BIRS Workshop 12w5073 Spectral Analysis, Stability and Bifurcation in Modern Nonlinear Physical Systems

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organized by O. N. Kirillov *et al.*

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Lagrangian and Eulerian hybrid method for symmetric breaking bifurcation of a rotating flow

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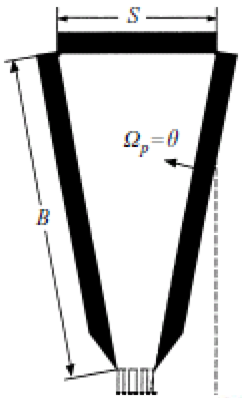
Instability of an anti-parallel vortex pair

Leweke & Williamson: *J. Fluid Mech.* **360** (1998) 85



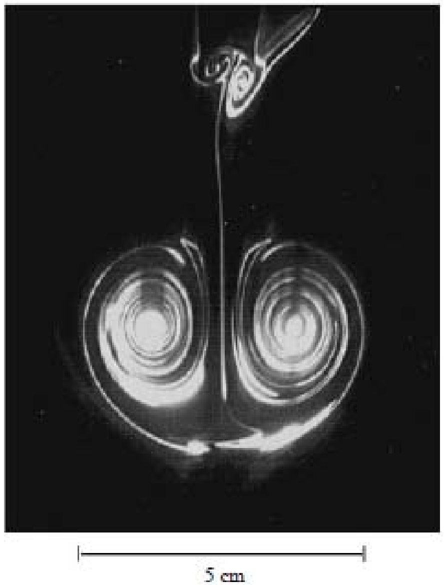


FIGURE 1. Fluorescent dye visualization of a symmetric counter-rotat
plane perpendicular to the vortex axes shortly after the end of

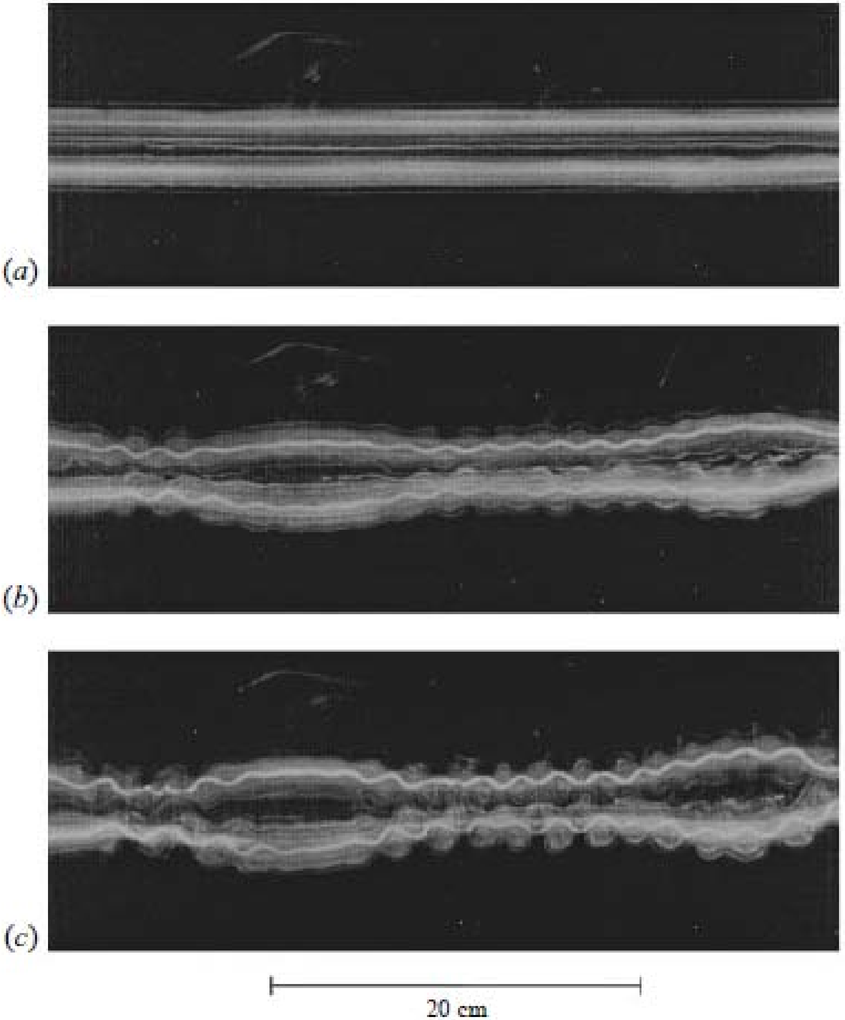


FIGURE 4. Visualization of vortex pair evolution under the combined action of long-wavelength (Crow) and short-wavelength instabilities. $Re = 2750$. The pair is moving towards the observer. (*a*) $t^* = 1.7$, (*b*) $t^* = 5.6$, (*c*) $t^* = 6.8$.

Close-up views of the short-wave instability

Leweke & Williamson: *J. Fluid Mech.* **360** (‘98) 85

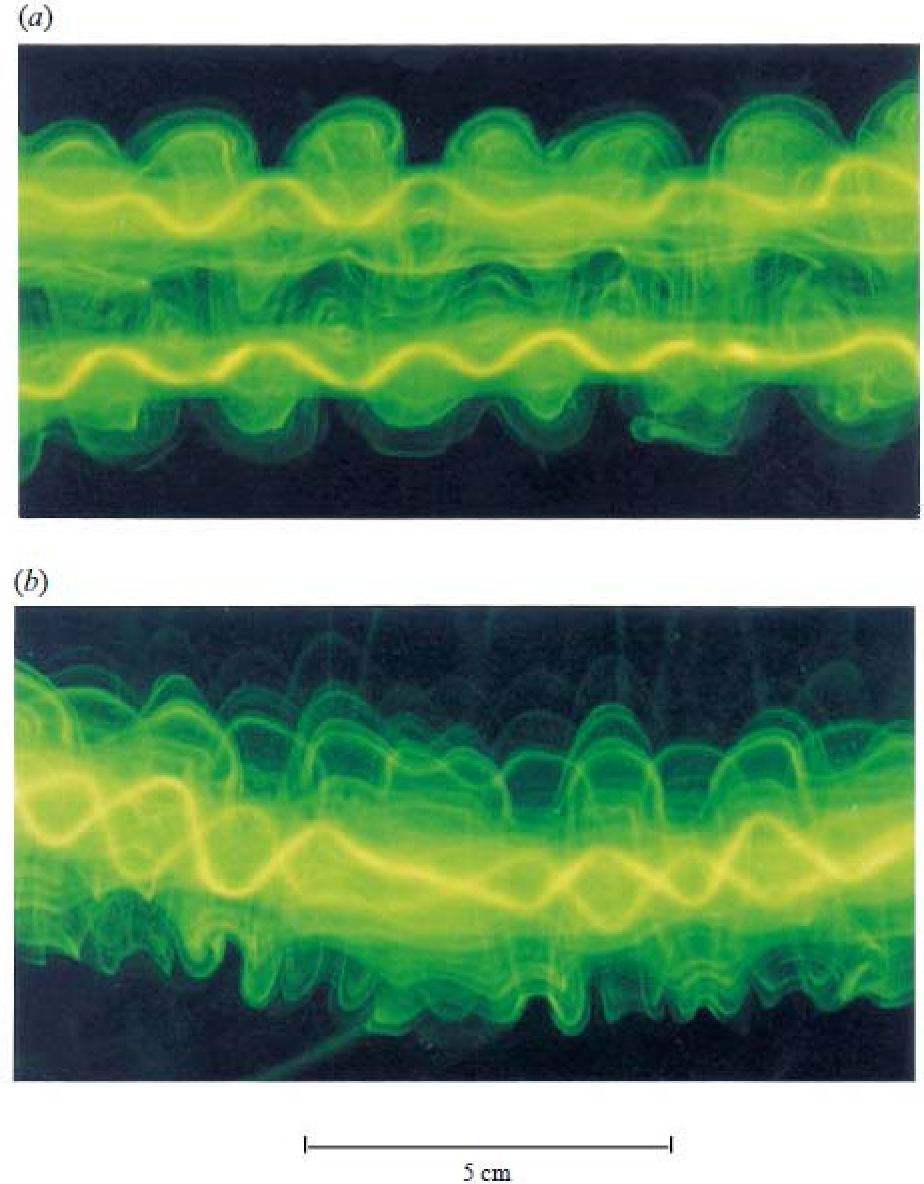


FIGURE 5. Simultaneous close-up views of the short-wavelength vortex pair perturbation in figure 4(*c*) from two perpendicular directions. $Re = 2750$, $t^* = 6.8$. (*a*) Front view (pair moving towards observer), (*b*) side view (pair moving down). The phase relation between the two vortices is clearly visible.

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Moore-Saffman-Tsai-Widnall instability

Moore & Saffman ('75), Tsai & Widnall ('76)

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Eulerian approach: (adaptation of) Cairn's formula

Lagrangian approach

3. Mean flow induced by nonlinear interaction of waves

↓

4. Weakly nonlinear evolution of Kelvin waves in a cylinder of elliptic cross-section

5. Three-wave interaction

1. Three-dimensional instability of a strained vortex tube

3D Linear stability cf. talks by Le Dizés, Llewellyn Smith

Moore & Saffman: Proc. R. Soc. Lond. A 346 (‘75) 413-425
Tsai & Widnall: J. Fluid Mech. **73** ('76) 721-733
Eloy & Le Dizés: Phys. Fluids **13** (‘01) 660-676.

Fukumoto : J. Fluid Mech. **493** (‘03) 287-318

Krein’s theory of Hamilitonian spectra

Energy of Kelvin waves

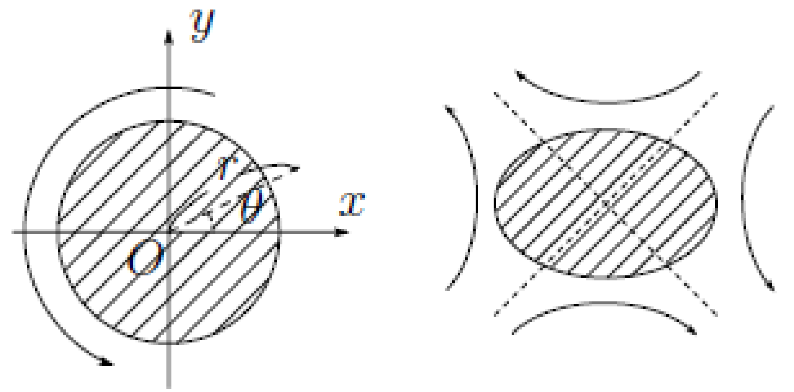
Hirota & Fukumoto & Hirota: J. Math. Phys. 49 (‘08) 083101
Fukumoto & Hirota: Physica Scripta T **132** (‘08) 014041
Fukumoto, Hirota & Mie: Math. Sci. Appl. **43** (‘12) 53-70

Elliptically strained vortex

$$U = \varepsilon U_1(r,\theta) + \cdots\,, \quad V = V_0(r) + \varepsilon V_1(r,\theta) + \cdots\,,$$
$$\Phi = \Phi_0(\theta) + \varepsilon \Phi_1(r,\theta) + \cdots\,.$$

$O(\varepsilon^0)$ **Rankine vortex**

$$V_0 = \begin{cases} r & (r \le 1) \\ 1/r & (r > 1)\,. \end{cases}$$



$O(\varepsilon^1)$ **Pure shear**

$$U_1 = -r\sin 2\theta\,, \quad V_1 = -r\cos 2\theta \quad \big(r < R(\theta,\varepsilon)\big)\,.$$

The boundary shape: $R(\theta,\varepsilon) \approx 1 + \frac{1}{2}\varepsilon\cos 2\theta$

Question: “Influence of **pure shear** upon *Kelvin waves* ?"

Expand infinitesimal disturbance in ε

Suppose that the core boundary is disturbed to

$$r = R(\theta, \varepsilon) + a(\theta; \varepsilon)e^{\mathrm{i}(kz-\omega t)}$$

We seek the disturbance velocity $\tilde{\boldsymbol{u}}$ in a power series of ε to first order:

$$\tilde{\boldsymbol{u}} \;=\; (\boldsymbol{u}_0 + \varepsilon \boldsymbol{u}_1 + \cdots)e^{\mathrm{i}(kz-\omega t)}\,,$$

with wavenumber k and frequency ω being

$$k = k_0 + \varepsilon k_1 + \cdots\,, \quad \omega = \omega_0 + \varepsilon \omega_1 + \cdots\,.$$

$O(\varepsilon^0)$: **Kelvin waves**

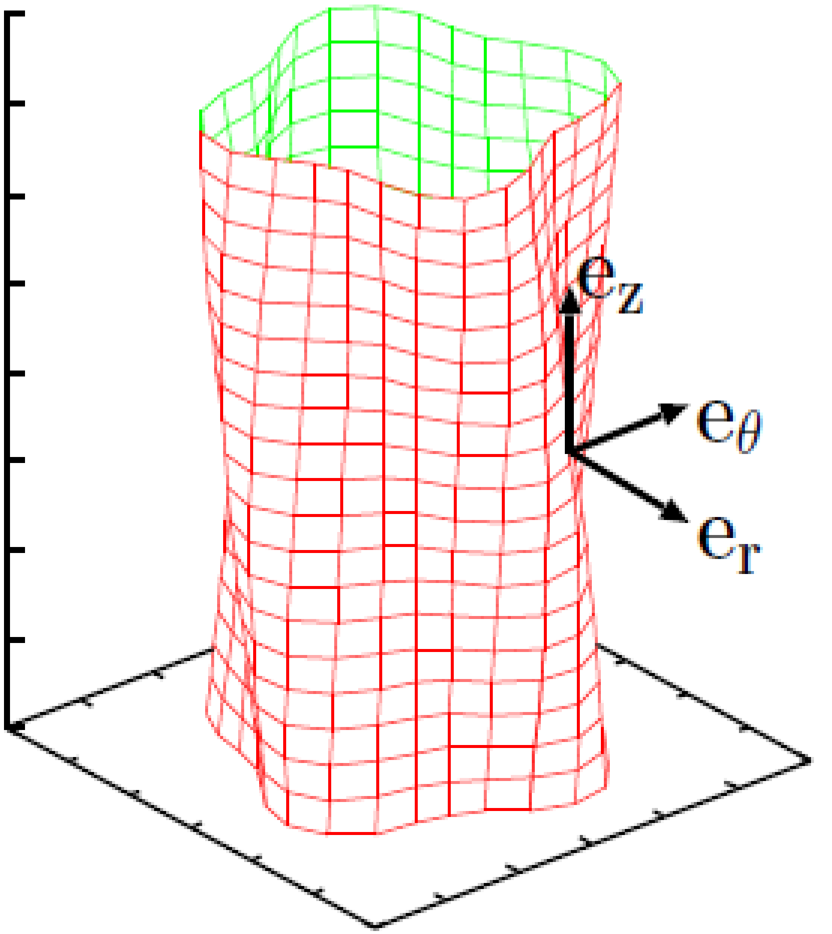
core: $\eta(\theta, z, t) = 1 + A_0^{(m)} \exp[\mathrm{i}(m\theta + k_0 z - \omega_0 t)]\,,$

$$\boldsymbol{u}_0 = \boldsymbol{u}_0^{(m)}(r)e^{\mathrm{i}m\theta}\,, \;\; \pi_0 = \pi_0^{(m)}(r)e^{\mathrm{i}m\theta}\,, \;\; \phi_0 = \phi_0^{(m)}(r)e^{\mathrm{i}m\theta}\,.$$

→ *the linearized Euler equations*

Example of a Kelvin wave *m=4*

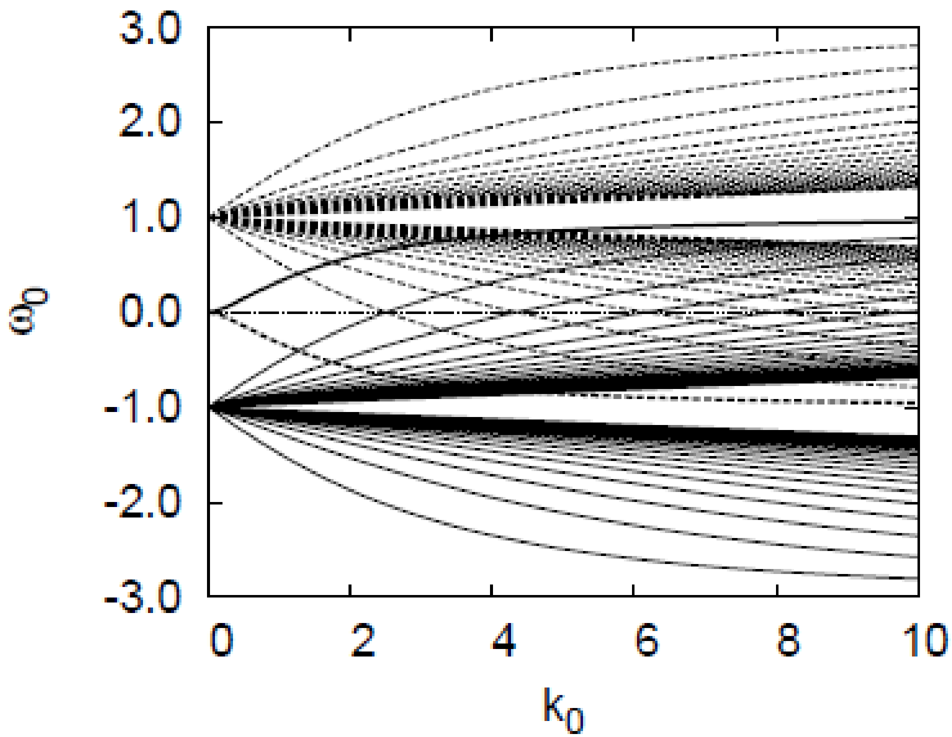
$$\tilde{\boldsymbol{u}} \propto \mathrm{e}^{\mathrm{i}(k_0 z + m\theta - \omega_0 t)}$$



Dispersion relation of Kelvin waves $m = \pm 1$

$$\eta_m J_{|m|}(\eta_m) K_{|m|-1}(k_0) - k_0 J_{|m|-1}(\eta_m) K_{|m|}(k_0) - \frac{2m(\eta_m/k_0)}{\omega_0 - m - \frac{2m}{|m|}} J_{|m|}(\eta_m) K_{|m|}(k_0) = 0$$

($J_{|m|}$ and $K_{|m|}$ are the (modified) Bessel functions)



m=-1 (solid lines) and *m=1* (dashed lines)

Equations for disturbance of $O(\varepsilon)$

$$\boldsymbol{u}_1 e^{\mathrm{i}(kz-\omega t)}; \qquad \boldsymbol{u}_1 = \{u_1, v_1, w_1, \pi_1, \phi_1\}$$

$$-\mathrm{i}\omega_0 u_1 + \frac{\partial u_1}{\partial \theta} - 2v_1 + \frac{\partial \pi_1}{\partial r} = i\omega_1 u_0 + \left(r\frac{\partial u_0}{\partial r} + u_0\right)\sin 2\theta + \frac{\partial u_0}{\partial \theta}\cos 2\theta\,,$$

$$\vdots$$

$$\frac{\partial u_1}{\partial r} + \frac{u_1}{r} + \frac{1}{r}\frac{\partial v_1}{\partial \theta} + ik_0 w_1 = -ik_1 w_0 \qquad (r < 1)\,.$$

$$\frac{\partial^2 \phi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi_1}{\partial \theta^2} - k_0^2\phi_1 = 2k_1 k_0 \phi_0 \qquad (r > 1)\,.$$

Disturbance field for the m, $m+2$ waves

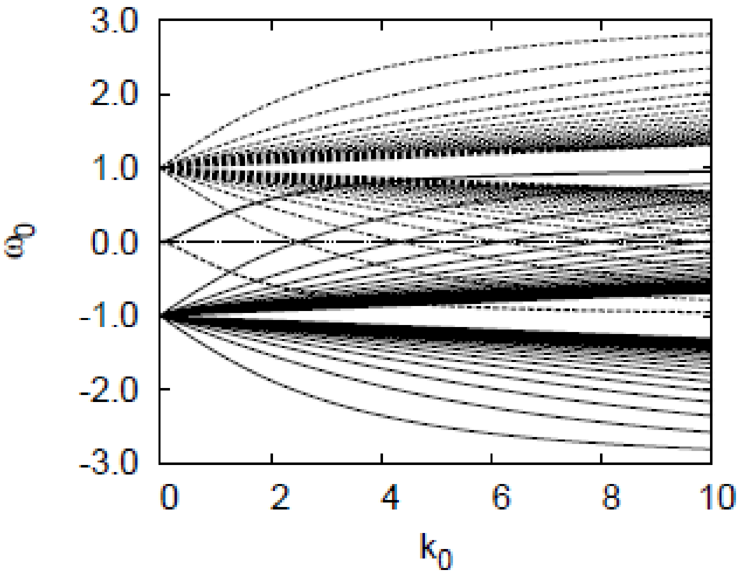
Pose to $O(\varepsilon^0)$

$$\boldsymbol{u}_0 = \boldsymbol{u}_0^{(1)} e^{\mathrm{i}m\theta} + \boldsymbol{u}_0^{(2)} e^{\mathrm{i}(m+2)\theta}\,.$$

Then at $O(\varepsilon^1)$

$$\Rightarrow \boxed{\boldsymbol{u}_1 = \boldsymbol{u}_1^{(1)} e^{\mathrm{i}m\theta} + \boldsymbol{u}_1^{(2)} e^{\mathrm{i}(m+2)\theta} + \boldsymbol{u}_1^{(3)} e^{\mathrm{i}(m-2)\theta} + \boldsymbol{u}_1^{(4)} e^{\mathrm{i}(m+4)\theta}}$$

Growth rate of helical waves *($m=\pm 1$)*



σ_{1max}: growth rate

Δk_1: unstable band width

stationary mode

$(\omega_0 = 0)$

⇓

k_0	σ_{1max}	Δk_1
0	0.5	∞
2.504982369	0.5707533917	2.145502816
4.349076726	0.5694562098	3.518286549
6.174012330	0.5681222780	4.883945142
7.993536550	0.5671646287	6.247280752
9.810807288	0.5664714116	7.609553122

Instability occurs at ***every*** intersection points of dispersion curves of (*m*, *m*+2) waves Why?

Moore-Saffman-Tsai-Widnall instability

Wave energy: Difficulty in Eulerian treatment

base flow

disturbance

$$\boldsymbol{u} = \boldsymbol{U} + \tilde{\boldsymbol{u}}; \quad \tilde{\boldsymbol{u}} = \alpha \tilde{\boldsymbol{u}}_{01} + \frac{1}{2}\alpha^2 \tilde{\boldsymbol{u}}_{02}$$

Excess energy:

$$\frac{1}{2}\int \boldsymbol{u}^2 \mathrm{d}V - \frac{1}{2}\int \boldsymbol{U}^2 \mathrm{d}V$$

$$= \alpha \delta H + \frac{1}{2}\alpha^2 \delta^2 H;$$

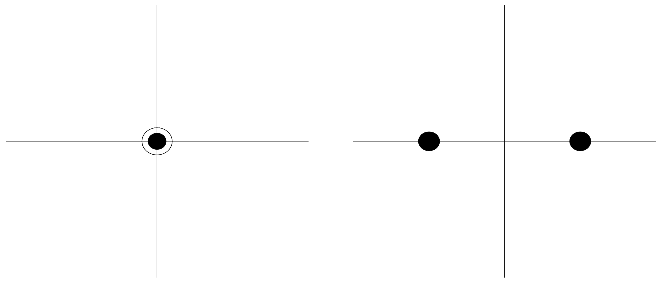
$$\delta H = \int \boldsymbol{U} \cdot \tilde{\boldsymbol{u}}_{01} \mathrm{d}V, \quad \delta^2 H = \int \left(\tilde{\boldsymbol{u}}_{01}^2 + \boldsymbol{U} \cdot \tilde{\boldsymbol{u}}_{02}\right) \mathrm{d}V$$

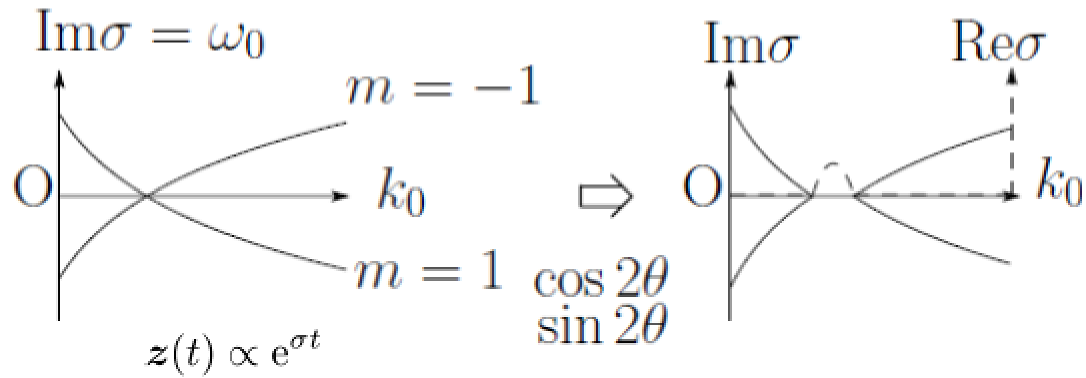
- $\ast$ $\delta H \neq \text{const.} \quad \delta^2 H \neq \text{const.}$
- $\ast$ $\tilde{\boldsymbol{u}}_{02}$ is to be defined

Krein's theory of Hamiltonian spectra

Hamiltonian pitchfork bifurcation

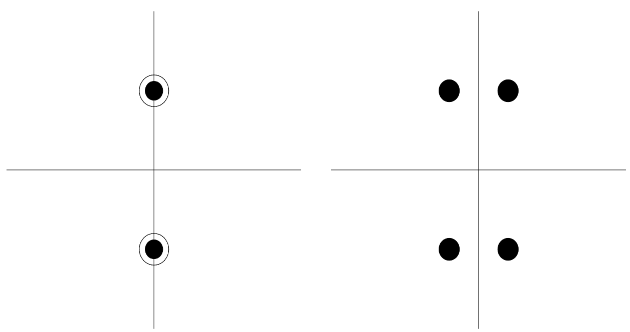
Stationary (-1,+1) mode $\omega_0 = 0$

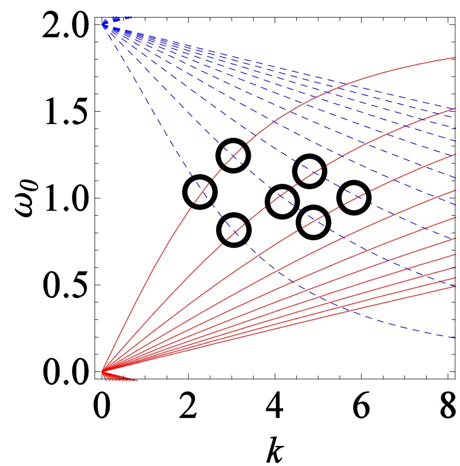




Hamiltonian Hopf bifurcation

Nonstationary (m,m+2) mode





Cairns' formula

R. A. Cairns: J. Fluid Mech. **92** ('79) 1-14

Boundary $\eta(\theta, z, t) = 1 + A_0^{(m)} \cos(m\theta + k_0 z - \omega_0 t)$.

Boundary pressure $p_< = p|_{r=\eta-}$, $p_> = p|_{r=\eta+}$;

$$p_> = D_>(k_0, \omega_0) A_0^{(m)} \cos(m\theta + k_0 z - \omega_0 t)\,, \quad p_< = D_<(k_0, \omega_0) A_0^{(m)} \cos(m\theta + k_0 z - \omega_0 t)\,.$$

$\Longrightarrow$ dispersion relation : $D(k_0, \omega_0) := D_>(k_0, \omega_0) - D_<(k_0, \omega_0) = 0$

Cairns' formula ('79) equates wave energy $E^{(m)}$, per unit length in z, to *work* W by 'external driving force:

$$-\overline{\dot{\eta}(p_> - p_<)} \approx \overline{i\omega_0 \eta(p_> - p_<)} = \frac{dW}{dt}\,.$$

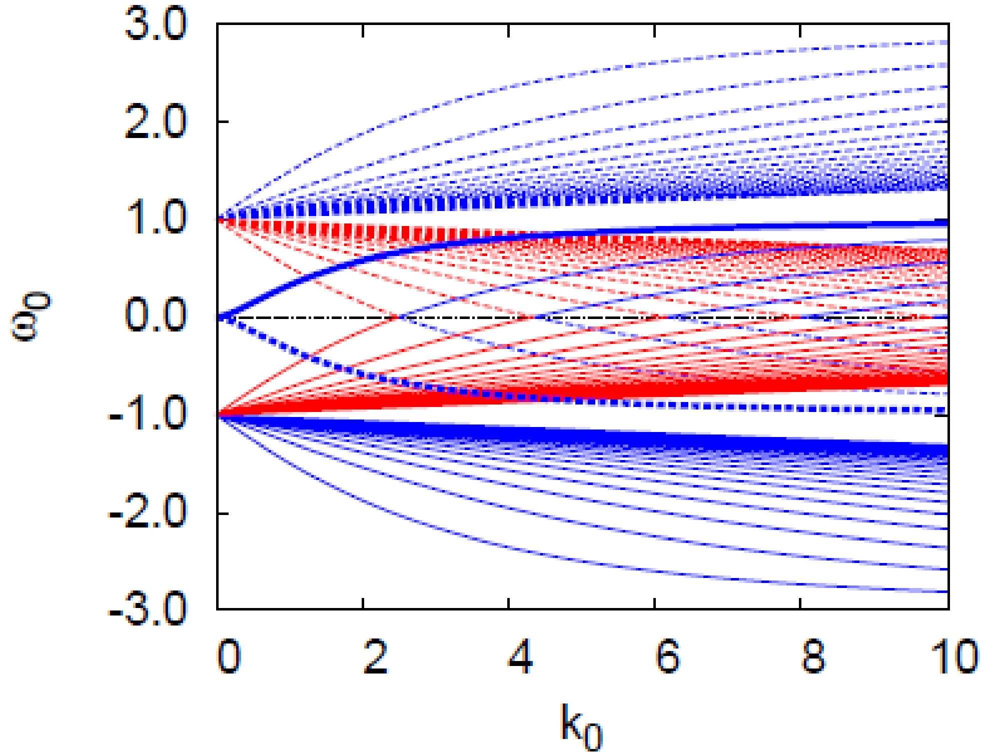
$$(W =)E^{(m)} = -\frac{\pi}{2}\omega_0 \frac{\partial D}{\partial \omega_0} (A_0^{(m)})^2\,,$$

$$\Rightarrow \quad E^{(m)} = \frac{2\pi\omega_0}{\omega_0 - m} \left\{ 1 + \frac{(k_0/\eta_m)^2 K_{|m|}}{k_0 K_{|m|-1} + |m| K_{|m|}} \left[\frac{2(\omega_0 + m)}{\omega_0 - m} + \left(\frac{m(\omega_0 + m)}{2} + k_0^2 \right) \frac{K_{|m|}}{k_0 K_{|m|-1} + |m| K_{|m|}} \right] \right\} \left(A_0^{(m)} \right)^2$$

Fukumoto : J. Fluid Mech. **493** ('03) 287-318

Energy signature of helical waves (m=±1)

- **Blue**: positive wave-energy
- **Red**: negative wave-energy



m=-1 (solid lines) and *m=1* (dashed lines)

2. Energy of waves

Justification of adapted Cairns' formula?

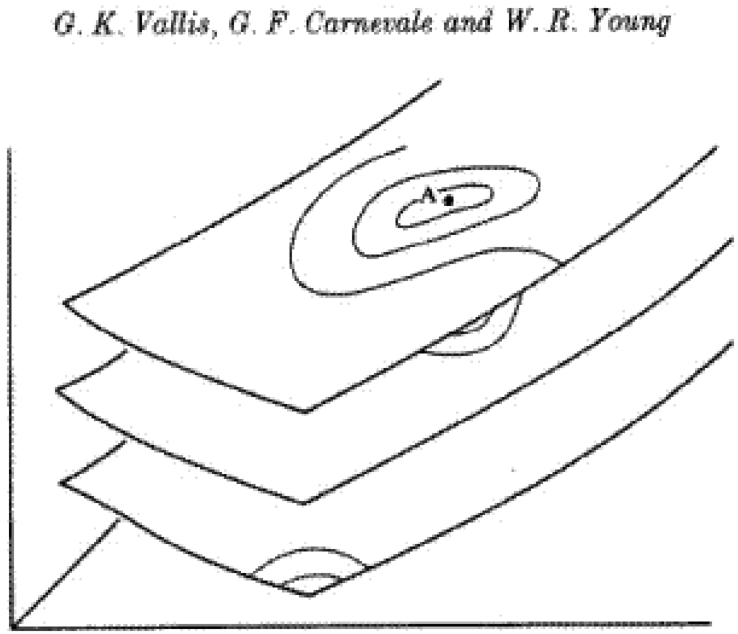
$$E^{(m)} = -\frac{\pi}{2}\omega_0 \frac{\partial D}{\partial \omega_0}\left(A_0^{(m)}\right)^2$$

This is different in sign from Cairns' original formula.

cf. Oliver Doaré (Nov. 7)

More systematic treatment?

Steady Euler flows



isovortical sheets

Kinematically accessible variation
(= preservation of local circulation)

$$\boldsymbol{x} \to \tilde{\boldsymbol{x}} \quad \Rightarrow \quad \omega = \tilde{\omega}$$

$$\frac{1}{2}\epsilon_{ijk}\omega_k(\boldsymbol{x},t)\mathrm{d}x_i \wedge \mathrm{d}x_j$$
$$= \frac{1}{2}\epsilon_{pqr}\tilde{\omega}_r(\tilde{\boldsymbol{x}},t)\mathrm{d}\tilde{x}_p \wedge \mathrm{d}\tilde{x}_q$$
$$(\tilde{\omega}_r = \omega_r + \delta\omega_r)$$

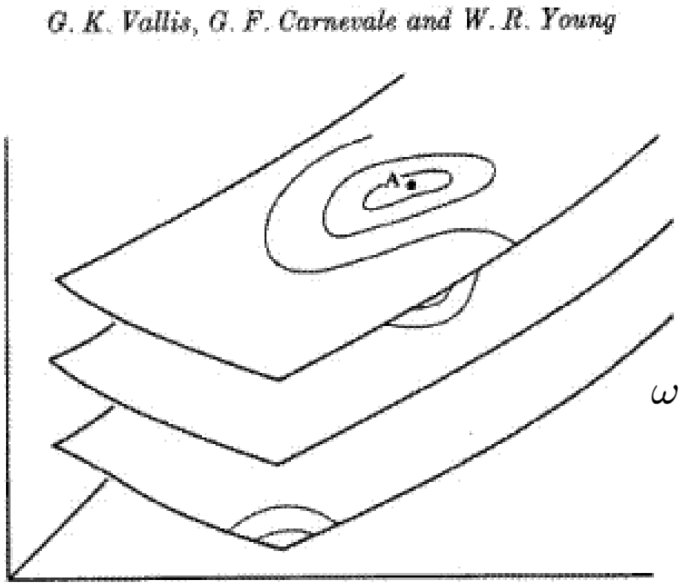
Theorem (Kelvin, Arnold 1965) A steady Euler flow is a coditional extremum of energy *H* w.r.t. kinematically accessible variations.

Isovortical disturbance on a steady Euler flow

$\varphi_{\alpha,t} \in \mathrm{SDiff}(\mathcal{D})$ *fluid flow map*

Lagrangian displacement

$$\varphi_{\alpha,t}(\boldsymbol{x}) = (\exp \boldsymbol{\xi}_\alpha(t))\, \boldsymbol{x}; \quad \boldsymbol{\xi}_\alpha = \alpha \boldsymbol{\xi}_1 + \frac{\alpha^2}{2}\boldsymbol{\xi}_2$$



isovortical sheets

$$\tilde{\boldsymbol{x}} = \varphi_{\alpha,t}(\boldsymbol{x}) = \boldsymbol{x} + \alpha \boldsymbol{\xi}_1 + \frac{\alpha^2}{2}[(\boldsymbol{\xi}_1 \cdot \nabla)\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2] + O(\alpha^3)$$

$$\omega := \frac{1}{2}\epsilon_{ijk}\omega_k(\boldsymbol{x}, t)\mathrm{d}x_i \wedge \mathrm{d}x_j$$

$$\boldsymbol{x} \to \tilde{\boldsymbol{x}} \quad \Rightarrow \quad \omega = \tilde{\omega};$$

$$\frac{1}{2}\epsilon_{ijk}\omega_k(\boldsymbol{x}, t)\mathrm{d}x_i \wedge \mathrm{d}x_j$$

$$= \frac{1}{2}\epsilon_{pqr}\tilde{\omega}_r(\tilde{\boldsymbol{x}}, t)\mathrm{d}\tilde{x}_p \wedge \mathrm{d}\tilde{x}_q$$

$$\left(\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} + \alpha \boldsymbol{\omega}_1 + \frac{\alpha^2}{2}\boldsymbol{\omega}_2\right)$$

$$\boldsymbol{\omega}_1 = \nabla \times (\boldsymbol{\xi}_1 \times \boldsymbol{\omega})$$

Fukumoto & Hirota '08

$$\boxed{\boldsymbol{\omega}_2 = \nabla \times (\boldsymbol{\xi}_1 \times \boldsymbol{\omega}_1) + \nabla \times (\boldsymbol{\xi}_2 \times \boldsymbol{\omega})}$$

Wave energy for kinematicaly accessible disturbance

Lagrangian displacement $\xi_\alpha = \alpha \xi_1 + \frac{\alpha^2}{2}\xi_2 + \cdots \quad \left[\varphi_{\alpha,t}(\boldsymbol{x}) = \exp\left(\boldsymbol{\xi}_\alpha(t)\right)\boldsymbol{x}\right]$

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$$

$$\boldsymbol{v}_1 = \mathcal{P}\left[\boldsymbol{\xi}_1 \times \boldsymbol{\omega}\right],$$
$$\boldsymbol{v}_2 = \mathcal{P}\left[\boldsymbol{\xi}_1 \times \left(\nabla \times (\boldsymbol{\xi}_1 \times \boldsymbol{\omega})\right) + \boldsymbol{\xi}_2 \times \boldsymbol{\omega}\right]$$

$$H(v_\epsilon) = H(v) + \epsilon H_1 + \frac{\alpha^2}{2} H_2 + \cdots$$

$$H_1 = \left\langle \frac{\delta H}{\delta v}, v_1 \right\rangle = \cdots = -\left\langle \xi_1, \frac{\partial v}{\partial t} \right\rangle = 0$$

If v is steady

$$H_2 = \left\langle \frac{\delta H}{\delta v}, v_2 \right\rangle + \left\langle \frac{\delta^2 H}{\delta v^2} v_1, v_1 \right\rangle = -\left\langle \xi_2, \frac{\partial v}{\partial t} \right\rangle - \left\langle \xi_1, \frac{\partial v_1}{\partial t} \right\rangle$$

For steady flow

$$H_2 = -\left\langle \xi_1, \frac{\partial v_1}{\partial t} \right\rangle = \int \boldsymbol{\omega} \cdot \left(\frac{\partial \boldsymbol{\xi}_1}{\partial t} \times \boldsymbol{\xi}_1 \right) \mathrm{d}V$$

Alternatively

$$H_2 = 2 \int \frac{\partial \xi_1}{\partial t} \cdot \left(\frac{\partial \xi_1}{\partial t} + (U \cdot \nabla)\xi_1 \right) \mathrm{d}V$$

Wave energy in terms of dispersion relation

Hirota & Fukumoto: J. Math. Phys. **49** ('08) 083101

For a rotating flow confined laterally in a circular cylinder of radius 1

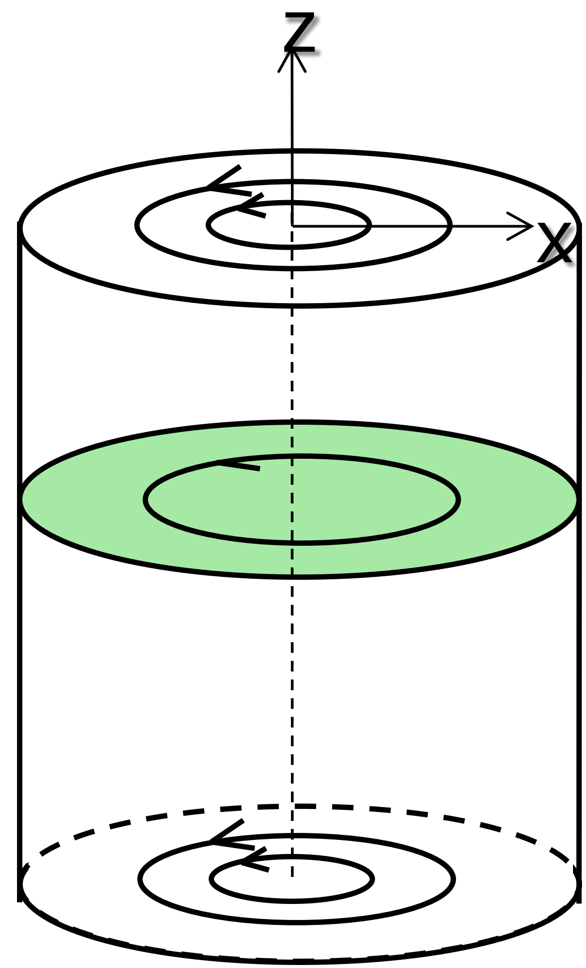
action

$$\begin{aligned} E_0 &= \omega_0 \mu_0; \\ \mu_0 &= \pi \frac{\partial D}{\partial \Omega}(\omega_0; m, k_0) \end{aligned}$$

where

$$D(\omega_0; m, k_0) = 0$$

Is the dispersion relation



(r, θ, z)

Wave energy in terms of dispersion relation: Derivation

Linearlized Euler equation for $v \in \mathfrak{g}^*$

$$i\frac{\partial v}{\partial t} = \mathcal{L}v, \quad \text{Define } \mathcal{A} : \mathfrak{g} \to \mathfrak{g}^* \text{ by } \mathcal{A}\xi = -\mathrm{ad}_\xi^* v_0$$

Equation for radial displacement $\widehat{\xi}_r(r; \omega_0, m, k_0)e^{-i\omega_0 t}$

$$\mathcal{E}(\omega_0) r\widehat{\xi}_r = 0, \quad \text{where } \mathcal{E}(\Omega) := i(\Omega - \mathcal{L})\mathcal{A}$$

One-sided Fourier transform (Laplace transform)

$$\Xi(r, \Omega) = \int_0^\infty [r\widehat{\xi}_r(r; \omega_0)e^{-i\omega_0 t}]e^{i\Omega t}\mathrm{d}t, \quad \mathrm{Im}(\Omega) > 0$$

Wave action

Hirota & Fukumoto: J. Math. Phys. **49** ('08) 083101

$$2\mu_0 = \frac{1}{2\pi i}\oint_{\Gamma(\omega_0)} \mathcal{D}(\Omega)\mathrm{d}\Omega,$$

$$\mathcal{D}(\Omega) := 2\pi \int_0^1 \overline{\Xi(r, \overline{\Omega})}\mathcal{E}(\Omega)\Xi(r, \Omega)\mathrm{d}r$$

$$\mathcal{E}(\Omega) = \mathcal{E}(\omega_0) + (\Omega - \omega_0)\frac{\partial \mathcal{E}}{\partial \Omega}(\omega_0) + \ldots,$$

3. Mean flow induced by nonlinear interaction of waves

Fukumoto & Hirota: Physica Scripta T **132** ('08) 014041
Fukumoto & Hirota: Physica Scripta T **142** ('10) 014049
Fukumoto & Mie: Physica Scripta ('12) *to appear*

$$\boldsymbol{u} = \boldsymbol{U}_0 + \varepsilon \boldsymbol{U}_1 + \alpha \boldsymbol{u}_{01} + \varepsilon\alpha \boldsymbol{u}_{01} + \alpha^2 \boldsymbol{u}_{02} + \cdots$$

Mean flow $O(\alpha^2)$

Fukumoto & Hirota '08, '10

Lagrangian displacement

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$$

$$\zeta_\alpha = \alpha\xi_1 + \frac{\alpha^2}{2}\xi_2 + \cdots,$$

$$\boldsymbol{v}_1 = \mathcal{P}\left[\boldsymbol{\xi}_1 \times \boldsymbol{\omega}\right],$$
$$\boldsymbol{v}_2 = \mathcal{P}\left[\boldsymbol{\xi}_1 \times \left(\nabla \times (\boldsymbol{\xi}_1 \times \boldsymbol{\omega})\right) + \boldsymbol{\xi}_2 \times \boldsymbol{\omega}\right]$$

Take the average over a long time

$$\Rightarrow \quad \overline{\boldsymbol{v}} = \boldsymbol{U} + \frac{1}{2}\alpha^2 \overline{\boldsymbol{v}_2} + O(\alpha^3)$$

$$\overline{\boldsymbol{v}_2} = \mathcal{P}\left(\overline{\boldsymbol{\xi}_1 \times [\nabla \times (\boldsymbol{\xi}_1 \times \boldsymbol{\omega})]} + \overline{\boldsymbol{\xi}_2} \times \boldsymbol{\omega}\right)$$

for the ***Rankine vortex***

Substitute the ***Kelvin wave*** $\boldsymbol{\xi}_1 = \mathrm{Re}\left[C_0 \hat{\boldsymbol{\xi}} \mathrm{e}^{\mathrm{i}(m\theta + +k_0 z - \omega_0 t)}\right]$

$$\mathcal{P}(\overline{\boldsymbol{\xi}_1 \times [\nabla \times (\boldsymbol{\xi}_1 \times \boldsymbol{\omega})]} = \begin{cases} \mathrm{i}k_0|C_0|^2(0,\, \widehat{\xi}_z^*\widehat{\xi}_r - \widehat{\xi}_r^*\widehat{\xi}_z,\, \widehat{\xi}_r^*\widehat{\xi}_\theta - \widehat{\xi}_\theta^*\widehat{\xi}_r) & (r \le 1) \\ 0 & (r > 1)\,. \end{cases}$$

$$J_z := \alpha^2 \int \overline{v_{2z}} \mathrm{d}A = \alpha^2 k_0 |C_0^2| \frac{\mathrm{i}}{2} \int \boldsymbol{\omega} \cdot (\boldsymbol{\xi}_1^* \times \boldsymbol{\xi}_1)\, \mathrm{d}A = k_0 \mu_0$$

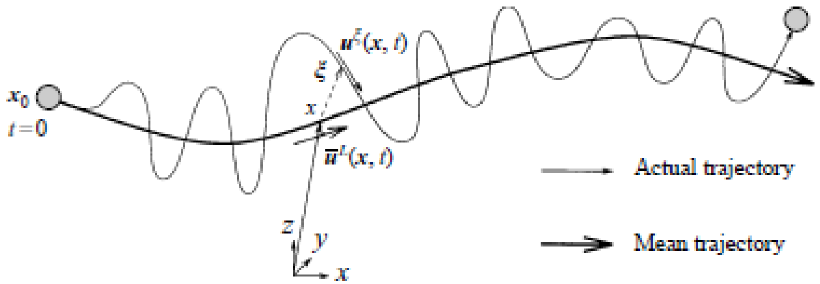
$k_0 = 0 \Rightarrow J_z = 0$ genuinely *3D* effect !! ***pseudomomentum***

Generalized Lagrangian mean (GLM) theory

Andrews & McIntyre: JFM ’78, Bühler ‘09

a closed loop

$$C \to C_\xi : \boldsymbol{x} \to \boldsymbol{x}^\xi = \boldsymbol{x} + \alpha\boldsymbol{\xi}$$



$$\begin{aligned}\Gamma &= \oint_{C_\xi} \boldsymbol{v}(\boldsymbol{x},t)\cdot \mathrm{d}\boldsymbol{x} = \oint_C \boldsymbol{v}(\boldsymbol{x}+\alpha\boldsymbol{\xi},t)\cdot \mathrm{d}\left(\boldsymbol{x}+\alpha\boldsymbol{\xi}\right)\\ &= \oint_C \left[v_i^\xi + \alpha v_j^\xi(\partial_i\xi_j)\right]\mathrm{d}x_i \qquad \left(\boldsymbol{v}^\xi := \boldsymbol{v}(\boldsymbol{x}+\alpha\boldsymbol{\xi},t)\right)\end{aligned}$$

Pseudomomentum $\quad \mathsf{p}_i := -\alpha\overline{(\partial_i\xi_j)v_j^\xi} = -\alpha\overline{(\partial_i\xi_j)v_j^\ell} \qquad (\boldsymbol{v}^\ell = \boldsymbol{v}^\xi - \bar{\boldsymbol{v}}^{\mathsf{L}})$

$$\bar{\Gamma} = \oint_C \left(\bar{v}_i^{\mathsf{L}} - \mathsf{p}_i\right)\mathrm{d}x_i = \oint_C \left(U_i + \frac{\alpha^2}{2}\bar{v}_{2i} + \bar{v}_i^{\mathsf{S}} - \mathsf{p}_i\right)\mathrm{d}x_i$$

Isovortical dusturbance $\quad \oint_C \boldsymbol{U}\cdot \mathrm{d}\boldsymbol{x} = \oint_C \left(\boldsymbol{U} + \frac{\alpha^2}{2}\overline{v_2} + \bar{\boldsymbol{v}}^{\mathsf{S}} - \mathsf{p}\right)\cdot \mathrm{d}\boldsymbol{x}$

$$\mathsf{p} \sim \mathcal{P}\left[\frac{\alpha^2}{2}\overline{v_2} + \bar{\boldsymbol{v}}^{\mathsf{S}}\right] \qquad \bar{\boldsymbol{v}}^{\mathsf{S}} = 0$$

Mean flow induced by Kelvin waves

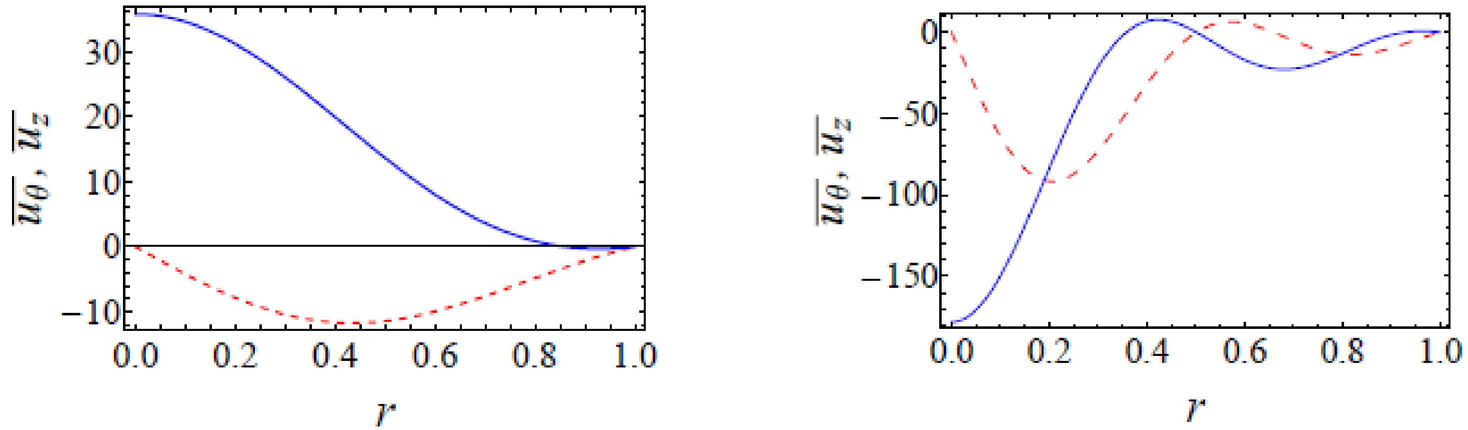


Figure 2: The mean flow $\overline{u_{02}}$ induced by nonlinear interaction of Kelvin waves at $(k_0, \omega_0) = (2.2, 0.3)$ for $m = -1$ (left) and $m = 1$ (right). The solid and dashed lines represent the axial ($\overline{u_{02z}}$) and the azimuthal ($\overline{u_{02\theta}}$) components, respectively.

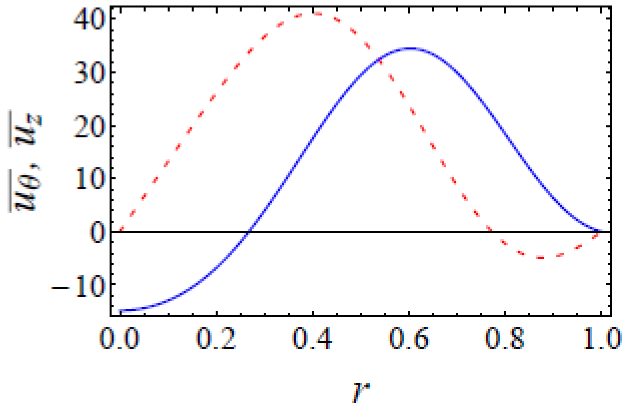


Figure 3: The same as figure 2, but at $(k_0, \omega_0) = (3.0, 2.0)$ for $m = 1$.

4. Weakly nonlinear evolution of Kelvin waves in a cylinder of elliptic cross-section

Malkus ('89), Eloy, Le Gal & Le Dizés ('00)

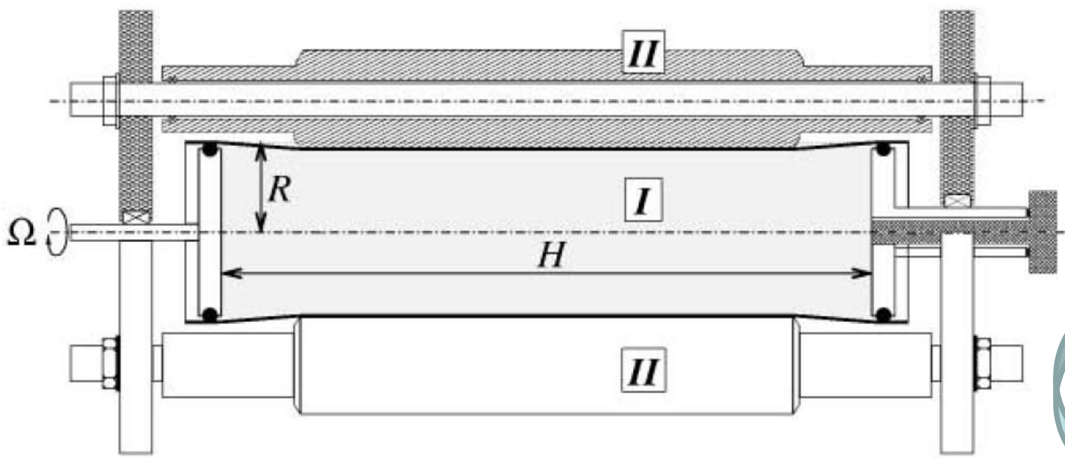
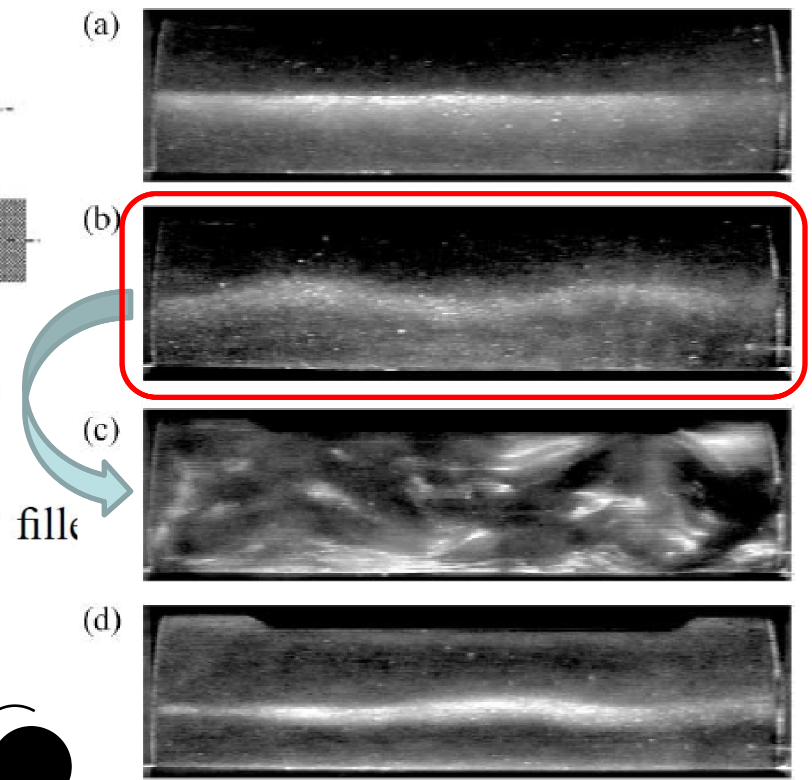
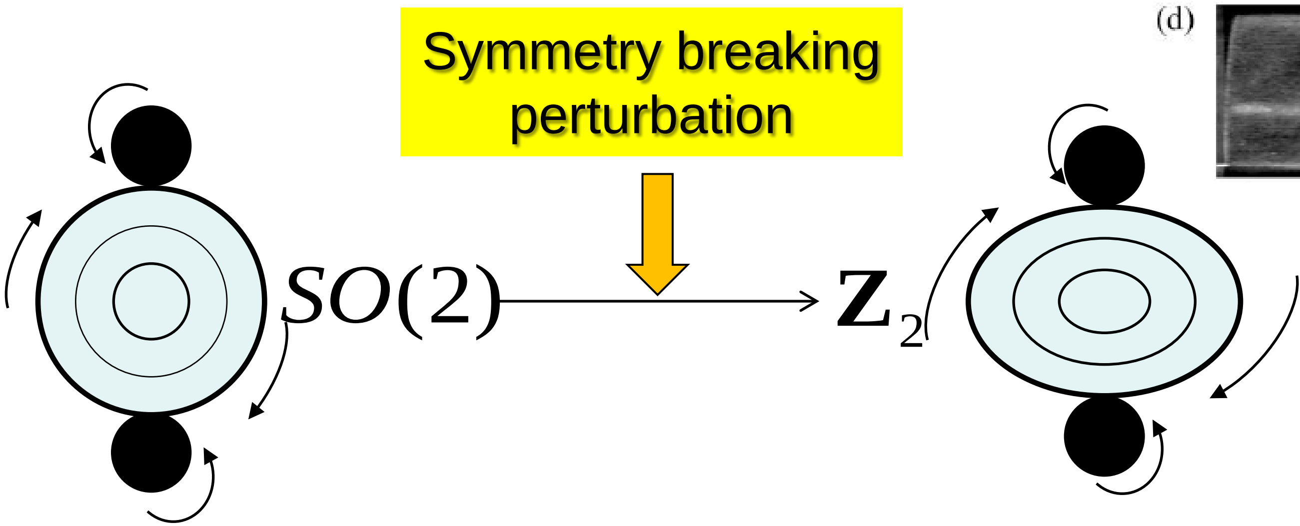


FIG. 1. Experimental setup: (I) plastic elastic cylinder fill
with water; (II) rollers.





Excitation of unstable modes for a rotating flow in a elliptically strained cylinder

Three-dimensional linear instability of an elliptically strained vortex

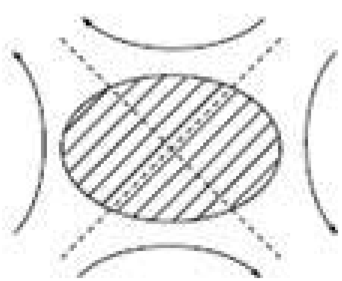
Cylindrical coordinates (r,θ,z)

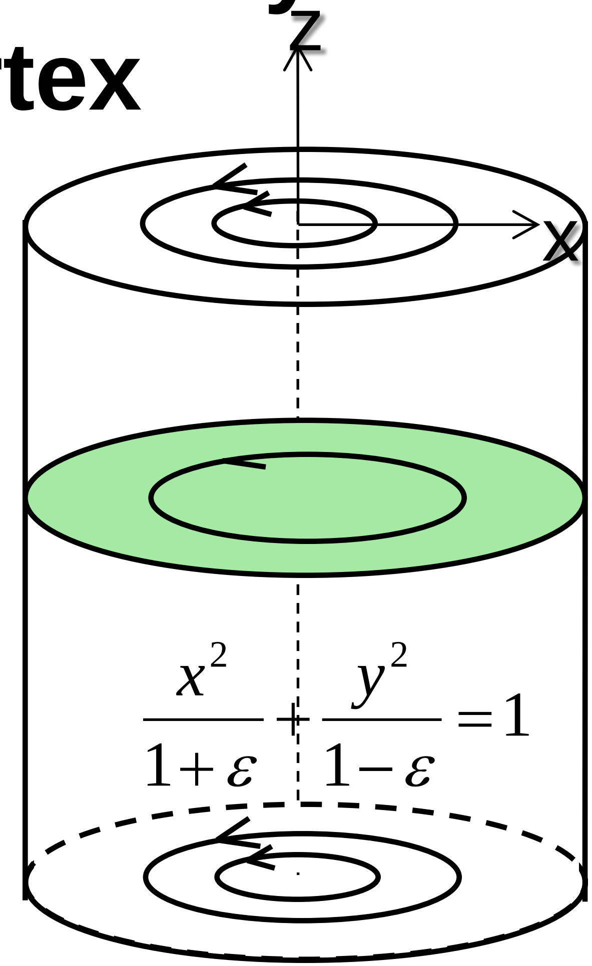
Boundary shape $r = 1 + \varepsilon \cos 2\theta / 2$

Basic flow $\mathbf{U} = \mathbf{U}_0 + \varepsilon \mathbf{U}_1$

$$\mathbf{U}_0 = (0, r, 0)$$

$$\mathbf{U}_1 = (-r\sin 2\theta, -r\cos 2\theta, 0)$$





Question: “Influence of pure shear upon Kelvin waves?”

Resonance between (m,m+2)=(0,2) modes

C. Eloy

in

Kerswell: Annu. Rev. Fluid Mech. (2002)

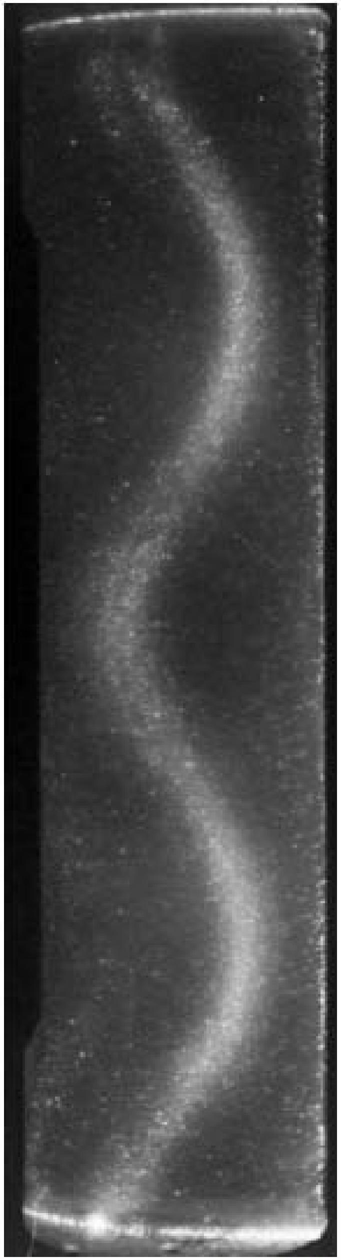


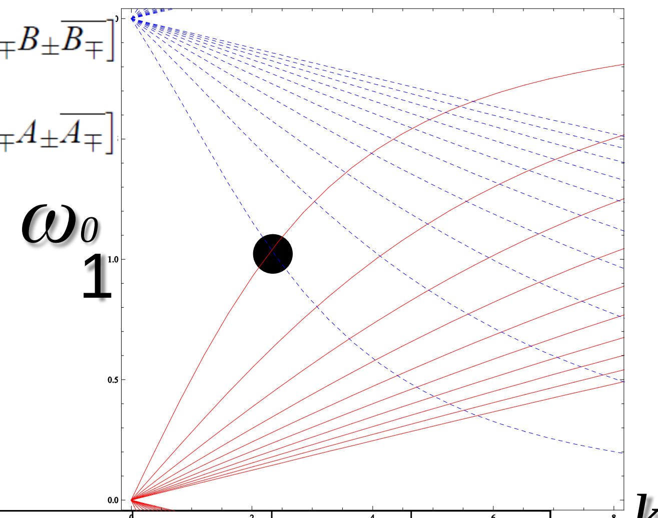
Figure 4 A snapshot of the $(m, m+2) = (0, 2)$ elliptical instability with resonant (inviscid) axial wavelength of 2.7009 in a container of height-to-radius ratio of 8.20. The Reynolds number is 2500 and the strain $\epsilon = 0.1$ (courtesy of C. Eloy).

Amplitude equations for (0,2)mode

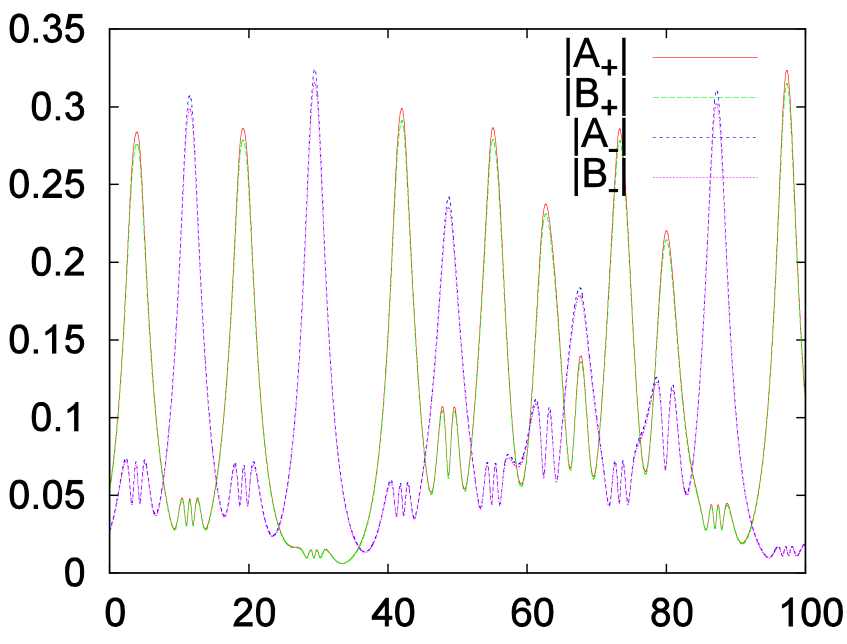
$$u_{01} = \mathrm{Re}\left[A_+ u^{(A+)} e^{ikz} + B_+ u^{(B+)} e^{i(2\theta+kz)} + A_- u^{(A-)} e^{-ikz} + B_- u^{(B-)} e^{i(2\theta-kz)}\right]$$

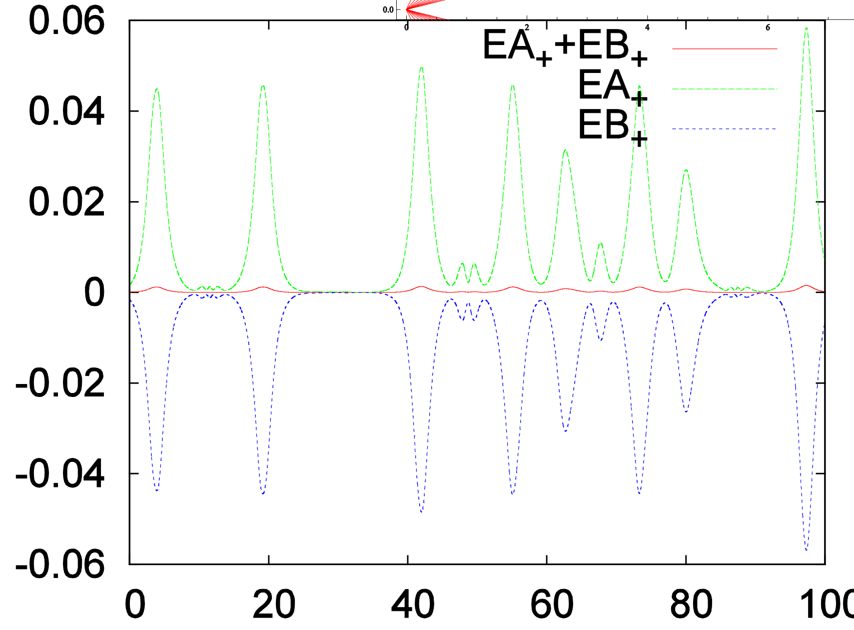
$$\frac{dA_\pm}{dt} = i\left[-\omega_0 A_\pm + \varepsilon p B_\pm + \alpha^2 A_\pm \left(s_{11}|A_\pm|^2 + s_{12}|B_\pm|^2 + s_{13}|A_\mp|^2 + s_{14}|B_\mp|^2\right) + \alpha^2 s_{15} A_\mp B_\pm \overline{B_\mp}\right]$$

$$\frac{dB_\pm}{dt} = i\left[-\omega_0 B_\pm + \varepsilon q A_\pm + \alpha^2 B_\pm \left(s_{21}|A_\pm|^2 + s_{22}|B_\pm|^2 + s_{23}|A_\mp|^2 + s_{24}|B_\mp|^2\right) + \alpha^2 s_{25} B_\mp A_\pm \overline{A_\mp}\right]$$



$$|A_+|,\ |B_+|,\ |A_-|,\ |B_-|$$





Canonical Hamilton equations

$$z_{1\pm} = A_{\pm}/\sqrt{p}, \quad z_{2\pm} = \overline{B_{\pm}}/\sqrt{q}$$

$$\frac{\mathrm{d}z_{1\pm}}{\mathrm{d}t} = i\left[-\omega_0 z_{1\pm} + \varepsilon\sigma\overline{z_{2\pm}} + z_{1\pm}\left(c_{11}|z_{1\pm}|^2 + c_{12}|z_{2\pm}|^2 + c_{13}|z_{1\mp}|^2 + c_{14}|z_{2\mp}|^2\right) + c_{15} z_{1\mp}\overline{z_{2\pm}} z_{2\mp}\right],$$

$$\frac{\mathrm{d}z_{2\pm}}{\mathrm{d}t} = -i\left[-\omega_0 z_{2\pm} - \varepsilon\sigma\overline{z_{1\pm}} + z_{2\pm}\left(c_{21}|z_{1\pm}|^2 + c_{22}|z_{2\pm}|^2 + c_{23}|z_{1\mp}|^2 + c_{24}|z_{2\mp}|^2\right) + c_{25}\overline{z_{1\pm}} z_{1\mp} z_{2\mp}\right]$$

If $c_{12} = -c_{21}$, $c_{14} = -c_{23}$ and $c_{15} = -c_{25}$,

$$H(z_{1+}, z_{2+}, z_{1-}, z_{2-}) = \frac{\omega_0}{2}\left(|z_{1+}|^2 - |z_{2+}|^2 + |z_{1-}|^2 - |z_{2-}|^2\right) - \varepsilon\sigma \mathrm{Re}\left[z_{1+}z_{2+} + z_{1-}z_{2-}\right]$$

$$-\frac{1}{4}c_{11}\left(|z_{1+}|^4 + |z_{1-}|^4\right) + \frac{1}{4}c_{22}\left(|z_{2+}|^4 + |z_{2-}|^4\right) - \frac{1}{2}c_{13}|z_{1+}|^2|z_{1-}|^2 + \frac{1}{2}c_{24}|z_{2+}|^2|z_{2-}|^2$$

$$-\frac{1}{2}c_{12}\left(|z_{1+}|^2|z_{2+}|^2 + |z_{1-}|^2|z_{2-}|^2\right) - \frac{1}{2}c_{14}\left(|z_{1+}|^2|z_{2-}|^2 + |z_{1-}|^2|z_{2+}|^2\right) - c_{15}\mathrm{Re}\left[z_{1+}z_{2+}\overline{z_{1-}z_{2-}}\right]$$

$$z_i = q_i + i p_i, \qquad \frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}$$

Why chaos? First integrals

Hamilton equations with 4 degrees of freedom

$$\frac{\mathrm{d}z_{1\pm}}{\mathrm{d}t} = -2i\frac{\partial H_2}{\partial \overline{z_{1\pm}}}, \quad \frac{\mathrm{d}z_{2\pm}}{\mathrm{d}t} = -2i\frac{\partial H_2}{\partial \overline{z_{2\pm}}}$$

3 first integrals

Energy of Kelvin waves

$$H_{02} = \frac{\omega_0}{2}\left[|z_{1+}|^2 + |z_{1-}|^2 - \left(|z_{2+}|^2 + |z_{2-}|^2\right)\right]$$

Axial flow flux of Kelvin waves

$$J_{02} = \frac{k_0}{2}\left[|z_{1+}|^2 - |z_{1-}|^2 - \left(|z_{2+}|^2 - |z_{2-}|^2\right)\right]$$

Hamiltonian

$$H(z_{1+}, z_{2+}, z_{1-}, z_{2-}) = \frac{\omega_0}{2}\left(|z_{1+}|^2 - |z_{2+}|^2 + |z_{1-}|^2 - |z_{2-}|^2\right) - \varepsilon\sigma \mathrm{Re}\left[z_{1+}z_{2+} + z_{1-}z_{2-}\right]$$
$$-\frac{1}{4}c_{11}\left(|z_{1+}|^4 + |z_{1-}|^4\right) + \frac{1}{4}c_{22}\left(|z_{2+}|^4 + |z_{2-}|^4\right) - \frac{1}{2}c_{13}|z_{1+}|^2|z_{1-}|^2 + \frac{1}{2}c_{24}|z_{2+}|^2|z_{2-}|^2$$
$$-\frac{1}{2}c_{12}\left(|z_{1+}|^2|z_{2+}|^2 + |z_{1-}|^2|z_{2-}|^2\right) - \frac{1}{2}c_{14}\left(|z_{1+}|^2|z_{2-}|^2 + |z_{1-}|^2|z_{2+}|^2\right) - c_{15}\mathrm{Re}\left[z_{1+}z_{2+}\overline{z_{1-}z_{2-}}\right]$$

Weakly nonlinear stability of an elliptically strained vortex tube: Eulerian treatment

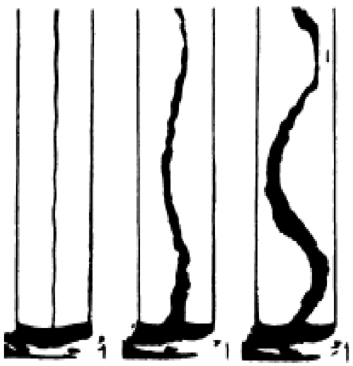
Sipp: *Phys. Fluids* **12** (2000) 1715
Waleffe: *PhD Thesis* (1989)

$$\mathbf{u} = \underbrace{\mathbf{u}_0 + \epsilon \mathbf{u}_1 + \cdots}_{\text{Steady 2D strained vortex}}$$

$$\underbrace{+ \alpha \mathbf{u}_{01} + \alpha^2 \mathbf{u}_{02} + \alpha^3 \mathbf{u}_{03} + \epsilon\alpha \mathbf{u}_{11} + \alpha^4 \mathbf{u}_{04} + \epsilon\alpha^2 \mathbf{u}_{12} + \cdots,}_{\text{Unsteady 3D perturbation}}$$

A combination of two helical waves

$$\mathbf{u}_{01}(r,\theta,z,t) = A e^{-i\theta} e^{ikz} \mathbf{u}_A(r) + B e^{+i\theta} e^{ikz} \mathbf{u}_B(r) + \text{c.c.},$$



A vortex tube in strain field

Amplitude equations

$$(\alpha^2 = \epsilon)$$

$$\frac{dA}{dt} = +i\epsilon a B - i\epsilon A(b|A|^2 + c|B|^2 + dC),$$

$$\frac{dB}{dt} = -i\epsilon a A + i\epsilon B(c|A|^2 + b|B|^2 + dC),$$

$$\frac{dC}{dt} = +i\epsilon(A\bar{B} - \bar{A}B),$$

undetermined const.
Influences linear term!

Solvability condition at $O(\alpha^2\epsilon)$

mean flow

$$\alpha^2 \mathbf{u}_C = \alpha^2 C \mathbf{t}(r)$$

Hamiltonian normal form for $SO(2) \times O(2) \rightarrow Z_2 \times O(2)$

Guckenheimer & Mahalov '92

Eulerian treatment Sipp: *Phys. Fluids* **12** (2000) 1715

Subspace

$$B = \bar{A}$$

$$\frac{dA}{d\tau} = +i\bar{A} + iA(D|A|^2 - D_0),$$

in which

$$D = D_{MF} - D_{NL}, \quad E_0 = |A_0|^2$$

$$D_0 = D_{MF}|A_0|^2.$$

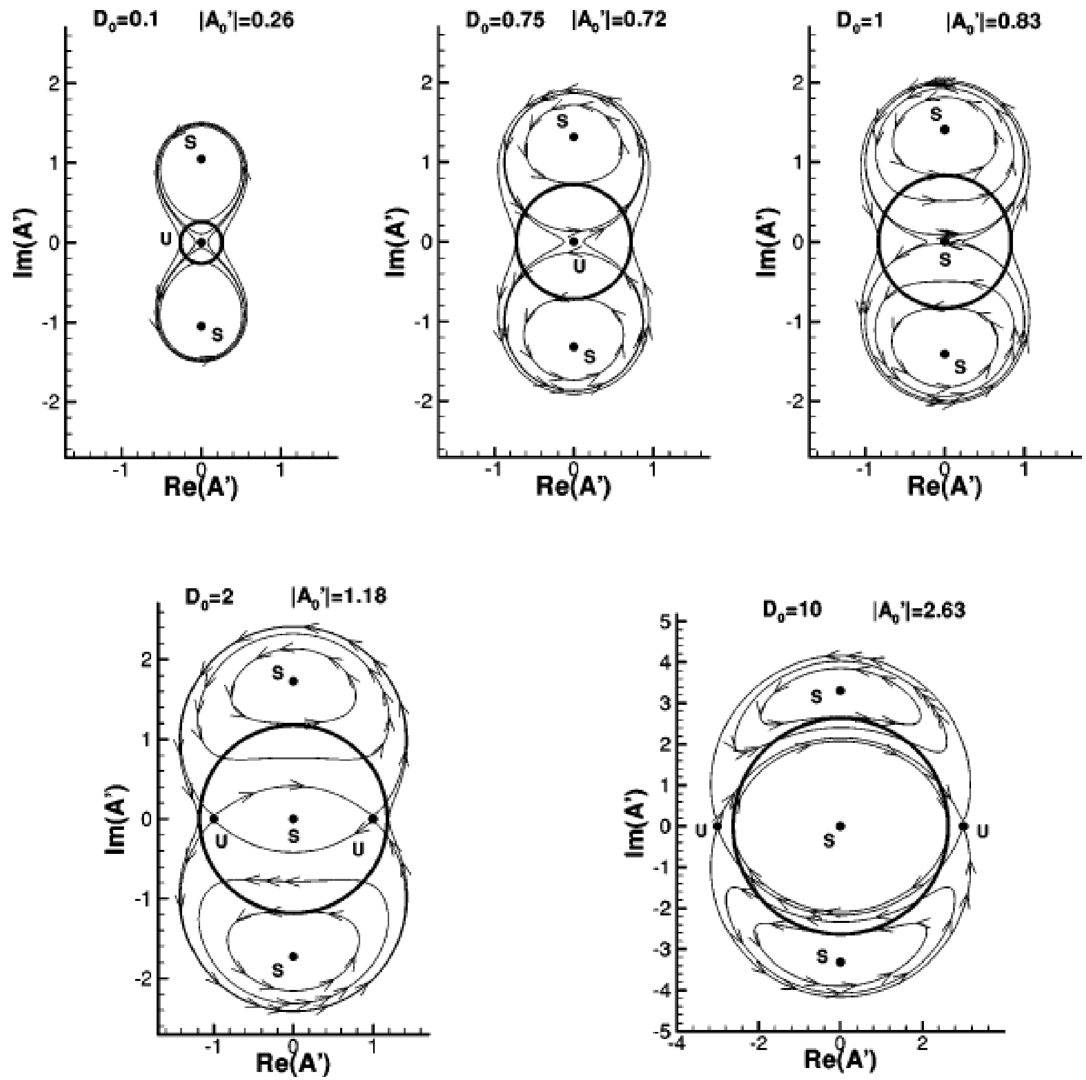


FIG. 6. Trajectories in the phase space projected on a plane $C''=cte$ in the cases $D_0=0.1$, 0.75, 1,2, 10. The circle in each figure represents the initial allowable conditions A_0'. Case $k=2.261$.

Hamiltonian normal form

Knobloch, Mahalov & Marsden
Physica D **73** (1994) 49

Energy of excited wave at $O(\alpha^2)$ $\quad |A|^2 + \dot{C}' = E_0.$

$E_0 = 0$ ⟶ $D_0 = 0$

Mie & Y. F. (2010)

Summary

Linear stability of an *strained vortex tube*, a straight vortex tube subject to a pure shear, to three-dimensional disturbances is reconsidered from the viewpoint of *Krein's theory of Hamiltonian spectra.*

1. **Lagrangian approach**: ***Energy*** of the Kelvin waves is calculated by restricting disturbances to ***kinematically accessible field***
   geometric formulation helps

   ⇩

2. ***Wave-induced mean flow*** is available as a byproduct

   Axial current: For the *Rankine vortex*, 2nd-order drift current includes not only azimuthal $\overline{u_\theta}$ but also *axial* component $\overline{u_z}$

energy $\xleftarrow{\times\omega_0}$ ***wave action*** $\xrightarrow{\times k_0}$ ***pseudomomentum***

3. **Weakly nonlinear amplitude equation:** Its coefficients all determined
   Canonical Hamilton equations
   Nonlinear saturation, though chaotic

⇨ Secondary instability (*three-wave interaction*)

How are many modes excited?

Secondary instability

↑

Three-wave interactions

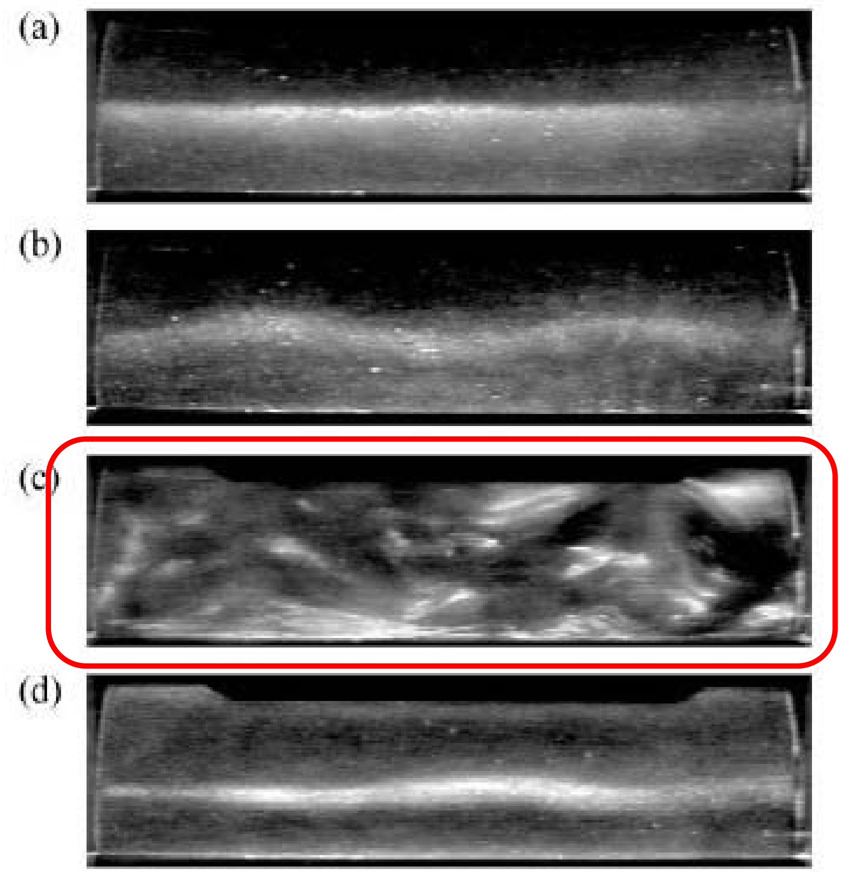


FIG. 4. Four successive images of the flow for $n = 2$, $\mathrm{Re} = 5000$, $H/R = 7.96$, and (a) $\Omega t = 294$, solid body rotation; (b) $\Omega t = 715$, appearance of mode $(-1, 1, 1)$; (c) $\Omega t = 943$, vortex breakup; (d) $\Omega t = 1113$, relaminarization.

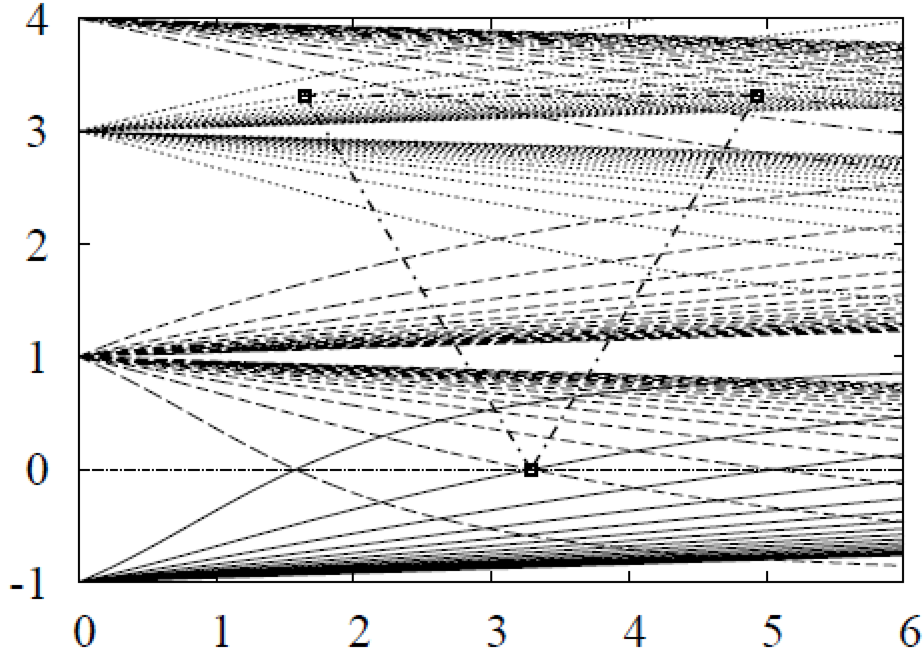
Three wave interactions

Energy

$(m, k_0, \omega_0) \approx (3, \beta/2, 3.32)$ **+**

Energy

$(4, 3\beta/2, 3.32)$ **-**



$(\pm 1, \beta, 0)$

$\beta \approx 3.286$

Energy 0

Amplitude equations for three-wave interaction

Kerswell: JFM '99

energy

$O(\alpha)$

$$u_{01} = A_+ \boldsymbol{u}_{A_+} \mathrm{e}^{i(\theta+k_0 z)} + A_- \boldsymbol{u}_{A_-} \mathrm{e}^{i(\theta-k_0 z)} \qquad 0$$

$$+ B_+ \boldsymbol{u}_{B_+} \mathrm{e}^{i(3\theta+k_0 z/2)} + B_- \boldsymbol{u}_{B_-} \mathrm{e}^{i(3\theta-k_0 z/2)} \qquad +$$

$$+ C_+ \boldsymbol{u}_{C_+} \mathrm{e}^{i(4\theta+3k_0 z/2)} + C_- \boldsymbol{u}_{C_-} \mathrm{e}^{i(4\theta-3k_0 z/2)} + c.c. \qquad -$$

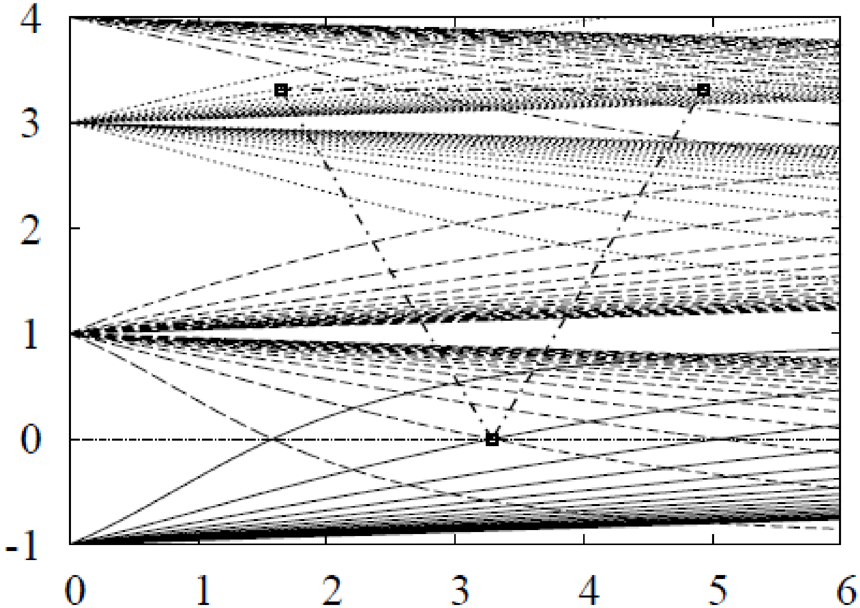
$O(\alpha^2)$

$$\frac{dA_\pm}{dt} = i\left[\varepsilon\left(p_1\overline{A_\mp} + p_{21}A_\pm\right) + \alpha q_1 \overline{B_\pm} C_\pm\right.$$

$$\frac{dB_\pm}{dt} = i\left[-\omega_0 B_\pm + \varepsilon p_{22} B_\pm + \alpha q_2 \overline{A_\pm} C_\pm\right.$$

$$\frac{dC_\pm}{dt} = i\left[-\omega_0 C_\pm + \varepsilon p_{23} C_\pm + \alpha q_3 A_\pm B_\pm\right]$$

$$(k = k_0 + \varepsilon k_1)$$



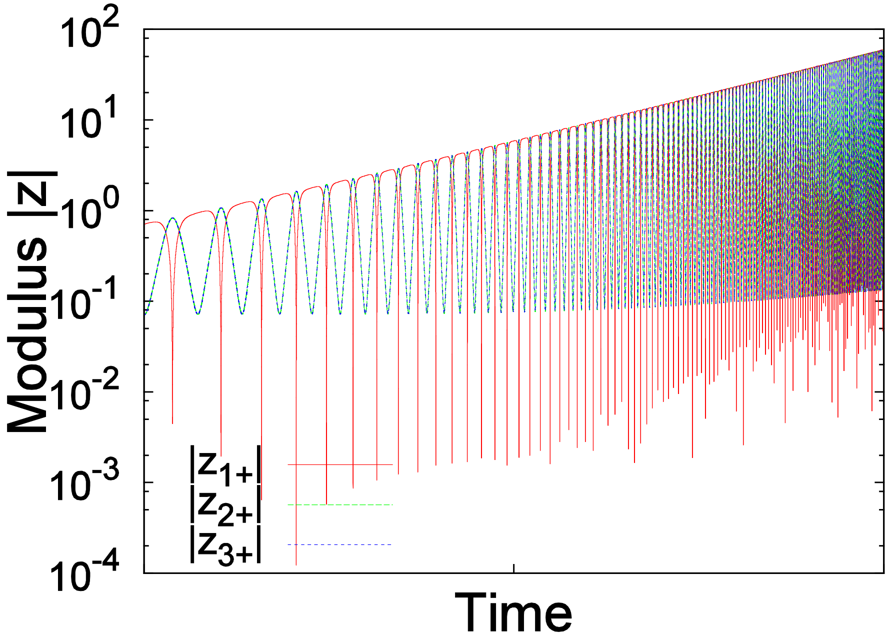
Fukumoto, Hattori & Fujimura '05

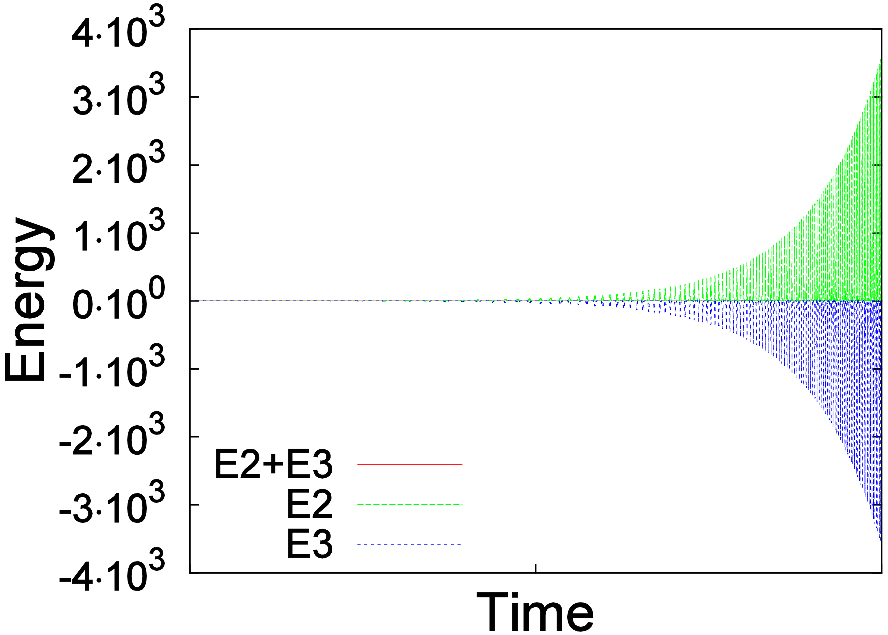
Amplitude evolutions for three-wave interaction

$$\frac{dz_{1\pm}}{dt} = -i\left[\varepsilon\left(p_1\overline{z_{1\mp}} + p_{21}z_{1\pm}\right) - \alpha\sigma z_{2\pm}z_{3\pm}\right]$$

$$\frac{dz_{2\pm}}{dt} = i\left[\varepsilon p_{22}z_{2\pm} + \alpha\sigma z_{1\pm}\overline{z_{3\pm}}\right]$$

$$\frac{dz_{3\pm}}{dt} = -i\left[\varepsilon p_{23}z_{3\pm} - \alpha\sigma z_{1\pm}\overline{z_{2\pm}}\right]$$





Hamilton equations for three-wave interaction

$$z_{1\pm}=\overline{A_\pm}/\sqrt{|q_1|},\quad z_{2\pm}=B_\pm e^{i\omega_0 t}/\sqrt{q_2},\quad z_{3\pm}=\overline{C_\pm}e^{-i\omega_0 t}/\sqrt{|q_3|},$$

$$\frac{dz_{1\pm}}{dt}=-i\left[\varepsilon\left(p_1\overline{z_{1\mp}}+p_{21}k_1z_{1\pm}\right)-\alpha\sigma z_{2\pm}z_{3\pm}\right]$$

$$\frac{dz_{2\pm}}{dt}=i\left[\varepsilon p_{22}k_1z_{2\pm}+\alpha\sigma z_{1\pm}\overline{z_{3\pm}}\right]$$

$$\frac{dz_{3\pm}}{dt}=-i\left[\varepsilon p_{23}k_1z_{3\pm}-\alpha\sigma z_{1\pm}\overline{z_{2\pm}}\right]$$

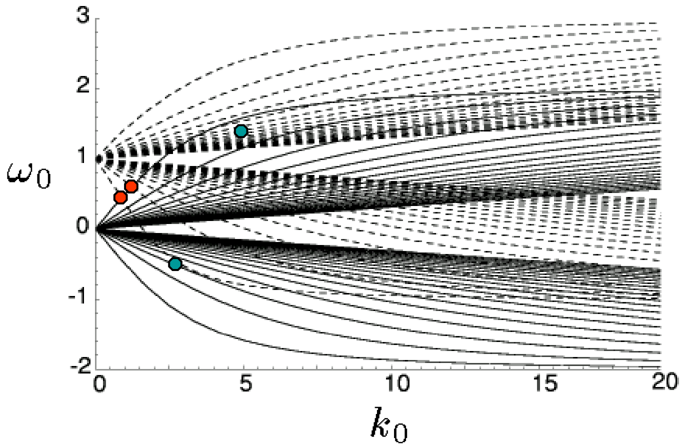
$$p_1=-0.554202,$$
$$p_{21}=-0.238347,$$
$$p_{22}=-0.180616,$$
$$p_{23}=0.127444,$$
$$\sigma=11.6724$$

Hamiltonian $H=H_+ + H_-$

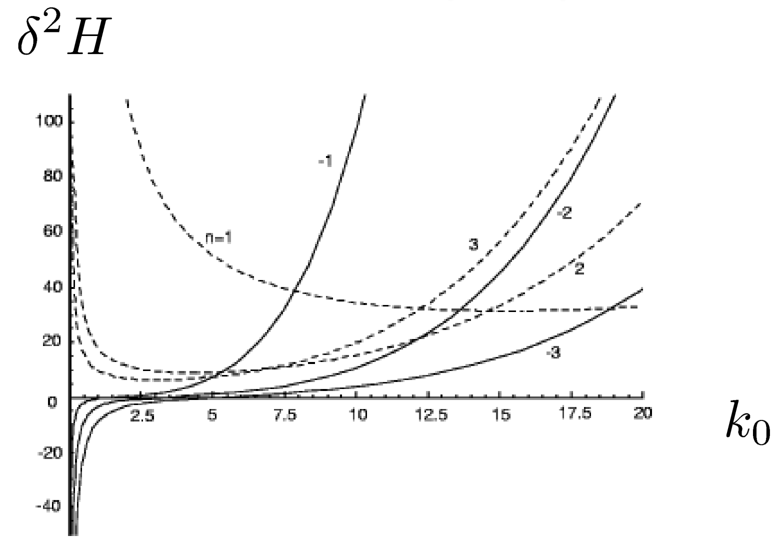
$$H_\pm=\alpha\sigma\mathrm{Re}\left(z_{1\pm}\overline{z_{2\pm}z_{3\pm}}\right)+\frac{1}{2}\varepsilon\left[p_1\mathrm{Re}\left(z_{1+}z_{1-}\right)+p_{21}|z_{1\pm}|^2-p_{22}|z_{2\pm}|^2+p_{23}|z_{3\pm}|^2\right]$$

First integrals $|z_{2+}|^2-|z_{3+}|^2,\qquad |z_{2-}|^2-|z_{3-}|^2$

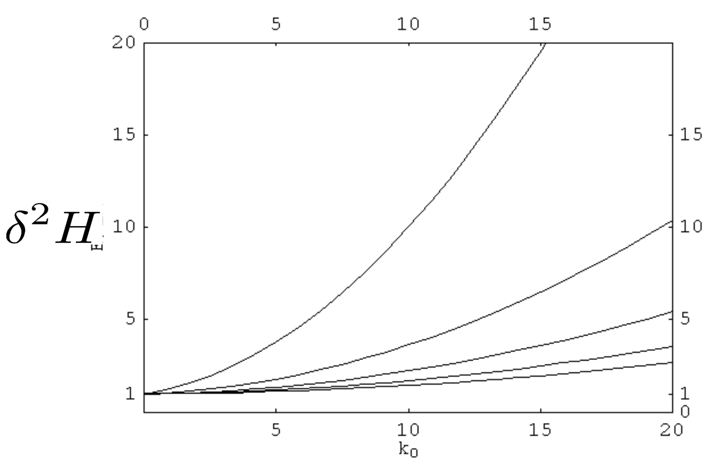
Energy of Kelvin waves



Helical wave *(m=1)*



Buldge wave *(m=0)*



$$E_0^{(m)} = 2\pi\omega_0 \left|C_0\right|^2 \frac{\omega_0 - m}{2} \int_0^1 |\hat{\boldsymbol{\xi}}|^2 \mathrm{d}r$$

The sign of **wave action** $\mu_0 = E_0/\omega_0$ is essential !