Maximal ideals in the algebra of bounded linear operators on Orlicz sequence spaces

Bentuo Zheng
University of Memphis, TN

Banff International Research Station, March, 2012

Background and Notations

This is a joint work with Peikee Lin and Bunyamin Sari.

Background and Notations

This is a joint work with Peikee Lin and Bunyamin Sari.

A classical result of Calkin says that the only nontrivial proper closed ideal in the algebra $L(\ell_2)$ of bounded linear operators on a separable Hilbert space is the ideal of compact operators. The same was shown to be true for ℓ_p $(1 \leq p < \infty)$ an c_0 by Gohberg, Markus and Feldman. Apart from these, there are only few Banach spaces for which the closed ideals in the algebra of bounded linear operators are completely determined.

Theorem (Argyros and Haydon, 2011)

There is a H.I. space X on which every bounded linear operator is a scalar multiple of the identity plus a compact operator.

As a consequence, the only nontrivial norm closed ideal in $L(X)$ is the space of compact operators.

Theorem (Argyros and Haydon, 2011)

There is a H.I. space X on which every bounded linear operator is a scalar multiple of the identity plus a compact operator.

As a consequence, the only nontrivial norm closed ideal in $L(X)$ is the space of compact operators.

Theorem (Laustsen, Loy and Read, 2004)

Let $X = (\oplus \ell_2^n)_{c_0}$. Then there are exactly two nontrivial closed ideals in $L(X)$, namely the ideal of compact operators and the closure of the ideal of operators that factor through c_0.

Theorem (Laustsen, Schlumprecht and Zsak, 2006)

Let $X = (\oplus \ell_2^n)_{\ell_1}$. Then there are precisely two nontrivial closed ideals in $L(X)$, namely the ideal of compact operators and the closure of the ideal of operators that factor through ℓ_1.

Theorem (Laustsen, Schlumprecht and Zsak, 2006)

Let $X = (\oplus \ell_2^n)_{\ell_1}$. Then there are precisely two nontrivial closed ideals in $L(X)$, namely the ideal of compact operators and the closure of the ideal of operators that factor through ℓ_1.

Theorem (Gramsch, Luft and Daws)

Let I be an infinite set, and let $X = \ell_p(I)$ for $1 \leq p < \infty$, or $X = c_0(I)$. If J is a closed ideal in $L(X)$, then $J = K_\alpha(X)$ for some cardinal α.

Theorem (Laustsen, Schlumprecht and Zsak, 2006)

Let $X = (\oplus \ell_2^n)_{\ell_1}$. Then there are precisely two nontrivial closed ideals in $L(X)$, namely the ideal of compact operators and the closure of the ideal of operators that factor through ℓ_1.

Theorem (Gramsch, Luft and Daws)

Let I be an infinite set, and let $X = \ell_p(I)$ for $1 \leq p < \infty$, or $X = c_0(I)$. If J is a closed ideal in $L(X)$, then $J = K_\alpha(X)$ for some cardinal α.

An operator $T \in L(X)$ is in $K_\alpha(X)$ if for every $\epsilon > 0$, there is a subset E of B_X with $|E| < \alpha$ so that for all $x \in B_X$

$$\inf\{\|Tx - Ty\| : y \in E\} < \epsilon.$$

Theorem (Laustsen, Schlumprecht and Zsak, 2006)

Let $X = (\oplus \ell_2^n)_{\ell_1}$. Then there are precisely two nontrivial closed ideals in $L(X)$, namely the ideal of compact operators and the closure of the ideal of operators that factor through ℓ_1.

Theorem (Gramsch, Luft and Daws)

Let I be an infinite set, and let $X = \ell_p(I)$ for $1 \leq p < \infty$, or $X = c_0(I)$. If J is a closed ideal in $L(X)$, then $J = K_\alpha(X)$ for some cardinal α.

An operator $T \in L(X)$ is in $K_\alpha(X)$ if for every $\epsilon > 0$, there is a subset E of B_X with $|E| < \alpha$ so that for all $x \in B_X$

$$\inf\{\|Tx - Ty\| : y \in E\} < \epsilon.$$

Theorem (Tarbard)

For each natural number n, there is a Banach space X so that $L(X)$ contains exactly n nontrivial closed ideals generated by the powers of a single nilpotent, strictly singular, non-compact operator.

Although it is extremely difficult to classify all norm closed ideals in the Banach algebra of bounded linear operators on a given Banach space, it is possible to find the maximal ideals which turn out to be crucial in characterizing commutators in the algebra.

Although it is extremely difficult to classify all norm closed ideals in the Banach algebra of bounded linear operators on a given Banach space, it is possible to find the maximal ideals which turn out to be crucial in characterizing commutators in the algebra.

Let X be a Banach space. We denote by M_X the set of all bounded linear operators T on X so that the identity operator on X does not factor through T. There are quite a number of spaces for which M_X is the unique maximal ideal. The following is a list of spaces recently found to be in the family.

Although it is extremely difficult to classify all norm closed ideals in the Banach algebra of bounded linear operators on a given Banach space, it is possible to find the maximal ideals which turn out to be crucial in characterizing commutators in the algebra.

Let X be a Banach space. We denote by M_X the set of all bounded linear operators T on X so that the identity operator on X does not factor through T. There are quite a number of spaces for which M_X is the unique maximal ideal. The following is a list of spaces recently found to be in the family.

1. L_p $(1 \leq p < \infty)$ (Dosev, Johnson and Schechtman, 2011)

Although it is extremely difficult to classify all norm closed ideals in the Banach algebra of bounded linear operators on a given Banach space, it is possible to find the maximal ideals which turn out to be crucial in characterizing commutators in the algebra.

Let X be a Banach space. We denote by M_X the set of all bounded linear operators T on X so that the identity operator on X does not factor through T. There are quite a number of spaces for which M_X is the unique maximal ideal. The following is a list of spaces recently found to be in the family.

1. L_p $(1 \leq p < \infty)$ (Dosev, Johnson and Schechtman, 2011)
2. $C([0, \omega_1])$ (Kania and Laustsen, 2011)

Although it is extremely difficult to classify all norm closed ideals in the Banach algebra of bounded linear operators on a given Banach space, it is possible to find the maximal ideals which turn out to be crucial in characterizing commutators in the algebra.

Let X be a Banach space. We denote by M_X the set of all bounded linear operators T on X so that the identity operator on X does not factor through T. There are quite a number of spaces for which M_X is the unique maximal ideal. The following is a list of spaces recently found to be in the family.

1. L_p $(1 \leq p < \infty)$ (Dosev, Johnson and Schechtman, 2011)
2. $C([0, \omega_1])$ (Kania and Laustsen, 2011)
3. $(\sum \ell_q)_{\ell_p}$ $(1 \leq q < p < \infty)$ (Chen, Johnson and Zheng, 2011)

Although it is extremely difficult to classify all norm closed ideals in the Banach algebra of bounded linear operators on a given Banach space, it is possible to find the maximal ideals which turn out to be crucial in characterizing commutators in the algebra.

Let X be a Banach space. We denote by M_X the set of all bounded linear operators T on X so that the identity operator on X does not factor through T. There are quite a number of spaces for which M_X is the unique maximal ideal. The following is a list of spaces recently found to be in the family.

1. L_p $(1 \leq p < \infty)$ (Dosev, Johnson and Schechtman, 2011)
2. $C([0, \omega_1])$ (Kania and Laustsen, 2011)
3. $(\sum \ell_q)_{\ell_p}$ $(1 \leq q < p < \infty)$ (Chen, Johnson and Zheng, 2011)
4. $d_{w,p}$ $(1 \leq p < \infty, w_1 = 1, w_n \to 0, \sum_{i=1}^{\infty} w_i = \infty)$ (Kaminska, Popov, Spinu, Tcaciuc and Troisky, 2011)

Recall that an *Orlicz function* $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing and convex function such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$.

Recall that an *Orlicz function* $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing and convex function such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$.

To any Orlicz function M we associate a sequence space ℓ_M of all sequences of scalars $x = (a_1, a_2, \ldots)$ such that $\sum_{n=1}^{\infty} M(|a_n|/\rho) < \infty$ for some $\rho > 0$. The space ℓ_M equipped with the norm

$$\|x\| = \inf\Big\{\rho > 0 : \sum_{n=1}^{\infty} M(|a_n|/\rho) \le 1\Big\}$$

is a Banach space usually called an Orlicz sequence space.

Recall that an *Orlicz function* $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing and convex function such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$.

To any Orlicz function M we associate a sequence space ℓ_M of all sequences of scalars $x = (a_1, a_2, \ldots)$ such that $\sum_{n=1}^{\infty} M(|a_n|/\rho) < \infty$ for some $\rho > 0$. The space ℓ_M equipped with the norm

$$\|x\| = \inf\Big\{\rho > 0 : \sum_{n=1}^{\infty} M(|a_n|/\rho) \leq 1\Big\}$$

is a Banach space usually called an Orlicz sequence space.

An Orlicz function M is said to satisfy the Δ_2-condition if $\sup_{0<t<1} M(2t)/M(t) < \infty$.

Let M be an Orlicz function. Then the following are equivalent.
1. M satisfies the Δ_2 condition.

Let M be an Orlicz function. Then the following are equivalent.

1. M satisfies the Δ_2 condition.
2. ℓ_M is separable.

Let M be an Orlicz function. Then the following are equivalent.

1. M satisfies the Δ_2 condition.
2. ℓ_M is separable.
3. ℓ_M contains no copy of ℓ_∞.

Let M be an Orlicz function. Then the following are equivalent.

1. M satisfies the Δ_2 condition.
2. ℓ_M is separable.
3. ℓ_M contains no copy of ℓ_∞.
4. The unit vector basis of ℓ_M is a symmetric basis for ℓ_M.

Let M be an Orlicz function. Then the following are equivalent.

1. M satisfies the Δ_2 condition.
2. ℓ_M is separable.
3. ℓ_M contains no copy of ℓ_∞.
4. The unit vector basis of ℓ_M is a symmetric basis for ℓ_M.

Recall that for $\Lambda > 0$, $C_{M,\Lambda}$ is the norm-closed convex hull in $C[0,1]$ of the set

$$E_{M,\Lambda} = \overline{\left\{ \frac{M(\lambda t)}{M(\lambda)}; 0 < \lambda < \Lambda \right\}}.$$

Let M be an Orlicz function. Then the following are equivalent.
1. M satisfies the Δ_2 condition.
2. ℓ_M is separable.
3. ℓ_M contains no copy of ℓ_∞.
4. The unit vector basis of ℓ_M is a symmetric basis for ℓ_M.

Recall that for $\Lambda > 0$, $C_{M,\Lambda}$ is the norm-closed convex hull in $C[0,1]$ of the set

$$E_{M,\Lambda} = \overline{\left\{ \frac{M(\lambda t)}{M(\lambda)}; 0 < \lambda < \Lambda \right\}}.$$

We let $E_M = \bigcap_{\Lambda > 0} E_{M,\Lambda}$ and $C_M = \bigcap_{\Lambda > 0} C_{M,\Lambda}$.

Let M be an Orlicz function. Then the following are equivalent.
1. M satisfies the Δ_2 condition.
2. ℓ_M is separable.
3. ℓ_M contains no copy of ℓ_∞.
4. The unit vector basis of ℓ_M is a symmetric basis for ℓ_M.

Recall that for $\Lambda > 0$, $C_{M,\Lambda}$ is the norm-closed convex hull in $C[0,1]$ of the set

$$E_{M,\Lambda} = \overline{\left\{ \frac{M(\lambda t)}{M(\lambda)}; 0 < \lambda < \Lambda \right\}}.$$

We let $E_M = \bigcap_{\Lambda>0} E_{M,\Lambda}$ and $C_M = \bigcap_{\Lambda>0} C_{M,\Lambda}$.

If M satisfies the Δ_2-condition then an Orlicz sequence space ℓ_N is isomorphic to a subspace of ℓ_M if and only if N is equivalent to some function in $C_{M,1}$.

Let $1 < p < \infty$ and let M be an Orlicz function. We say that the Orlicz sequence space ℓ_M is *close to* ℓ_p if the following conditions hold:

Let $1 < p < \infty$ and let M be an Orlicz function. We say that the Orlicz sequence space ℓ_M is *close to* ℓ_p if the following conditions hold:

(1) The unit vector basis of ℓ_M and the unit vector basis of ℓ_p are the only, up to equivalence, symmetric basic sequences in ℓ_M;

Let $1 < p < \infty$ and let M be an Orlicz function. We say that the Orlicz sequence space ℓ_M is *close to* ℓ_p if the following conditions hold:

(1) The unit vector basis of ℓ_M and the unit vector basis of ℓ_p are the only, up to equivalence, symmetric basic sequences in ℓ_M;

(2) Normalized block basic sequences of ℓ_M uniformly dominate the unit vector basis of ℓ_p; i.e. there is a $C > 0$ so that every normalized block bases (x_i) of ℓ_M satisfies for all $(a_i) \subset \mathbb{R}$

$$\|\sum a_i x_i\| \geq C(\sum |a_i|^p)^{1/p}.$$

Let $1 < p < \infty$ and let M be an Orlicz function. We say that the Orlicz sequence space ℓ_M is *close to* ℓ_p if the following conditions hold:

(1) The unit vector basis of ℓ_M and the unit vector basis of ℓ_p are the only, up to equivalence, symmetric basic sequences in ℓ_M;

(2) Normalized block basic sequences of ℓ_M uniformly dominate the unit vector basis of ℓ_p; i.e. there is a $C > 0$ so that every normalized block bases (x_i) of ℓ_M satisfies for all $(a_i) \subset \mathbb{R}$

$$\|\sum a_i x_i\| \geq C(\sum |a_i|^p)^{1/p}.$$

M is said to be p-regular if $\lim_{\lambda \to 0} \frac{M(\lambda t)}{M(\lambda)} = t^p, 0 < t \leq 1$.

Remark. It immediate from the definition that if ℓ_M is close to ℓ_p, then M satisfies the Δ_2-condition and ℓ_M is reflexive. Actually since ℓ_M does not contain c_0, M satisfies the Δ_2-condition. Since ℓ_M does not contain ℓ_1, ℓ_M is reflexive. Moreover, if M is p-regular, then $E_M = C_M = \{t^p\}$.

Remark. It immediate from the definition that if ℓ_M is close to ℓ_p, then M satisfies the Δ_2-condition and ℓ_M is reflexive. Actually since ℓ_M does not contain c_0, M satisfies the Δ_2-condition. Since ℓ_M does not contain ℓ_1, ℓ_M is reflexive. Moreover, if M is p-regular, then $E_M = C_M = \{t^p\}$.

Theorem (Lin, Sari and Zheng, 2012)

Let $1 < p < \infty$. Let ℓ_M be an Orlicz space close to ℓ_p and M be p-regular. Then M_{ℓ_M} is the unique maximal ideal in $L(\ell_M)$.

Lemma (1)

Let $1 < p < \infty$ and ℓ_M be an Orlicz sequence space close to ℓ_p. Let (u_j) be a normalized block basis in ℓ_M. Then (u_i) is K-dominated by the unit vector basis of ℓ_M for some constant K independent of (u_i) and there exists a subsequence of (u_j) which is either equivalent to the unit vector basis of ℓ_M or to the unit vector basis of ℓ_p. If, in addition, M is regular, and $\lim_j \|u_j\|_\infty = 0$, then a subsequence of (u_j) is equivalent to the unit vector basis of ℓ_p.

Lemma (1)

Let $1 < p < \infty$ and ℓ_M be an Orlicz sequence space close to ℓ_p. Let (u_j) be a normalized block basis in ℓ_M. Then (u_i) is K-dominated by the unit vector basis of ℓ_M for some constant K independent of (u_i) and there exists a subsequence of (u_j) which is either equivalent to the unit vector basis of ℓ_M or to the unit vector basis of ℓ_p. If, in addition, M is regular, and $\lim_j \|u_j\|_\infty = 0$, then a subsequence of (u_j) is equivalent to the unit vector basis of ℓ_p.

Lemma (2)

Let $1 < p < \infty$. Suppose that ℓ_M is an Orlicz sequence space close to ℓ_p, and M is p-regular. Let (u_i) be a normalized block basis of ℓ_M which is either equivalent to the unit vector basis of ℓ_p or the unit vector basis of ℓ_M. Then $[(u_i)]$ is complemented in ℓ_M.

Lemma (3)

Let $1 < p < \infty$. Let ℓ_M be an Orlicz space close to ℓ_p and M be p-regular. Then M_{ℓ_M} is an ideal in $L(\ell_M)$.

Lemma (3)

Let $1 < p < \infty$. Let ℓ_M be an Orlicz space close to ℓ_p and M be p-regular. Then M_{ℓ_M} is an ideal in $L(\ell_M)$.

Proof. An operator T is in M_{ℓ_M} if and only if T does not preserve a copy of ℓ_M (i.e. T is ℓ_M-strictly singular).
Let S and T be two ℓ_M-strictly singular operators. Suppose that $S + T$ is not ℓ_M-strictly singular. Then we can find a normalized sequence (x_i) in ℓ_M so that both (x_i) and $(Sx_i + Tx_i)$ are equivalent to the unit vector basis of ℓ_M. By passing to a subsequence of (x_i) and perturbing, without loss of generality we assume that both (x_i) and $(Sx_i + Tx_i)$ are block bases in ℓ_M.
By Lemma (1), there exists a $\delta > 0$ so that $\|Sx_i + Tx_i\|_\infty > \delta$.
By passing to a further subsequence (y_i) of (x_i), we get either $\|Sy_i\|_\infty > \delta/2$ for all $i \in \mathbb{N}$ or $\|Ty_i\|_\infty > \delta/2$ for all $i \in \mathbb{N}$. But this implies that either (Sy_i) or (Ty_i) is equivalent to the unit vector basis of ℓ_M since the unit vector basis of ℓ_M dominates every block basis of ℓ_M (by Lemma (1) again). Hence either S or T is preserve a copy of ℓ_M which contradicts our hypothesis.

Theorem (Dosev and Johnson, 2010)

Let X be a Banach space. If M_X is an ideal, then it is automatically maximal.

Theorem (Dosev and Johnson, 2010)

Let X be a Banach space. If M_X is an ideal, then it is automatically maximal.

Theorem (Lin, Sari and Zheng, 2012)

Let $1 < p < \infty$. Let ℓ_M be an Orlicz space close to ℓ_p and M be p-regular. Then M_{ℓ_M} is the unique maximal ideal in $L(\ell_M)$.

Let ℓ_M be an Orlicz space. We use Γ^{ℓ_p} to denote the ideal of all operators in $L(\ell_M)$ which factor through ℓ_p. Let $\bar{\Gamma}^{\ell_p}$ be the closure of Γ^{ℓ_p}.

Let ℓ_M be an Orlicz space. We use Γ^{ℓ_p} to denote the ideal of all operators in $L(\ell_M)$ which factor through ℓ_p. Let $\bar{\Gamma}^{\ell_p}$ be the closure of Γ^{ℓ_p}.

Theorem

Let $1 < p \leq 2$ and M be p-regular. Let ℓ_M be an Orlicz sequence space close to ℓ_p but not isomorphic to ℓ_p. Then $\bar{\Gamma}^{\ell_p}$ is a proper subset of M_{ℓ_M}.

Let ℓ_M be an Orlicz space. We use Γ^{ℓ_p} to denote the ideal of all operators in $L(\ell_M)$ which factor through ℓ_p. Let $\bar{\Gamma}^{\ell_p}$ be the closure of Γ^{ℓ_p}.

Theorem

Let $1 < p \leq 2$ and M be p-regular. Let ℓ_M be an Orlicz sequence space close to ℓ_p but not isomorphic to ℓ_p. Then $\bar{\Gamma}^{\ell_p}$ is a proper subset of M_{ℓ_M}.

Lemma (4)

Let $1 \leq p < \infty$ and let X be a complemented subspace of an Orlicz space ℓ_M and let P be a projection from ℓ_M onto X. If P is in $\bar{\Gamma}^{\ell_p}$, then X is isomorphic to ℓ_p.

Lemma (5)

Let $1 < p \leq 2$ and M be p-regular. Let ℓ_M be an Orlicz sequence space close to ℓ_p but not isomorphic to ℓ_p. Then there exists an infinite dimensional complemented subspace X of ℓ_M which is not isomorphic to ℓ_p so that ℓ_M does not embed into X.

Lemma (5)

Let $1 < p \leq 2$ and M be p-regular. Let ℓ_M be an Orlicz sequence space close to ℓ_p but not isomorphic to ℓ_p. Then there exists an infinite dimensional complemented subspace X of ℓ_M which is not isomorphic to ℓ_p so that ℓ_M does not embed into X.

Theorem

Let $1 < p \leq 2$ and M be p-regular. Let ℓ_M be an Orlicz sequence space close to ℓ_p but not isomorphic to ℓ_p. Then the $\bar{\Gamma}^{\ell_p}$ is a proper subset of the ℓ_M-strictly singular operators.

Lemma (5)

Let $1 < p \leq 2$ and M be p-regular. Let ℓ_M be an Orlicz sequence space close to ℓ_p but not isomorphic to ℓ_p. Then there exists an infinite dimensional complemented subspace X of ℓ_M which is not isomorphic to ℓ_p so that ℓ_M does not embed into X.

Theorem

Let $1 < p \leq 2$ and M be p-regular. Let ℓ_M be an Orlicz sequence space close to ℓ_p but not isomorphic to ℓ_p. Then the $\bar{\Gamma}^{\ell_p}$ is a proper subset of the ℓ_M-strictly singular operators.

Proof. Let X be the complemented subspace of ℓ_M as in Lemma (5). Let P be a projection from ℓ_M onto X. Since X is not isomorphic to ℓ_p, by Lemma (4), P is not in $\bar{\Gamma}^{\ell_p}$. Since ℓ_M does not embed into X, P is ℓ_M strictly singular.

Let $I_{\ell_M \to \ell_p}$ be the formal identity from ℓ_M into ℓ_p. By composing with an isomorphic embedding of ℓ_p into ℓ_M, it is considered as an operator on ℓ_M. It is easy to prove that if ℓ_M is close to ℓ_p, then the closed ideal $\bar{\Gamma}^{I_{\ell_M \to \ell_p}}$ generated by $I_{\ell_M \to \ell_p}$ is an immediate successor of the compacts.

Let $I_{\ell_M \to \ell_p}$ be the formal identity from ℓ_M into ℓ_p. By composing with an isomorphic embedding of ℓ_p into ℓ_M, it is considered as an operator on ℓ_M. It is easy to prove that if ℓ_M is close to ℓ_p, then the closed ideal $\bar{\Gamma}^{I_{\ell_M \to \ell_p}}$ generated by $I_{\ell_M \to \ell_p}$ is an immediate successor of the compacts.

Theorem

Let ℓ_M be an Orlicz space close to ℓ_p. If $T \in L(\ell_M)$ is not compact, then $\bar{\Gamma}^{I_{\ell_M \to \ell_p}}$ is a subspace of $\bar{\Gamma}^T$.

Let $I_{\ell_M \to \ell_p}$ be the formal identity from ℓ_M into ℓ_p. By composing with an isomorphic embedding of ℓ_p into ℓ_M, it is considered as an operator on ℓ_M. It is easy to prove that if ℓ_M is close to ℓ_p, then the closed ideal $\bar{\Gamma}^{I_{\ell_M \to \ell_p}}$ generated by $I_{\ell_M \to \ell_p}$ is an immediate successor of the compacts.

Theorem

Let ℓ_M be an Orlicz space close to ℓ_p. If $T \in L(\ell_M)$ is not compact, then $\bar{\Gamma}^{I_{\ell_M \to \ell_p}}$ is a subspace of $\bar{\Gamma}^T$.

Question: Is $\bar{\Gamma}^{I_{\ell_M \to \ell_p}}$ a proper subspace of $\bar{\Gamma}^{\ell_p}$?