On model-theoretic connected components in some group extensions 2

Jakub Gismatullin (joint work with Krzysztof Krupiński)

School of Mathematics, University of Leeds
Instytut Matematyczny
Uniwersytetu Wrocławskiego

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- $(G, \cdot, \ldots)$ – a group with some first order structure
- G^* – saturated extension of $(G, \cdot, \ldots)$ (model monstrum, $\overline{\kappa}$-saturated, $\overline{\kappa}$-strongly homogeneus)
- $B \subset G^*$ some small set of parameters ($|B| < \overline{\kappa}$)

Definition

- ${G^*}^0_B = \bigcap\{H < G^* : H \text{ is } B\text{-def. and } [G^* : H] < \omega\}$
- ${G^*}^{00}_B = \bigcap\{H < G^* : H \text{ is } B\text{-type def. and } [G^* : H] < \overline{\kappa}\}$
- ${G^*}^{000}_B = \bigcap\{H < G^* : H \text{ is } \mathrm{Aut}(G^*/B)\text{-inv. and } [G^* : H] < \overline{\kappa}\}$

We say, that ${G^*}^{000}$ exists, if for every small $B \subset G^*$,

$$ {G^*}^{000}_B = {G^*}^{000}_\emptyset . $$

E.g. when G has NIP, ${G^*}^{000}$, ${G^*}^{00}$ and ${G^*}^0$ exist.

The main theorem

Let G be a group acting by automorphisms on an abelian group A, where G, A and the action of G on A are $\emptyset$-definable in a structure $\mathcal{G}$. Suppose

$$h\colon G \times G \to A$$

is a 2-cocycle which is B-definable in $\mathcal{G}$ and with finite image $\mathrm{Im}(h) \subset B$ (for some finite parameter set $B \subset \mathcal{G}$). Denote $A_0 = \langle \mathrm{Im}(h) \rangle$.

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Theorem

Assume that:

(i) *the induced 2-cocycle* $\overline{h}\colon {G^*}^{00}_B \times {G^*}^{00}_B \to A_0/\left({A^*}^0 \cap A_0\right)$ *is non-splitting,*

(ii) $A_0/\left({A^*}^0 \cap A_0\right)$ *is torsion free (and so* $\cong \mathbb{Z}^n$ *for some natural* n*).*

Then $\widetilde{G^*}^{000}_B \neq \widetilde{G^*}^{00}_B$.

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Then $\widetilde{G^*}^{000}_B \neq \widetilde{G^*}^{00}_B$.
Suppose furthermore that ${G^*}^{000}_B = G^*$*, and for every proper, type-definable over* B *in* $\mathcal{G}^*$ *and invariant under the action of* G^* *subgroup* H *of* A^* *with bounded index, the induced 2-cocycle* $\overline{h}\colon G^* \times G^* \to A_0/(H \cap A_0)$ *is non-splitting. Then* $\widetilde{G^*}^{00}_B = \widetilde{G^*}$.

Corollary

Assume that:

(1) *The 2-cocycle* $h\colon G \times G \to A_0$ *is non-splitting (via a function taking values in* A_0*).*

(2) $A^{*0} \cap A_0$ *is trivial and* A_0 *is torsion free (and so* $A_0 \cong \mathbb{Z}^n$ *for some* n*).*

(3) ${G^*}_B^{00} = G^*$.

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Under some additional assumptions, we also get $\widetilde{G^*}_B^{00} = \widetilde{G^*}$.

2-cocycles $h\colon G \times G \to \mathbb{Z}$ with finite image

Notation: for $c, d \in \mathbb{R}$ define $c(d) = \begin{cases} c & : c \neq 0 \\ d & : c = 0 \end{cases}$.

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Fact (Asai, '70)

The topological universal cover $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ *of* $\mathrm{SL}_2(\mathbb{R})$ *is defined by means of the following 2-cocycle* $h\colon \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \to \{-1, 0, 1\} \subset \mathbb{Z}$.

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Let $A_1 \cdot A_2 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$. *Then*

$$h(A_1, A_2) = \begin{cases} 1 & : c_1(d_1) > 0 \wedge c_2(d_2) > 0 \wedge c_3(d_3) < 0 \\ -1 & : c_1(d_1) < 0 \wedge c_2(d_2) < 0 \wedge c_3(d_3) > 0 \\ 0 & : \textit{otherwise} \end{cases}.$$

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Clearly, h is definable in $(\mathbb{R}, +, \cdot, 0, 1, <)$.

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Fact (Hrushovski-Peterzil-Pillay, '11)

Let G *be a definable Lie group, i.e. a definably connected group definable in an o-minimal expansion of RCF. The 2-cocycle* $h\colon G \times G \to \pi_1(G)$ *corresponding to the topological universal cover* $\widehat{G}$ *of* G *has finite image.*

General theory of central extensions

Definition (Classical)

A central extension $\ker(\pi) \hookrightarrow \widetilde{G} \overset{\pi}{\twoheadrightarrow} G$ is called *universal* if for any central extension $\pi'\colon G' \to G$ of G by A, there exists a unique homomorphism $f\colon \widetilde{G} \to G$ such that $\pi' \circ f = \pi$, that is the following diagram commutes

$$\begin{array}{ccccc} \ker(\pi) & \hookrightarrow & \widetilde{G} & \overset{\pi}{\twoheadrightarrow} & G\,. \\ \Big\downarrow {\scriptstyle f|_{\ker(\pi)}} & & \Big\downarrow {\scriptstyle f} & \nearrow {\scriptstyle \pi'} & \\ A & \hookrightarrow & G' & & \end{array}$$

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Each perfect group G possesses a universal central extension, which is unique up to isomorphism over G.

Let k be an arbitrary infinite field.
$\mathrm{SL}_2(k)$ has a universal central extension

$$\ker(\pi) = \mathrm{K}_2^{\mathrm{sym}}(k) \hookrightarrow \mathrm{St}_2(k) \xrightarrow{\pi} \mathrm{SL}_2(k).$$

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Theorem (Moore '68, Matsumoto '69)

The group $\mathrm{K}_2^{\mathrm{sym}}(k)$ *can be presented abstractly as*

$$\langle c(x, y) \mid (S1), (S2), (S3)\rangle_{x,y\in k^\times} ,$$

where $c(x, y)$ *for* $x, y \in k^\times$ *are generators satisfying the following relations*

(S1) $c(x, y)\, c(xy, z) = c(x, yz)\, c(y, z)$,

(S2) $c(1, 1) = 1,\ c(x, y) = c(x^{-1}, y^{-1})$,

(S3) $c(x, y) = c(x, (1 - x)y)$ *for* $x \neq 1$.

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Let A be an abelian group. Then every homomorphism $\mathrm{K}_2^{\mathrm{sym}}(k) \to A$ corresponds to a *symplectic Steinberg symbol*, that is a mapping $c' \colon k^\times \times k^\times \to A$ satisfying (S1), (S2) and (S3).

Corollary

Let $H\colon \mathrm{SL}_2(k)\times \mathrm{SL}_2(k)\to \mathrm{K}_2^{\mathrm{sym}}(k)$ *be a 2-cocycle defining the universal central extension* $\mathrm{K}_2^{\mathrm{sym}}(k) \hookrightarrow \mathrm{St}_2(k) \xrightarrow{\pi} \mathrm{SL}_2(k)$, *for a suitable section* $s\colon \mathrm{SL}_2(k)\to \mathrm{St}_2(k)$.

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The formula $H_{c'}(a,a') = f_{c'}(H(a,a'))$ *defines a 2-cocycle* $H_{c'}\colon \mathrm{SL}_2(k)\times\mathrm{SL}_2(k)\to A$ *and an extension* $\widetilde{\mathrm{SL}_2(k)}$ *of* $\mathrm{SL}_2(k)$ *by* A*:*

$$\begin{array}{ccccc} \mathrm{K}_2^{\mathrm{sym}}(k) & \hookrightarrow & \mathrm{St}_2(k) & \xrightarrow{\pi} & \mathrm{SL}_2(k) \\ \downarrow{\scriptstyle f_{c'}} & & \downarrow & \nearrow{\scriptstyle \pi'} & \\ A & \hookrightarrow & \widetilde{\mathrm{SL}_2(k)} & & \end{array}$$

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Then there exists a unique homomorphism $f_{c'}\colon \mathrm{K}_2^{\mathrm{sym}}(k) \to A$ *satisfying* $f_{c'}(c(x,y)) = c'(x,y)$ *for all* $x, y \in k^\times$.
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Fact (Important for finite image)

Every value of the 2-cocycle H *from the corollary is a linear combination of two Steinberg symbols. For example, if* $d_1c_2^2 + c_1a_2c_2 \neq 2$,

$$H\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) = c\left(-\frac{c_2}{c_1}, \frac{c_1}{d_1c_2^2 + c_1a_2c_2}\right) - c\left(-\frac{c_2}{c_1}, \frac{1}{c_2}\right).$$

Non-splittingness

Theorem

Suppose that $c'\colon k^\times \times k^\times \to \mathbb{Z}$ *is a symplectic Steinberg symbol such that* $c'(-1,-1) = 1$, $\mathrm{char}(k) = 0$ *and*

$$\mathrm{SL}_2(\mathbb{Q}) < G < \mathrm{SL}_2(k).$$

Then $H_{c'}$ *restricted to* G *is a non-splitting 2-cocycle (actually a stronger result about* $H_{c'}$ *is true).*

Examples

Example (1)

Suppose $(k, +, \cdot, <)$ is a an ordered field. The following mapping $c' : k^{\times} \times k^{\times} \to \mathbb{Z}$ is a symplectic Steinberg symbol

$$c'(x, y) = \begin{cases} 1 & \text{if } x < 0 \text{ and } y < 0 \\ 0 & \text{otherwise} \end{cases} .$$

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$$\widetilde{G^*}_B^{000} \neq \widetilde{G^*}_B^{00} = \widetilde{G^*}.$$

Moreover, the quotient $\widetilde{G^*}_B^{00} / \widetilde{G^*}_B^{000}$ is abelian. In fact, $\widetilde{G^*}_B^{000} = \left(\mathbb{Z}^{*0} + \mathbb{Z}\right) \times G^*$, and $\widetilde{G^*}_B^{00} / \widetilde{G^*}_B^{000}$ is isomorphic to $\widehat{\mathbb{Z}}/\mathbb{Z}$, where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$.

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Example (1) generalizes to any group G such that $\mathrm{SL}_2(\mathbb{Q})<G<\mathrm{SL}_2(k)$.

More examples

There exists an extension $\widetilde{G}$ of $\mathrm{SL}_2(k)$ by $\mathrm{SO}_2(k)$ which is definable in $\mathcal{G} := (k, +, \cdot, <)$ and such that $\widetilde{G^*}^{00}_B \neq \widetilde{G^*}^{000}_B$.

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Example

Let $g \in \mathrm{SO}_2(k)$ be of infinite order and $B := \{-g, 0, g\}$. Consider the following 2-cocycle $H' \colon \mathrm{SL}_2(k) \times \mathrm{SL}_2(k) \to \mathrm{SO}_2(k)$,

$$H'(x, y) = H_{c'}(x, y) \cdot g.$$

Then $\widetilde{G}$ is definable in $(k, +, \cdot, <)$ and $\widetilde{G^*}_B^{00} \neq \widetilde{G^*}_B^{000}$.

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$$\mathcal{H}^* \succ \mathcal{H}$$

be a monster model. Assume additionally that $\mathrm{Hom}(\ker(f^*), \mathbb{Z})$ is trivial.

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is an extension of G by $\ker(f)$.
Let $\mathcal{H}$ be any expansion of $\mathcal{G}$ in which H and f are $\emptyset$-definable (e.g. $\mathcal{H}$ is the expansion of $\mathcal{G}$ by the new sort H together with the function f), and let

$$\mathcal{H}^* \succ \mathcal{H}$$

be a monster model. Assume additionally that $\mathrm{Hom}(\ker(f^*), \mathbb{Z})$ is trivial.
Put

$$h' := H_{c'} \circ (f, f) \colon H \times H \to \mathbb{Z}$$

a 2-cocycle definable in $\mathcal{H}$ over B. Let $\widetilde{H}$ be the extension of H by $\mathbb{Z}$ corresponding to h'. Then

$$h'_{|{H^*}_B^{00} \times {H^*}_B^{00}} \colon {H^*}_B^{00} \times {H^*}_B^{00} \to \mathbb{Z}$$

is non-splitting, and so $\widetilde{H^*}_B^{000} \neq \widetilde{H^*}_B^{00}$.

- If k is not formally real and $\mathrm{char}(k) \neq 2$, then one can prove that every symplectic Steinberg symbol $c' \colon k^\times \times k^\times \to \mathbb{Z}$ with finite image is trivial.

- If k is not formally real and $\mathrm{char}(k) \neq 2$, then one can prove that every symplectic Steinberg symbol $c' \colon k^\times \times k^\times \to \mathbb{Z}$ with finite image is trivial.
- We have almost generalized our result from $\mathrm{SL}_2(k)$ to all symplectic groups $\mathrm{Sp}_n(k)$, where k is an arbitrary ordered field.

- *G-compactness and groups* J. G., L. Newelski,
  Archive for Mathematical Logic, 47 (2008), no. 5, p. 479-501
- *Model theoretic connected components of groups* J. G.,
  Israel J. Math. 184, 251 — 274 (2011)
- *Absolute connectedness and classical groups* J. G., `arXiv:1002.1516`
- *Connected components of definable groups and o-minimality I*,
  A. Conversano, A. Pillay, `arXiv:1101.5705`
- *On model-theoretic connected components in some group extensions*
  J. G., K. Krupiński, `arXiv:1201.5221`