Chain conditions in dependent groups

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Contents

- Chain conditions for dependent groups.
- Connected components.
- A motivational question: Artin-Schreier closed fields in dependent theories.
- An example.
- Baldwin-Saxl type lemmas.
- Strongly dependent theories.
- Dp-minimal theories, theories of bounded dp-rank.
- κ-intersection.
- Strongly^2 dependent theories.

Chain conditions

What do I mean by a chain condition?

Definition

If G is a group, and for all $c \in C$, $\varphi(x,c)$ defines a subgroup, then $\{\varphi(\mathfrak{C},c) \mid c \in C\}$ is a family of *uniformly defined subgroups*.

Lemma

[Baldwin-Saxl] Let G be a dependent group. Given a family of uniformly defined subgroups, there is a number $n < \omega$ such that any finite intersection of groups from this family is an intersection of n of them.

Type definable groups

A lot of my motivation lies in type definable groups.

Definition

A *type definable group* for a theory T is a type — a collection $\Sigma(x)$ of formulas (maybe over parameters), and a formula $\nu(x,y,z)$, such that in the monster model $\mathfrak{C}$ of T, $\langle \Sigma(\mathfrak{C}), \nu \rangle$ is a group with ν defining the group operation (without loss of generality, $T \models \forall xy \exists^{\leq 1} z \, (\nu(x,y,z))$).

Paradise

Under the assumption of stability, everything is better. Several books have been written on stable groups and much is known (generic types, connected component, etc.).
In stable theories, we have the following, which makes life much easier:

Fact

If T is stable, then a type definable groups is an intersection of definable groups.

Paradise

Example

Let $T = Th(\mathbb{R}, +, \cdot, 0, 1)$. Then the group of infinitesimal elements is type definable, but it is not an intersection of groups.

Connected components

Definition

Let G be a type definable group.

1. A type definable subgroup H is said to have bounded index if $[G : H] < |\mathfrak{C}|$ (equivalently, $[G : H] \leq 2^{|T|+\mathrm{dom}(H)}$).
2. For a set A, G_A^{00} is the minimal A-type definable subgroup of G of bounded index.
3. We say that G^{00} exists if $G_A^{00} = G_\emptyset^{00}$ for all A.

Theorem

[Shelah] If G is a type definable group in a dependent theory, then G^{00} exists.

Motivating question

Theorem

[Artin-Schreier] Let k be a field of characteristic $p > 0$. Let $\varrho(X)$ be the polynomial $X^p - X$.

1. *Given $a \in k$, either the polynomial $\varrho - a$ has a root in k, in which case all its root are in k, or it is irreducible. In the latter case, if α is a root then $k(\alpha)$ is cyclic of degree p over k.*
2. *Conversely, let K be a cyclic extension of k of degree p. Then there exists $\alpha \in K$ such that $K = k(\alpha)$ and for some $a \in k$, $\varrho(\alpha) = a$.*

Such extensions are called Artin-Schreier extensions.

Theorem

[K., Scanlon, Wagner] Let K be an infinite dependent field of characteristic $p > 0$. Then K is Artin-Schreier closed — ϱ is onto.

Motivating question

The following question is still open:

Question

What about the type definable case? What if K is an infinite type definable field in a dependent theory, is it still AS-closed?

In the simple case, we have:

Theorem

[Wagner] Let K be a type definable field in a simple theory. Then K has boundedly many AS extensions.

Motivating question

Theorem

For an infinite type definable field K in a dependent theory there are either unboundedly many Artin-Schreier extensions, or none.

Proof.

It is easy to see that $(K,+)^{00} = K$, and it is known that the number of AS extension is finite iff the index $[K : \varrho(K)]$ is finite, and otherwise it is in bijection with $[K : \varrho(K)]$. If this index is bounded, then $\varrho(K) \supseteq K^{00} = K$ and so there are no AS extensions. □

Corollary

If T is stable, then type definable fields are AS closed.

Motivating question

Fact

In the proof of the theorem, it is enough to find a number n, and $n+1$ algebraically independent elements, $\langle a_i \mid i \leq n \rangle$ in $k := K^{p^\infty}$, such that

$$\bigcap_{i<n} a_i \varrho(K) = \bigcap_{i \leq n} a_i \varrho(K).$$

So the Baldwin-Saxl applies in the case where the field K is definable.
If K is type definable, we may want something similar.

Motivating Question

A conjecture of Frank Wagner is the main motivation question

Question

Call the following property "Property A":

- Suppose G is a type definable group. Suppose $p(x, y)$ is a type and $\langle a_i \mid i < \omega \rangle$ is an indiscernible sequence such that $G_i = p(x, a_i) \leq G$. Then there is some n, such that for all finite sets, $v \subseteq \omega$, the intersection $\bigcap_{i \in v} G_i$ is equal to a sub-intersection of size n.

Suppose T is dependent. Does it have Property A?

Fact

If Property A is true for a theory T, then type definable fields are Artin-Schreier closed.

Counterexample

Let $S = \{u \subseteq \omega \mid |u| < \omega\}$, and $V = \{f : S \to 2 \mid |\mathrm{supp}(f)| < \infty\}$ where $\mathrm{supp}(f) = \{x \in S \mid f(x) \neq 0\}$. This has a natural group structure as a vector space over $\mathbb{F}_2$.

For $n, m < \omega$, define the following groups:

- $G_n = \{f \in V \mid u \in \mathrm{supp}(f) \Rightarrow |u| = n\}$
- $G_\omega = \prod_n G_n$
- $G_{n,m} = \{f \in V \mid u \in \mathrm{supp}(f) \Rightarrow |u| = n \,\&\, m \in u\}$ (so $G_{0,m} = 0$)

Counterexample

Let M be the following $\{P,Q\} \cup \{R_n \mid n < \omega\} \cup L_{AG}$-structure:

- $P^M = G_\omega$ (with the group structure),
- $Q^M = \omega$ and
- $R_n = \{(\eta, m) \mid \eta(n) \in G_{n,m}\}$.

Let $T = Th(M)$.

Counterexample

Let $p(x, y)$ be the type $\bigcup \{R_n(x, y) \mid n < \omega\}$. Since $H_{n,m}$ is a subgroup of G_ω, $p(M, m)$ is a subgroup of G_ω.

Claim

Let $N \models T$ be $\aleph_1$-saturated. For any m, and any distinct $\alpha_0, \ldots, \alpha_m \in P^N$, $\bigcap_{i \leq m} p(N, \alpha_i)$ is different than any sub-intersection of size m.

Counterexample

Proof.

We show that $\bigcap_{i\leq m} p(N,\alpha_i) \subsetneq \bigcap_{i<m} p(N,\alpha_i)$. More specifically, we show that

$$\left(\bigcap_{i<m} p(N,\alpha_i)\right) \setminus R_m(N,\alpha_i) \neq \emptyset.$$

By saturation, it is enough to show that this is the case in M. Note that if $\eta \in \bigcap_{i\leq m} R_m(M,\alpha_i)$, then $\eta(m) \in G_{n,\alpha_i}$ for all $i \leq m$. So for all $i \leq m$, $u \in \operatorname{supp}(\eta(n)) \Rightarrow |u| = m \,\&\, \alpha_i \in u$. This implies that $\operatorname{supp}(\eta(m)) = \emptyset$, i.e. $\eta(m) = 0$. □

Counterexample

Definition

Let κ be a cardinal. A model M is called κ-resplendent if whenever

- $M \prec N$; N' is an expansion of N by less than κ many symbols; $\bar{c}$ is a tuple of elements from M and $\lg(\bar{c}) < \kappa$

There exists an expansion M' of M to the language of N' such that $\langle M', \bar{c}\rangle \equiv \langle N', \bar{c}\rangle$.

Theorem

[Sh363] Assume κ is regular and $\lambda = \lambda^{\kappa} + 2^{|T|}$. Then, if T is unstable then T has $> \lambda$ pairwise nonisomorphic κ-resplendent models of size λ. On the other hand, if T is stable and $\kappa \geq \kappa(T) + \aleph_1$ then every κ-resplendent model is saturated.

(Poizat, 1986, about the notion of resplendence) "Ce n'est rien d'autre qu'un gadget ... mais qui n'aura jamais de signification pour un mathematicien normal."

Counterexample

Claim

T is stable.

Proof.

The strategy is to prove that T has a unique model in size λ which is κ-resplendent where $\kappa = \aleph_0$, $\lambda = 2^{\aleph_0}$. The idea is that in these models, the size of every definable set is λ. □

Baldwin-Saxl type lemmas

Lemma

(T dependent) If $\{G_i \mid i < |T|^+\}$ is a family of type definable subgroups (maybe with parameters), then there is some $i_0 < |T|^+$ such that $\bigcap G_i = \bigcap_{i \neq i_0} G_i$.

Corollary

Suppose G is a type definable group in a dependent theory T. Given a family of uniformly type definable subgroups, defined by $p(x, y)$, and an indiscernible sequence $\langle a_i \mid i \in \mathbb{Z} \rangle$, $\bigcap_{i \neq 0} p(\mathfrak{C}, a_i) = \bigcap_{i \in \mathbb{Z}} p(\mathfrak{C}, a_i)$.

Strongly dependent theories

Definition

A theory T is *strongly dependent* if there is no sequence of formulas $\langle \varphi_i(x, y_i) \,|\, i < \omega \rangle$ and an array of sequences $\left\langle b_i^j \,|\, i, j < \omega \right\rangle$ such that for all functions $\eta : \omega \to \omega$, the following set is consistent $\left\{ \varphi_i\left(x, b_i^j\right)^{\eta(i)=j} \,|\, i, j < \omega \right\}$.

Lemma

Suppose G is a type definable group in a strongly dependent theory T. Given a family of type definable subgroups $\{p(x, a_i) \,|\, i < \omega\}$ such that $\langle a_i \,|\, i < \omega \rangle$ is an indiscernible sequence, there is some $i < \omega$ such that $\bigcap_{j \neq i} p(\mathfrak{C}, a_j) = \bigcap_{j < \omega} p(\mathfrak{C}, a_j)$.

Strongly dependent theories

Claim

Assume T is strongly dependent (strongness is enough), G a type definable group and $G_i \leq G$ are type definable normal subgroups for $i < \omega$. Then there is some i_0 such that $\left[\bigcap_{i \neq i_0} G_i : \bigcap_{i<\omega} G_i\right] < \infty$.

Corollary

If G is an abelian definable group in a strongly dependent theory and $S \subseteq \omega$ is an infinite set of pairwise co-prime numbers, then for almost all $n \in S$, $[G : G^n] < \infty$.
In particular, if K is a definable field in a strongly dependent theory, then for almost all primes p, $\left[K^\times : (K^\times)^p\right] < \infty$.
So if K is strongly dependent and stable, then for almost all p, $K^p = K$.

Strongly dependent theories

Fact

The theory T constructed in the counterexample above is not strongly dependent.

Question

Does Property A hold for strongly dependent theories?

Theories with bounded dp-rank

Definition

A theory is said to have bounded dp-rank if there is some $n < \omega$ such that there is no sequence of formulas $\langle \varphi_i(x, y_i) \,|\, i < n \rangle$ and an array of sequences $\left\langle b_i^j \,|\, i < n, j < \omega \right\rangle$ such that for all functions $\eta : \omega \to \omega$, the following set is consistent $\left\{ \varphi_i\left(x, b_i^j\right)^{\eta(i)=j} \,|\, i < n, j < \omega \right\}$. It is dp-minimal if $n = 2$.

Examples

All o-minimal theories, the p-adics, algebraically closed valued fields, and much more.

Theories with bounded dp-rank

Definition

The alternation rank of a formula $\varphi(x,y)$ is the maximal $n<\omega$ such that

$$\exists \langle a_i \,| i<\omega\rangle \text{ indiscernible, } \exists b : \varphi(b,a_i) \leftrightarrow \neg\varphi(b,a_{i+1}) \text{ for } i<n-1$$

Theorem

[K., Usvyatsov, Onshuus] If T has bounded dp-rank, then for every n, there is some $k(n)<\omega$ such that the alteration rank of $\varphi(x,y)$ is $\leq k(\lg(x))$.

Theories with bounded dp-rank

Corollary

If T has bounded dp-rank, then if $p(x,y)$ is a (finitary) type then any number $n<\omega$ such that

$$\exists \langle a_i \,|\, i<\omega\rangle \text{ indiscernible}, \exists b : p(a_i,b) \leftrightarrow \neg p(a_{i+1},b) \text{ for } i<n-1$$

is bounded by $k(\lg(x))$.

This is not true for general strongly dependent theories.

Corollary

If T has bounded dp-rank, then Property A holds in T.

Theories with bounded dp-rank

Proof.

Property A says: Suppose G is a type definable group. Suppose $p(x,y)$ is a type and $\langle a_i \mid i < \omega \rangle$ is an indiscernible sequence such that $G_i = p(x, a_i) \leq G$. Then there is some n, such that for all finite sets, $v \subseteq \omega$, the intersection $\bigcap_{i \in v} G_i$ is equal to a sub-intersection of size n.
Let n be $k(\lg(x))$. Towards a contradiction, for $i < n$, let $b_i \in \bigcap_{j \neq i} G_j$, and let $c = \prod_{2|i} b_i$. Then $p(c, a_i)$ iff i is odd. □

Corollary

If T is stable and has bounded dp-rank, then $\bigcap_{i<\omega} p(\mathfrak{C}, a_i) = \bigcap_{i<n} p(\mathfrak{C}, a_i)$.

κ-intersection

Definition

For a cardinal κ and a family $\mathfrak{F}$ of subgroups of a group G, the κ intersection $\bigcap_{\kappa} \mathfrak{F}$ is $\{g \in G \,|\, |\{F \in \mathfrak{F} \,|\, g \notin F\}| < \kappa\}$.

Lemma

[With Frank Wagner] Let G be a type definable group in a dependent theory. Suppose

- *$\mathfrak{F}$ is a family of uniformly type definable subgroups defined by $p(x, y)$.*

Then for any regular cardinal $\kappa > |T|$, and any subfamily $\mathfrak{G} \subseteq \mathfrak{F}$, there is some $\mathfrak{G}' \subseteq \mathfrak{G}$ such that

- *$|\mathfrak{G}'| < \kappa$ and $\bigcap \mathfrak{G}$ is $\bigcap \mathfrak{G}' \cap \bigcap_{\kappa} \mathfrak{G}$.*

Strongly^2 dependent theories

Definition

A theory T is said to be not *strongly^2 dependent* if there exists a sequence of formulas $\langle \varphi_i(x, y_i, z_i) \mid i < \omega \rangle$, an array $\langle a_{i,j} \mid i, j < \omega \rangle$ and $b_k \in \{a_{i,j} \mid i < k, j < \omega\}$ such that

- The array $\langle a_{i,j} \mid i, j < \omega \rangle$ is an indiscernible array (over $\emptyset$).
- For all functions $\eta : \omega \to \omega$, the following set is consistent $\left\{ \varphi_i(x, a_{i,j}, b_i)^{\eta(i)=j} \mid i, j < \omega \right\}$

So T is *strongly^2 dependent* when this configuration does not exist.

Strongly^2 dependent theories

Lemma

Suppose T is strongly^2 dependent, then it is impossible to have a sequence of type definable groups $\langle G_i \mid i < \omega \rangle$ such that $G_{i+1} \leq G_i$ and $[G_i : G_{i+1}] = \infty$.

Corollary

Assume T is strongly^2 dependent. If G is a type definable group and h is a definable homomorphism $h : G \to G$ with finite kernel then $[G : h(G)] < \infty$.
If K is a strongly^2 dependent field, then for all $n < \omega$, $\left[K^\times : (K^\times)^n\right] < \infty$.
If K is a strongly^2 stable field, then K is algebraically closed.

Proof.

Consider the sequence of groups $\left\langle h^{(i)}(G) \mid i < \omega \right\rangle$ (i.e. G, $h(G)$, $h(h(G))$, etc.). □

Strongly^2 dependent theories

Corollary

Let G be type definable group in a strongly^2 dependent theory T.

1. *Given a family of uniformly type definable subgroups $\{p(x, a_i) \mid i < \omega\}$ such that $\langle a_i \mid i < \omega \rangle$ is an indiscernible sequence, there is some $n < \omega$ such that $\bigcap_{j<\omega} p(\mathfrak{C}, a_j) = \bigcap_{j<n} p(\mathfrak{C}, a_j)$.*
   In particular, T has Property A.
2. *Given a family of uniformly definable subgroups $\{\varphi(x, c) \mid c \in C\}$, the intersection*

$$\bigcap_{c \in C} \varphi(\mathfrak{C}, c)$$

   is already a finite one. (As in the stable case)

Proof.

(1) Let $G_i = p(\mathfrak{C}, a_i)$, and let $H_i = \bigcap_{j<i} G_i$. For some $i_0 < \omega$, $[H_{i_0} : H_{i_0+1}] < \infty$. For $r \geq i_0$, let $H_{i_0,r} = \bigcap_{j<i_0} G_j \cap G_r$ (so $H_{i_0+1} = H_{i_0,i_0}$). By indiscerniblity, $[H_{i_0} : H_{i_0,r}] < \infty$.
This means (by definition of $H_{i_0}^{00}$) that $H_{i_0}^{00} \leq H_{i_0,r}$ for all $r > i_0$.
However, if $H_{i_0,i_0} \neq H_{i_0,r}$ for some $i_0 < r < \omega$, then by indiscerniblity $H_{i_0,r} \neq H_{i_0,r'}$ for all $i_0 \leq r < r'$, and by compactness and indiscerniblity we may increase the length ω of the sequence to any cardinality κ, so that the size of $H_{i_0}/H_{i_0}^{00}$ is unbounded — contradiction. This means that $H_{i_0+1} \subseteq G_r$ for all $r > i_0$, and so $\bigcap_{i<\omega} G_i = \bigcap_{i<i_0+1} G_i$. □

Strongly^2 dependent theories

Example

Suppose $\langle G, +, < \rangle$ is an ordered abelian group. Then its theory $Th(G, +, 0, <)$ is not strongly^2 dependent.
In particular, $Th(\mathbb{R}, +, \cdot, 0, 1)$ and $Th(\mathbb{Q}_p, +, \cdot, 0, 1)$ are strongly dependent but not strongly^2 dependent.

Example

Let $L = L_{\text{rings}} \cup \{P, <\}$ where L_{rings} is the language of rings $\{+, \cdot, 0, 1\}$, P is a unary predicate and $<$ is a binary relation symbol.
Let K be an algebraically closed field , and let $P \subseteq K$ be a countable set of algebraically independent elements, enumerated as $\{a_i \mid i \in \mathbb{Q}\}$. Let $M = \langle K, P, < \rangle$ where $a <^M b$ iff $a, b \in P$ and $a = a_i$, $b = a_j$ where $i < j$.
Then $Th(M)$ is strongly^2 dependent.

Questions

Question

Are strongly^2 dependent fields algebraically closed?

Question

Are strongly stable fields algebraically closed?

Question

Are all strongly^2 dependent groups stable?

The End

Thank You!