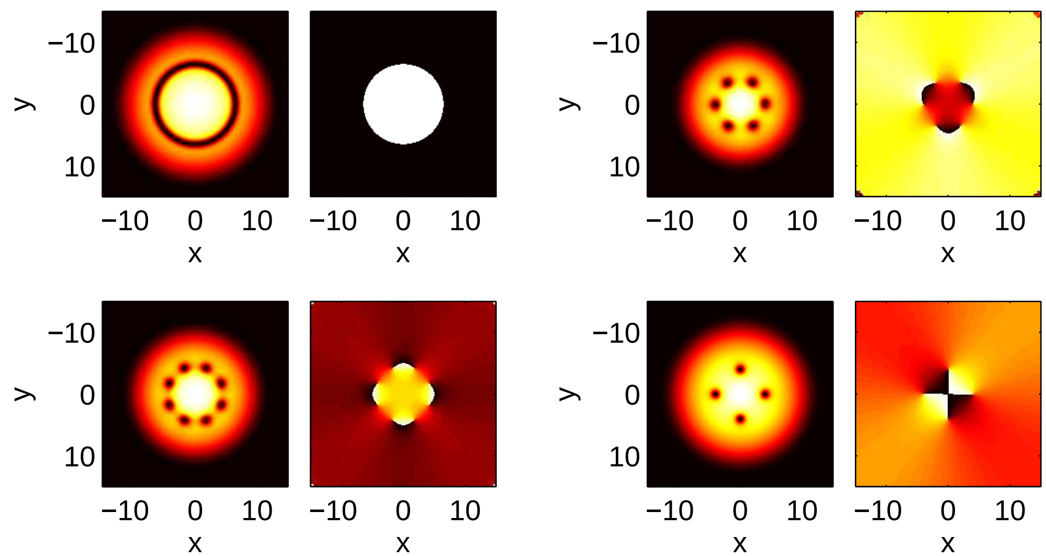
Part a: Rectangular States



Twist I: Generalizing the Bifurcation Picture

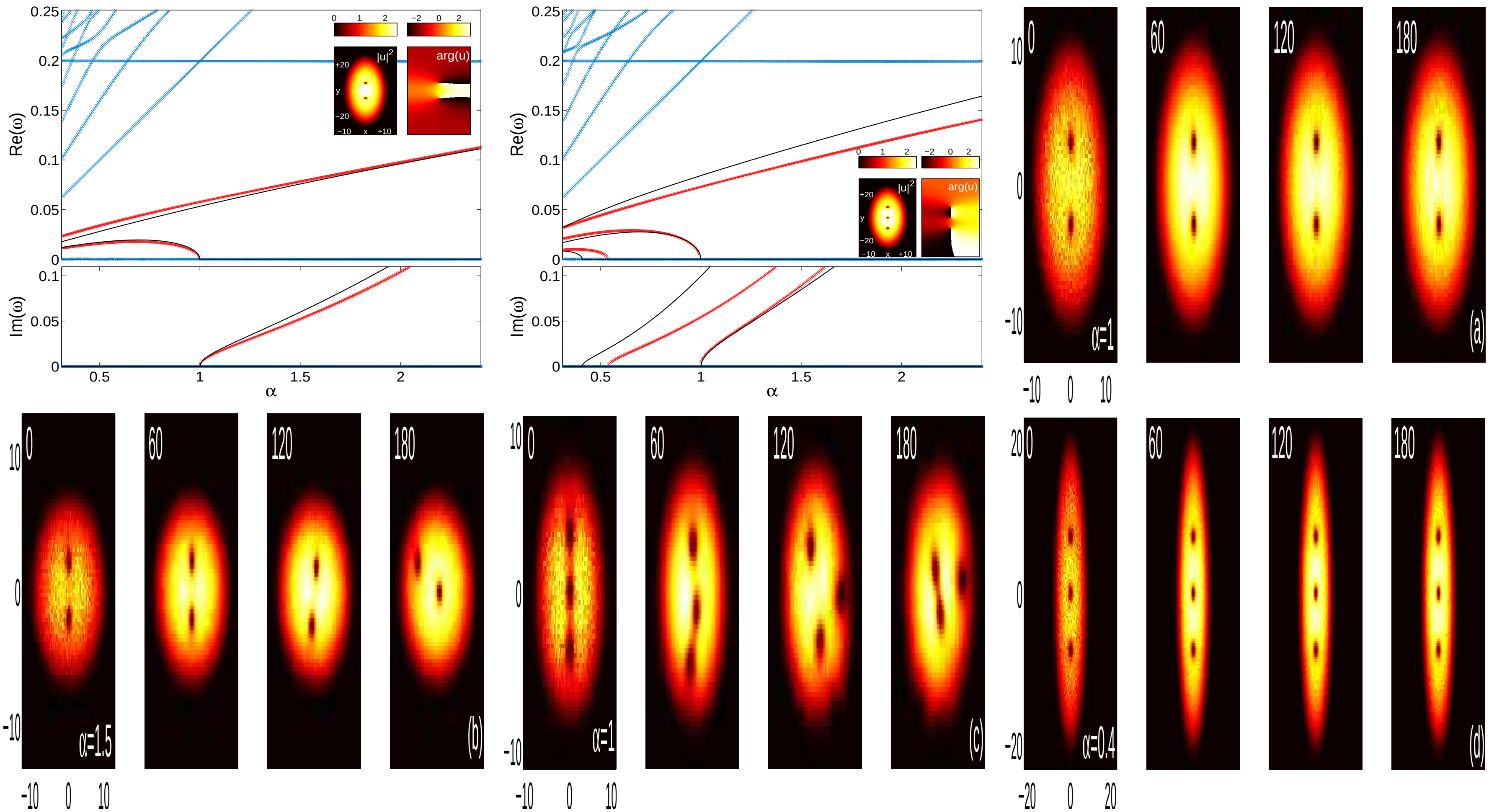
Part b: Radial States



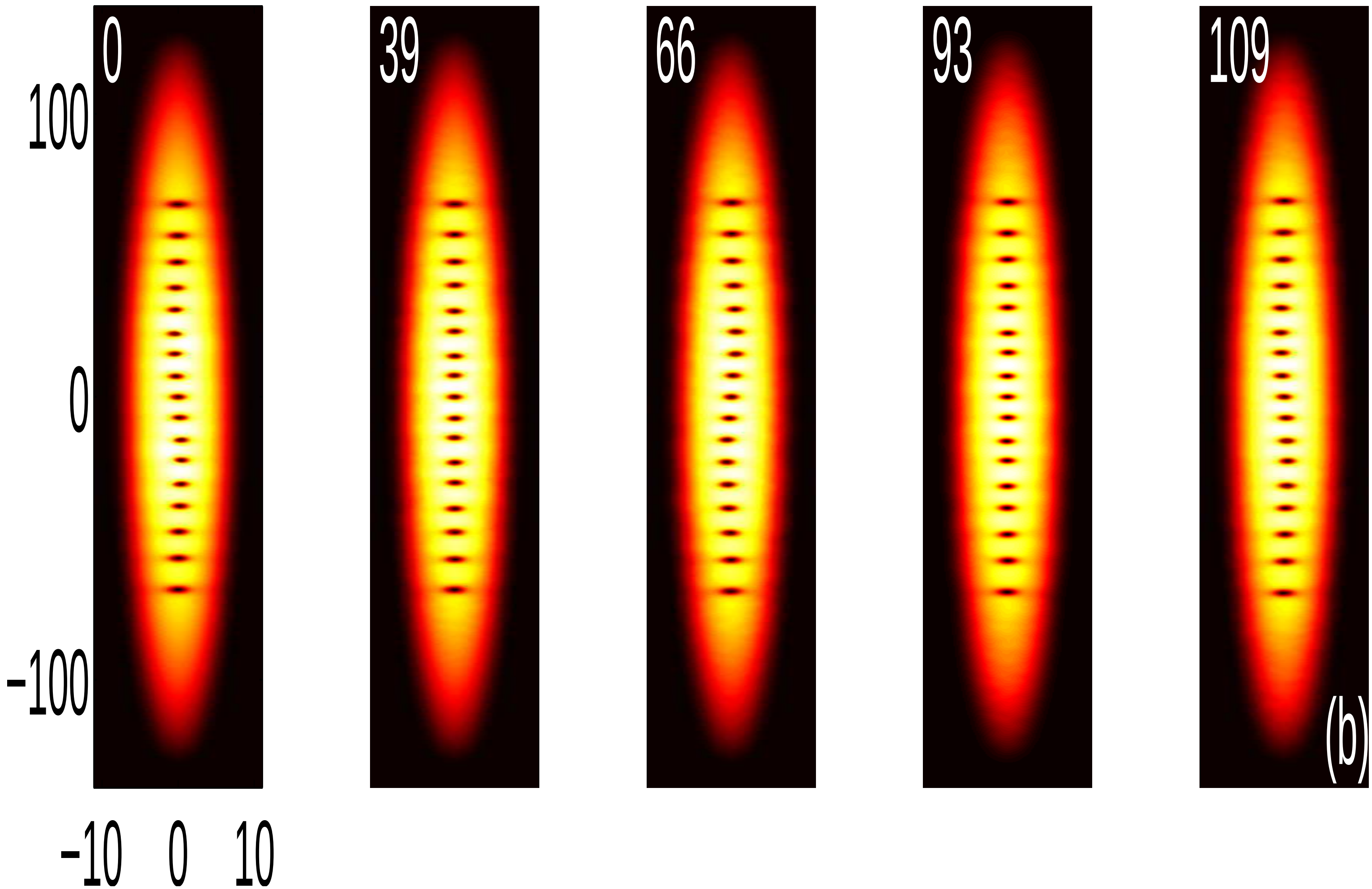


Twist II: The role of Anisotropy

- $\dot{x} = -\omega_y^2 Q y$ and $\dot{y} = \omega_x^2 Q x$, where $Q = \ln(A\mu/\omega_{\text{eff}})/(2\mu)$, $A \approx 2\sqrt{2}\pi$ and $\omega_{\text{eff}} = \sqrt{(\omega_x^2 + \omega_y^2)/2}$



Stabilizing Vortex States at Will



Twist III: The Case of Co-Rotating Vortices

- One can write Equations of Motion for 2 Vortices in the form:

$$\dot{r}_m = -\frac{c r_n \sin(\theta_m - \theta_n)}{\rho_{mn}^2}, \tag{53}$$

$$\dot{\theta}_m = -\frac{c r_n \cos(\theta_m - \theta_n)}{r_m \rho_{mn}^2} + \frac{c}{\rho_{mn}^2} + \frac{1}{1 - r_m^2}. \tag{54}$$

- Stationary Solutions then satisfy: $r_1 = r_2 = r_*$ and $\theta_1 - \theta_2 = \pi$, while:

$$\dot{\theta}_1 = \dot{\theta}_2 = \omega_{orb} = \frac{c}{2r_*^2} + \frac{1}{1 - r_*^2}. \tag{55}$$

- Linearization around this Stationary State yields the Epicyclic Frequency: $\omega_{\text{ep}}^2 = \frac{c^2}{2r_*^4} - \frac{2c}{(1-r_*^2)^2}$.
- This Changes Sign, causing an Instability at: $r_{\text{cr}}^2 = \sqrt{c}/(\sqrt{c} + 2)$.
- A New Asymmetric State emerges, satisfying:

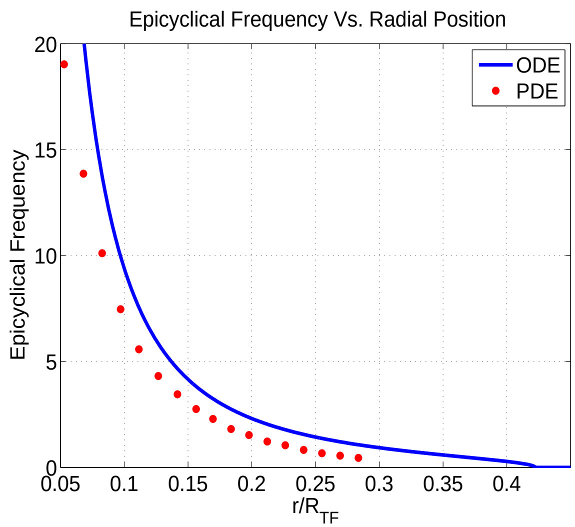
$$-r_1^* r_2^* (r_1^* + r_2^*)^2 + c\left(1 - r_1^{*2}\right)\left(1 - r_2^{*2}\right) = 0,$$

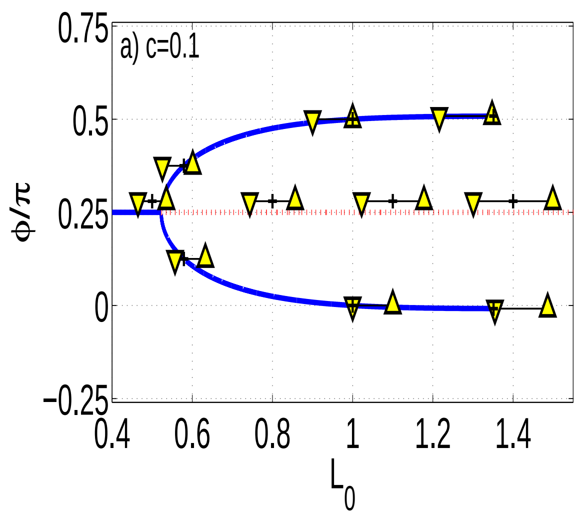
- Visualize the Instability using $L_0^2 = \sum_i r_i^2$ and

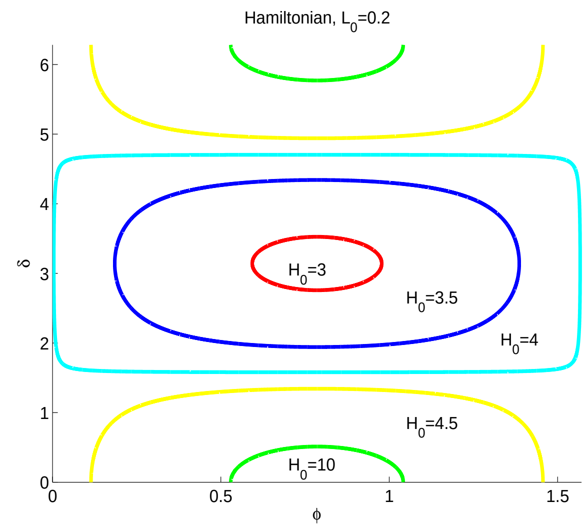
$$H = \frac{1}{2} \ln\left[\left(1 - r_1^2\right)\left(1 - r_2^2\right)\right] - \frac{c}{2} \ln\left[r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)\right].$$

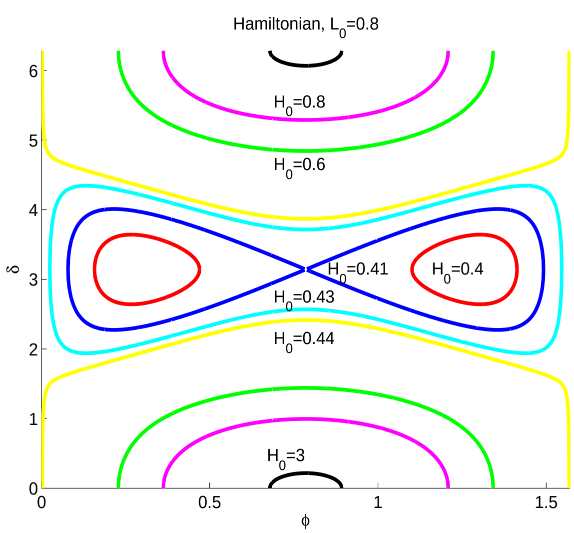
- Similar Instabilities arise for $N = 3$ and $N = 4$ vortices, respectively at: $r_{\text{cr}}^2 = \sqrt{c}/(\sqrt{c} + \sqrt{2})$ and $r_{\text{cr}}^2 = \sqrt{3c}/(\sqrt{3c} + 2)$.

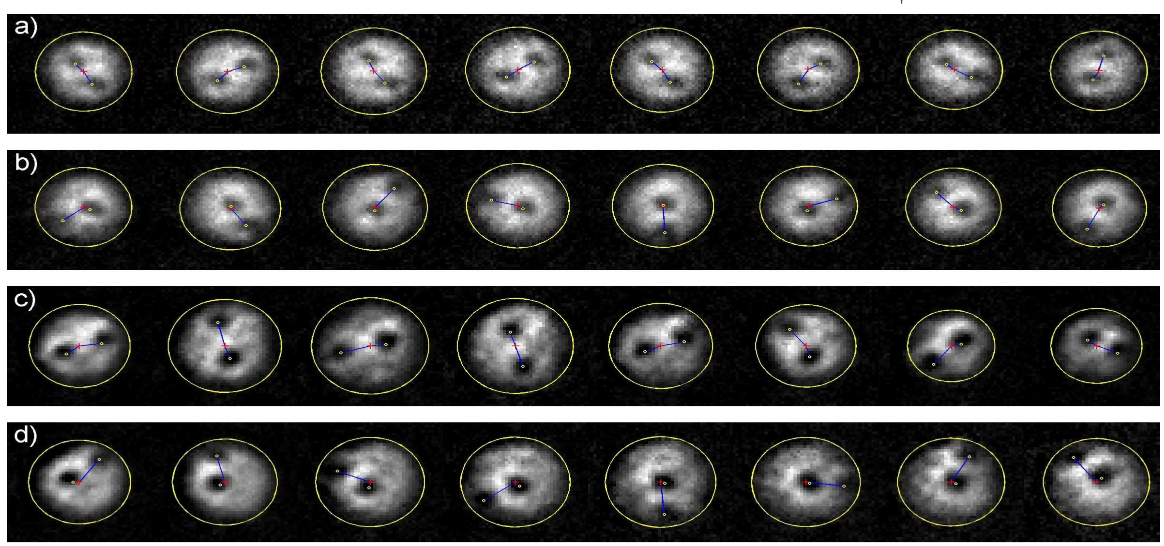
Twist III: Symmetry Breaking For 2, 3, 4 Co-rotating Vortices











Summary of Results

- Dark Solitons Oscillate and Interact.
- Their Oscillation and Interactions can be characterized using tools such Variational Theory, Integrability Results, Perturbation Theory.
- The accuracy of the ensuing ODEs can be tested Quantitatively against Recent Experiments.
- The 2d Generalization of Dark Solitons becomes Progressively More Unstable.
- Out of this Symmetry Breaking Emanate Multi-Vortex States
- Properties such as the Equilibrium Positions and Epicyclic Dynamics of such states can be Experimentally Monitored.
- For the Vortices similarly to the Dark Solitons, Particle Based Methods can be developed to monitor their Complex Dynamics.

Present/Future Challenges

- Still, on Dark Solitons
  - Effects of Temperature $\rightarrow$ Dissipative NLS Models
  - Generalization to N-Soliton States $\rightarrow$ Dark Soliton Crystal vs. Dark Soliton Gas ?
- Dark Solitons, Take 2
  - Effects of Optical Lattice
  - Single & Multi-Soliton States
- Generalization to Vortices
  - Painting a Similar Picture $\rightarrow$ Anomalous Modes, Precession, Characterization of Interactions
  - Generalization of the Picture $\rightarrow$ Including Effects of Temperature, N-Vortex States
- Generalization to Multi-Components
  - Characterize Dark-Bright Soliton States $\rightarrow$ Monitor their Oscillations and Interactions
  - Generalize these States $\rightarrow$ Crystals, Thermal Effects ...

One Example (Thermal Effects & Dark Solitons)

- Consider the Dissipative GPE as a way of modeling Finite Temperature Effects

$$(i-\gamma)\partial_t\psi = \left[\frac{1}{2}\partial_z^2 + V(z) + |\psi|^2 - \mu\right]\psi, \tag{56}$$

- Then, the motion of a Single Dark Soliton Characterized by:

$$\frac{d^2 z_0}{dt^2} = \left[\frac{2}{3}\gamma\mu\frac{dz_0}{dt} - \left(\frac{\Omega}{\sqrt{2}}\right)^2 z_0\right]\cdot\left[1-\left(\frac{dz_0}{dt}\right)^2\right]. \tag{57}$$

- At the Linearization level this yields:

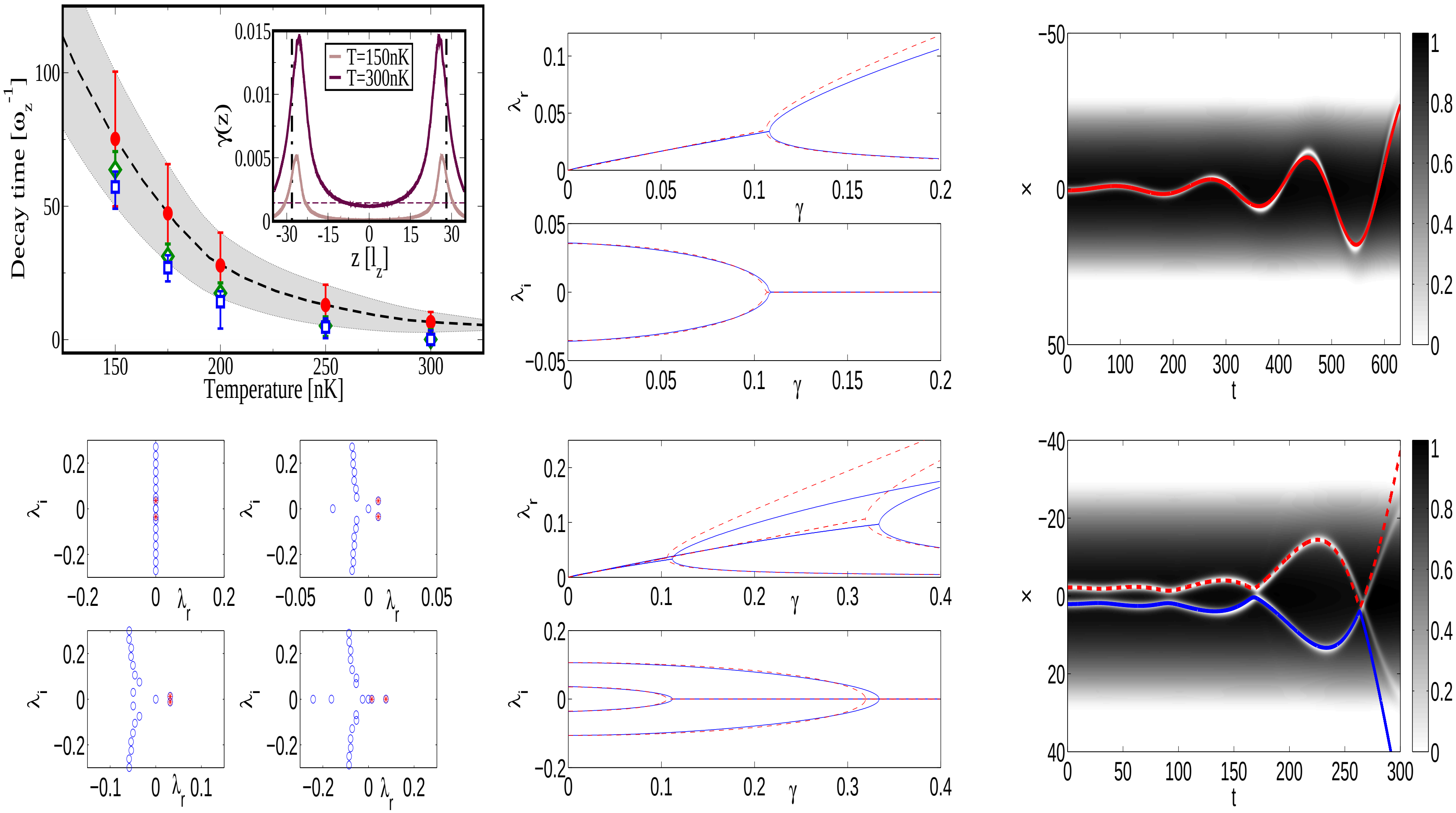
$$\omega_{1,2} = \frac{1}{3}\gamma\mu \pm \left(\frac{\Omega}{\sqrt{2}}\right)\sqrt{\Delta},$$

with $\Delta = \left(\frac{\gamma}{\gamma_{cr}}\right)^2 - 1$, and $\gamma_{cr} = (3/\mu)(\Omega/\sqrt{2})$.

- Also, this can be generalized to Multi-Soliton Dynamics e.g. for 2-solitons:

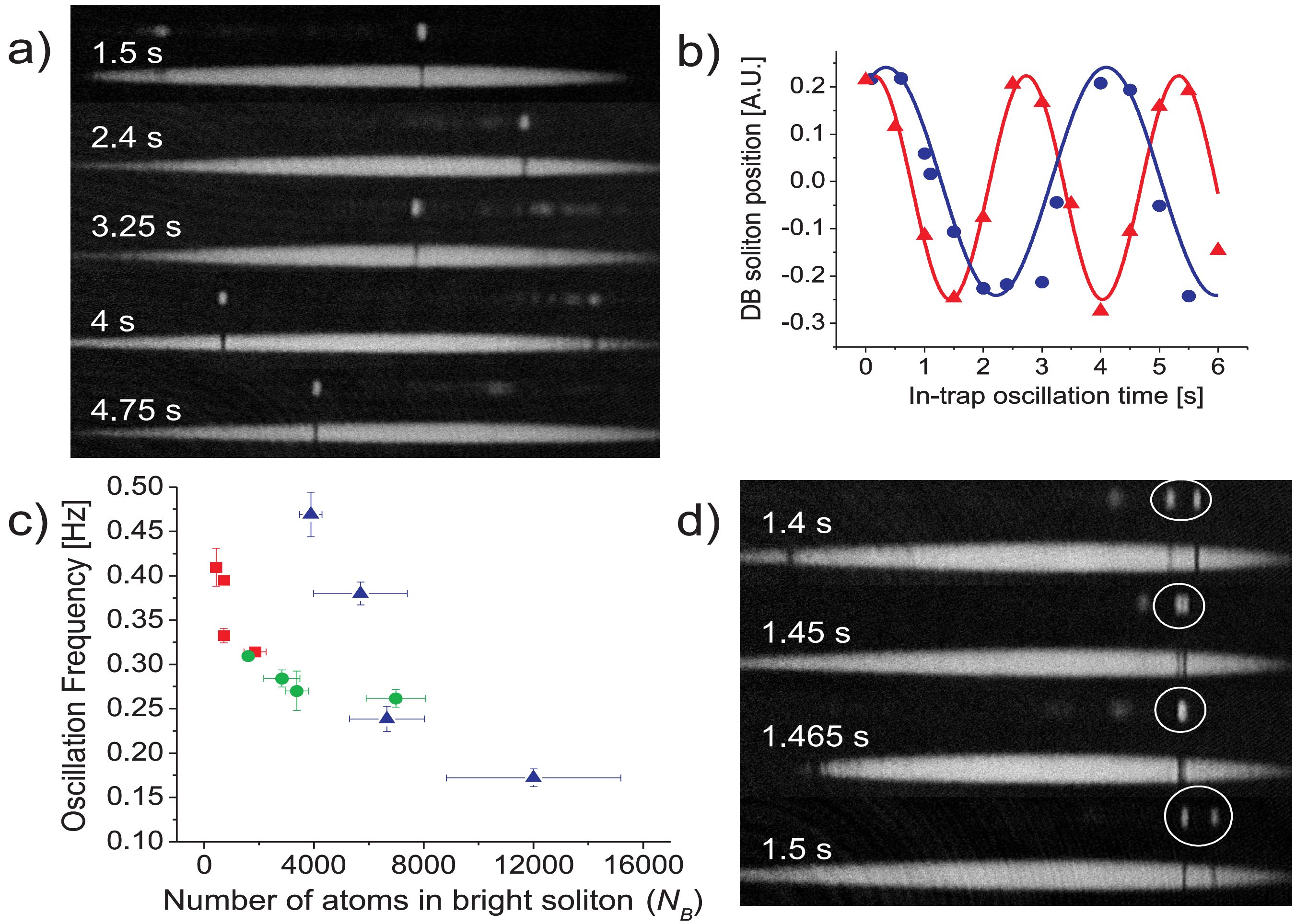
$$\frac{d^2 x_1}{dt^2} = \frac{2}{3}\gamma\frac{dx_1}{dt} - \left(\frac{\Omega}{\sqrt{2}}\right)^2 x_1 - 8\exp\left(-(x_2-x_1)\right), \tag{58}$$

$$\frac{d^2 x_2}{dt^2} = \frac{2}{3}\gamma\frac{dx_2}{dt} - \left(\frac{\Omega}{\sqrt{2}}\right)^2 x_2 + 8\exp\left(-(x_2-x_1)\right). \tag{59}$$


Decay time [ω_z^{-1}]
Temperature [nK]
T=150nK
T=300nK
γ(z)
z [l_z]
λ_r
λ_i
γ
x
t

Generalizations, Part II: Higher Components

2-Components, 1-dimension: Dark-Bright Solitons in Pullman



Extensions in 2-Components, 1-Dimension: Dark-Bright Solitons

- Model reads: $i\hbar\partial_t\psi_j = \left(-\frac{\hbar^2}{2m}\partial_x^2\psi_j + V(x) - \mu_j + \sum_{k=1}^{2} g_{jk}|\psi_k|^2\right)\psi_j.$

- Dark-Bright Soliton Solutions:

$$\psi_1(x,t) = \cos\phi \tanh\left[D(x - x_0(t))\right] + i\sin\phi, \tag{60}$$

$$\psi_2(x,t) = \eta\, \mathrm{sech}\left[D(x - x_0(t))\right] \exp\left[ikx + i\theta(t)\right], \tag{61}$$

- Interaction between 2 DB has 3 pieces (Stationary State Exists) + Restoring Force (Trap):

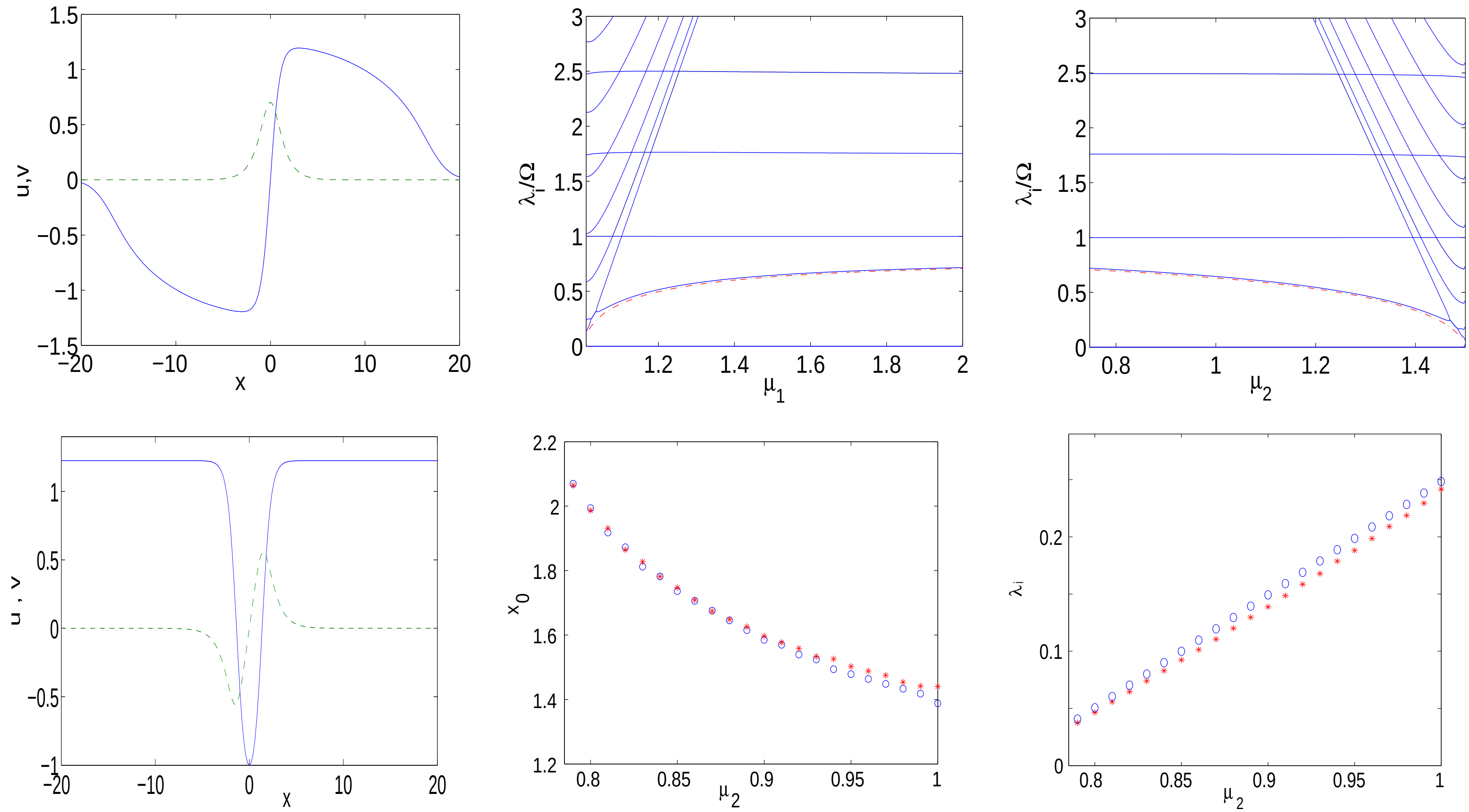
$$F_{\mathrm{DD}} = \frac{1}{\chi_o}\left[\frac{1}{3}(544 - 352D_0^2) + 128D_0\left(D_0^2 - 1\right)x_0\right] \mathrm{e}^{-4D_0x_0},$$

$$F_{\mathrm{BB}} = \frac{\chi}{\chi_o}\left[\left(4 - 2\chi D_0 - 6D_0^2\right)D_0 + 4D_0^2\left(D_0^2 + 1\right)x_0\right]\cos\Delta\theta \mathrm{e}^{-2D_0x_0} - 8\frac{\chi^2}{\chi_o}D_0^3x_0\cos^2\Delta\theta$$

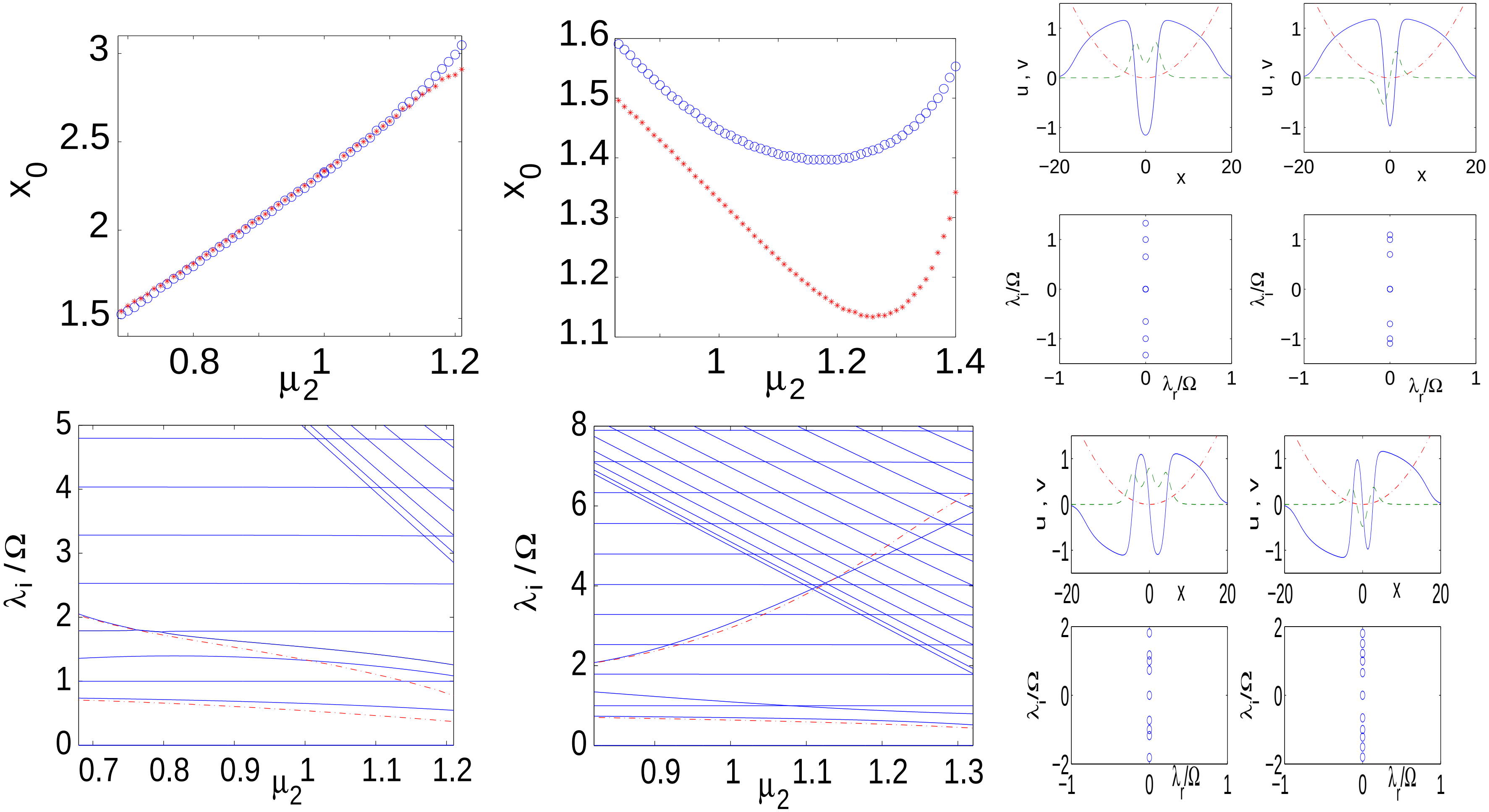
$$F_{\mathrm{DB}} = \frac{\chi}{2\chi_o}\left(6\chi D_0^2 + 12\chi D_0^2\cos\Delta\theta - \frac{214}{3}D_0 + 8\left(8D_0^2 - \chi D_0^3\right)x_0\right)\mathrm{e}^{-4D_0x_0},$$

$$F_{Trap} = -\Omega_{DB}^2 x_0 = -\Omega^2\left(\frac{1}{2} - \frac{\chi}{\chi_o}\right)x_0$$

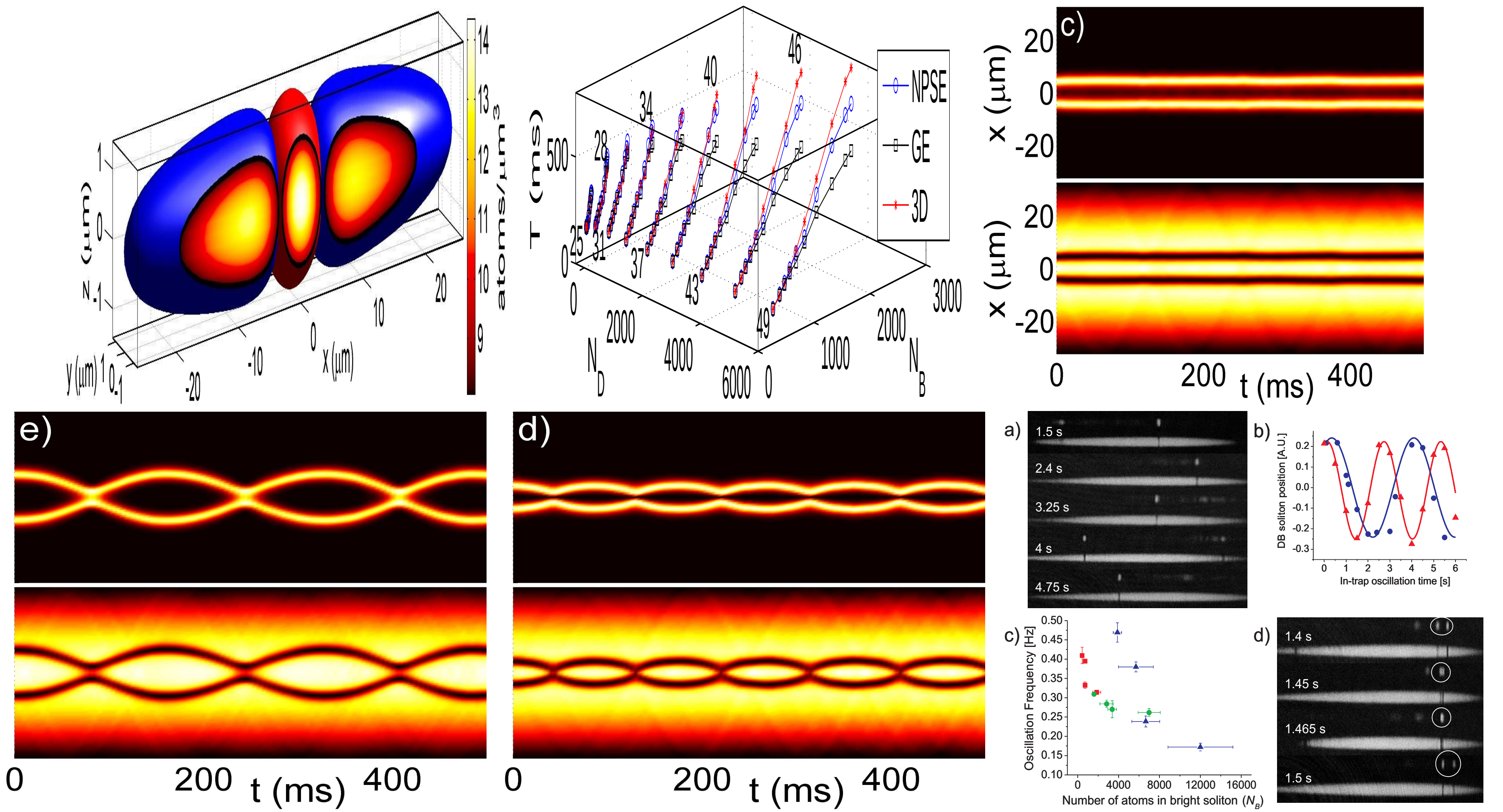
1 and 2 DBs: Analysis vs. Numerics



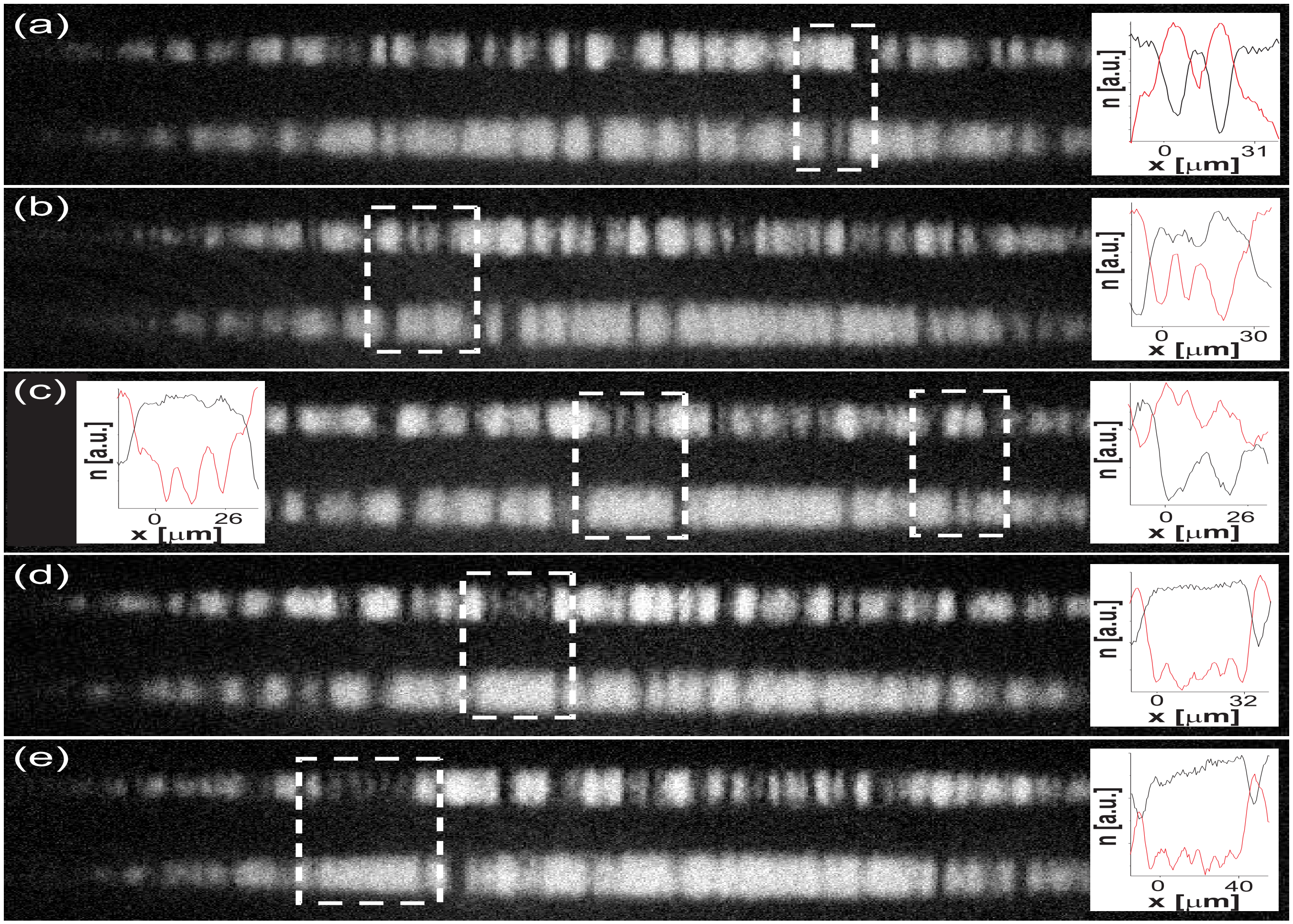
Multiple DBs: Analysis vs. Numerics



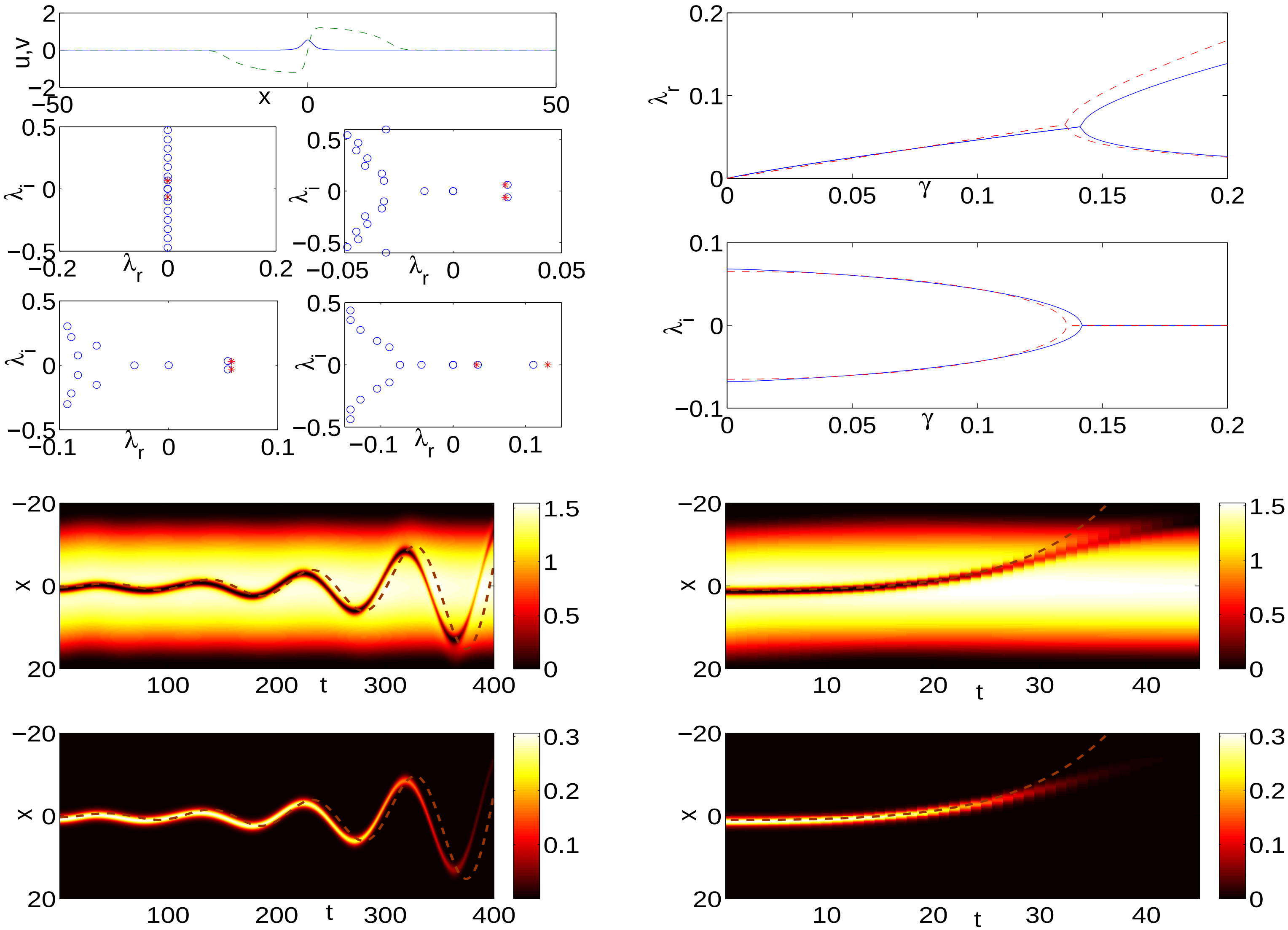
3d Version of Quasi-1d Results and Connection with P. Engels' Experiments



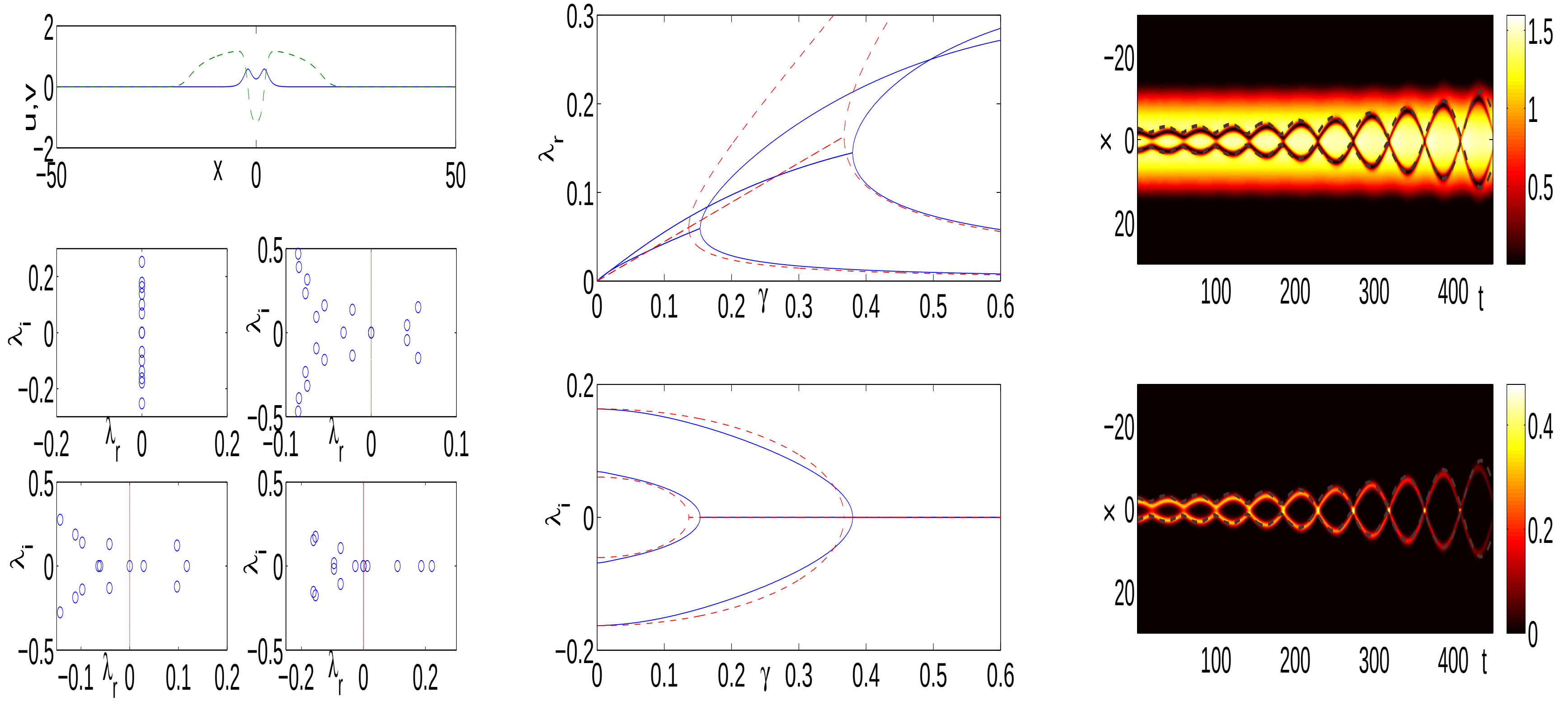
Connection with Peter Engels Experiments: Multi-DBs

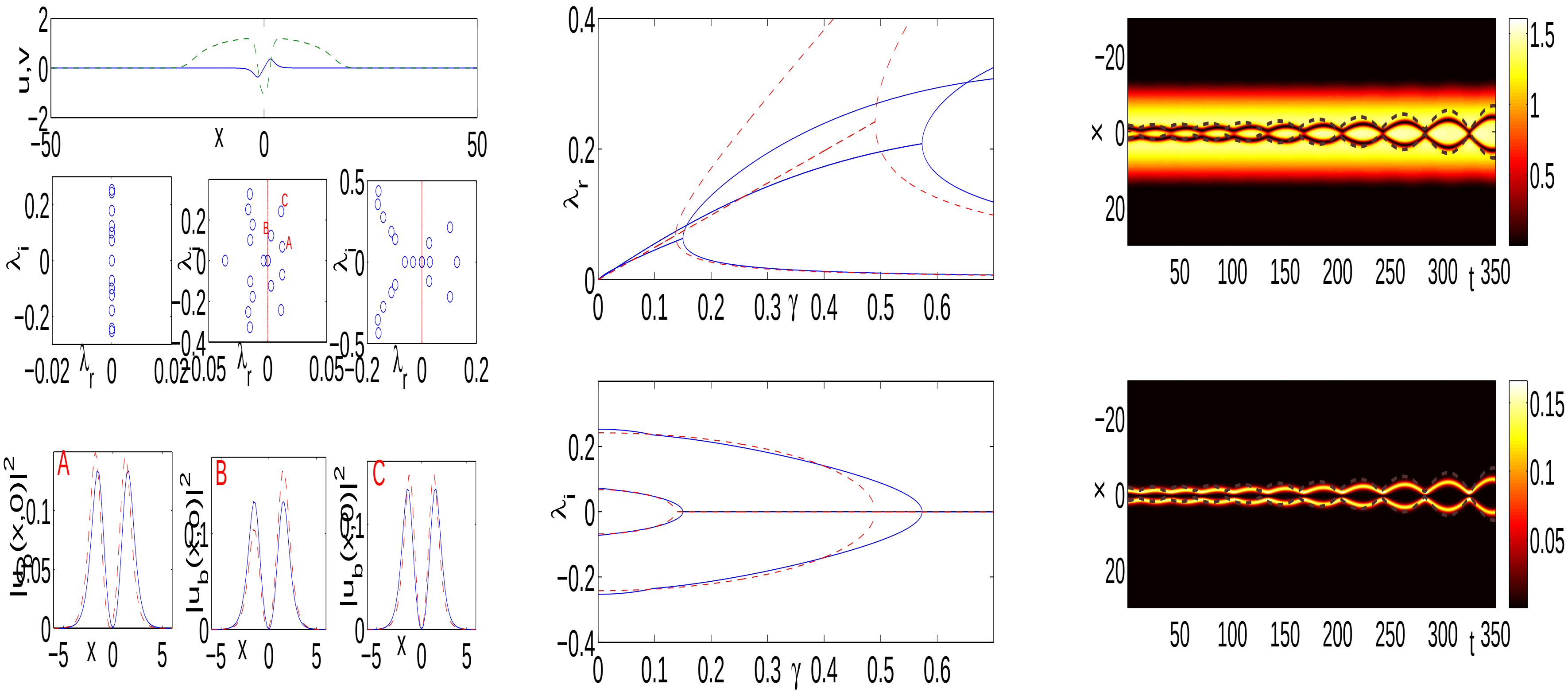


Thermal Effects on Single Dark-Bright Soliton



Thermal Effects on Multiple Dark-Bright Solitons




u,v
x
λi
λr
A
B
C
|ub(x,0)|2
λr
γ
λi
γ
x
t
x
t