“Tensor products” between metric spaces and Banach spaces

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Algebraic tensor products

Tensor products are normally used to linearize bilinear maps.

$$\begin{array}{ccc} E \times F & \xrightarrow{\varphi} & G \\ \downarrow & \nearrow_{\tilde{\varphi}} & \\ E \otimes F & & \end{array}$$

What sense could there possibly be in thinking about tensor products of a metric space with a Banach space?

Tensor products and Banach spaces

In Banach space theory, tensor products are used for more than linearizing bilinear maps.

There are many different choices for a “reasonable” norm on $E \otimes F$.

Most importantly, there are deep connections between tensor norms and operator ideals.

Duality relations

It often happens that

$$(E \otimes_\alpha F)^* \equiv \mathcal{A}(E, F^*)$$

for some tensor norm α and some operator ideal $\mathcal{A}$.

Examples:

1. $(E \otimes_\pi F)^* \equiv L(E, F^*)$.
2. $(E \otimes_{d_p} F)^* \equiv \Pi_{p'}(E, F^*)$.
3. $(E \otimes_{w_2} F)^* \equiv \Gamma_2(E, F^*)$.

All of these examples have nonlinear counterparts.

Tensoring with the identity

Another type of result takes the form

$$T \in \mathcal{A}(E,F) \quad \Leftrightarrow \quad \|T \otimes id_G : E \otimes_\alpha G \to F \otimes_\beta G\| < \infty \quad \forall G.$$

with $\mathcal{A}$ an operator ideal and α, β tensor norms.
Examples:

1. $T \in \Pi_p(E,F) \Leftrightarrow \left\|T \otimes id_G : E \otimes_{d_{p'}} G \to F \otimes_{d_1} G\right\| < \infty \forall G.$
2. $T \in M_{q,p}(E,F) \Leftrightarrow \left\|T \otimes id_G : E \otimes_{d_{p'}} G \to F \otimes_{d_{q'}} G\right\| < \infty \forall G.$

Again, these and other examples have nonlinear counterparts.

Duality results

A baby example for duality

Suppose we want to find a nonlinear version of $(E \otimes_\pi F)^* \equiv L(E, F^*)$.

In the nonlinear setting, Lipschitz maps play the role corresponding to that of linear bounded maps.

That means we want to find some sort of tensor product so that $(X \boxtimes_\pi F)^* \equiv \mathrm{Lip}_0(X, F^*)$.

The easiest instance of this would be when $F = \mathbb{R}$.

The Arens-Eells space

The Arens-Eells space of a metric space X (denoted Æ(X)), also known as the free Lipschitz space of X (denoted $\mathscr{F}(X)$) satisfies

$$\mathscr{F}(X)^* \equiv X^{\#} := \mathrm{Lip}_0(X, \mathbb{R}) = \{f : X \to \mathbb{R} \ : \ \mathrm{Lip}(f) < \infty, \ f(0) = 0\}.$$

It was introduced in [Arens/Eells 1956], and has been used in Banach space theory [Godefroy/Kalton 2003], [Kalton 2004].

Molecules and the Arens-Eells space

- A molecule on a metric space X is a finitely supported $m : X \to \mathbb{R}$ such that
$$\sum_{x\in X} m(x) = 0.$$
Note that the space of molecules is a vector space.
- Those of the form $am_{xx'}$ where
$$m_{xx'} := \chi_{\{x\}} - \chi_{\{x'\}}$$
with $a \in \mathbb{R}$ and $x, x' \in X$ are called *atoms*.
- The Arens-Eells space of X is the space of molecules with the norm
$$\|m\|_{\mathscr{F}(X)} := \inf\left\{ \sum_{j=1}^{n} |a_j| d(x_j, x_j') \ : \ m = \sum_{j=1}^{n} a_j m_{x_j x_j'} \right\}.$$

Properties of the Arens-Eells space

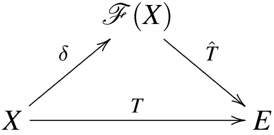
(a) $\|\cdot\|_{\mathscr{F}(X)}$ is a norm.

(b) $\delta : X \hookrightarrow \mathscr{F}(X)$ given by $\delta(x) = m_{x0}$ is an isometric embedding.

(c) $\mathscr{F}(X)^* = \mathrm{Lip}_0(X, \mathbb{R}) = X^\#$ via the duality pairing

$$\langle f, m \rangle = \sum_{x \in X} f(x) m(x)$$

(d) Whenever $T : X \to E$ is a Lipschitz map, there is a linear map $\hat{T} : \mathscr{F}(X) \to E$ such that $\|\hat{T}\| = \mathrm{Lip}(T)$ and $\hat{T} \circ \delta = T$.



Duality for $L(E, F)$

Theorem

$(E \otimes_\pi F)^* = L(E, F^*)$.

Where for $w \in E \otimes F$

$$\|w\|_\pi = \inf \left\{ \sum_{j=1}^{n} \|u_j\| \cdot \|v_j\| \ : \ w = \sum_{j=1}^{n} u_j \otimes v_j \right\}$$

and the identification is given via trace duality, considering an element in $E \otimes F$ as a map $F^* \to E$. That is, for $w = \sum_{j=1}^{n} u_j \otimes v_j \in E \otimes F$ and $T : E \to F^*$,

$$\langle T, w \rangle = \mathrm{tr}(w \circ T) = \sum_{j=1}^{n} \langle T x_j, y_j \rangle.$$

Vector valued molecules

Definition (C, 2011)

Let X be a metric space and E a Banach space.
An E-valued molecule on X is a function $m : X \to E$ such that

$$\sum_{x \in X} m(x) = 0.$$

- An E*-valued atom* is a function of the form $vm_{xx'}$ with $x, x' \in X$ and v in E.
- Every E-valued molecule on X can be expressed as a sum of E-valued atoms.

Projective norm for vector valued molecules

For an E-valued molecule m, let

$$\|m\|_\pi := \inf\left\{\sum_{j=1}^{n} \|v_j\|\, d(x_j, x_j') \;:\; m = \sum_{j=1}^{n} v_j m_{x_j x_j'}\right\}.$$

We will denote by $X \boxtimes_\pi E$ the space of E-valued molecules on X with the projective norm. It is not hard to show that

$$(X \boxtimes_\pi E)^* = \mathrm{Lip}_0(X, E^*)$$

with the duality given by the pointwise pairing

$$\langle T, m\rangle = \sum_{x\in X} \langle T(x), m(x)\rangle.$$

It was known that $\mathrm{Lip}_0(X, E^*)$ is a dual space [J. Johnson, 1970], but as far as I know the approach via molecules is new.

"Products" of operators

Proposition (C, 2012)

Let $S : X \to Z$ *be a Lipschitz map mapping 0 to 0, and* $T : E \to F$ *a bounded linear map. Then there is a unique operator* $S \boxtimes T : X \boxtimes_\pi E \to Z \boxtimes_\pi F$ *such that*

$$(S \boxtimes T)(vm_{xy}) = (Tv)m_{(Sx)(Sy)}, \qquad \text{for all } v \in E, x, y \in X.$$

Furthermore, $\|S \boxtimes_\pi T\| = \mathrm{Lip}(S)\,\|T\|$.

Justifying the "projective" name

Recall that a linear operator $T : E \to F$ is a *linear quotient* if it is surjective and

$$\|w\| = \inf\left\{ \|v\| \ : \ v \in E, \ Tv = w\right\} \text{ for every } w \in F.$$

On the other hand, a map $S : X \to Z$ is called a C*-co-Lipschitz* if for every $x \in X$ and $r > 0$,

$$f\big(B(x,r)\big) \supseteq B\big(f(x), r/C\big).$$

A map that is Lipschitz, co-Lipschitz and surjective is a *Lipschitz quotient*.

Theorem (C, 2012)

Let $S : X \to Z$ be a Lipschitz quotient with Lipschitz and co-Lipschitz constants equal to 1, and mapping 0 to 0, and let $T : E \to F$ be a linear quotient map. Then $S \boxtimes_\pi T : (X \boxtimes_\pi E) \to (Z \boxtimes_\pi F)$ is also a linear quotient map.

Example: $X =$ a graph-theoretic tree

Recall

$$\|m\|_\pi = \inf\left\{ \sum_{j=1}^{n} \|v_j\|\, d(x_j, x_j') : m = \sum_{j=1}^{n} v_j m_{x_j x_j'} \right\}$$

Note we can consider only representations where the pairs (x_j, x_j') are endpoints of edges. Since X is a tree, every molecule has only one such representation so

$$X \boxtimes_\pi E \equiv \ell_1^N(E)$$

where $N = \#$ of edges of X.
I suspect a similar result should work for more general metric trees as in [Godard 2010].

Reasonable tensor norms

A tensor norm α is called *reasonable if it satisfies*

(a) $\alpha(u \otimes v) \leq \|u\| \cdot \|v\|$ for every $u \in E$, $v \in F$.

(b) $\alpha^*(u^* \otimes v^*) \leq \|u^*\| \, \|v^*\|$ for every $u^* \in E^*$, $v^* \in F^*$.

Reasonable tensor norms are characterized by being between the projective and injective tensor norms: a tensor norm α is reasonable if and only if $\varepsilon \leq \alpha \leq \pi$, where

$$\|w\|_\varepsilon = \sup \left\{ \sum_{j=1}^{n} \langle u^*, u_j \rangle \langle v^*, v_j \rangle \ : \ w = \sum_{j=1}^{n} u_j \otimes v_j, \ u^* \in B_{E^*}, \ v^* \in B_F^* \right\}.$$

Reasonable molecular norms

A norm $\|\cdot\|$ on the space of E-valued molecules on a metric space X is called *reasonable* if

(i) $\|vm_{xx'}\| \leq \|v\| \, d(x,x')$ for all $x, x' \in X$, $v \in E$.

(ii) $|\langle v^* \circ m, f\rangle| \leq \|v^*\| \operatorname{Lip}(f) \, \|m\|$ for all $v^* \in E^*$, $m \in \mathcal{M}(X,E)$ and $f \in X^{\#}$.

Reasonable molecular norms are also characterized by being between the projective and injective norms: a molecular norm α is reasonable if and only if $\varepsilon \leq \alpha \leq \pi$, where

$$\|m\|_{\varepsilon} = \sup\left\{ \sum_{j=1}^{n} \left[f(x_j) - f(x_j')\right] v^*(v_j) \right.$$

$$\left. : m = \sum_{j=1}^{n} v_j m_{x_j x_j'}, \; f \in B_{X^{\#}}, \;\; v^* \in B_{E^*} \right\}.$$

The injective norm

The injective norm is also deserving of its name: it behaves well under injections.

However, it is not so interesting for us because it “forgets” about the metric space and only takes into account the structure of $\mathscr{F}(X)$. In fact,

$$X \boxtimes_\varepsilon E \equiv \mathscr{F}(X) \otimes_\varepsilon E.$$

p-summing operators

E, F Banach spaces, $T : E \to F$ a linear map, $1 \leq p \leq \infty$.

T is called p-summing if there exists $C > 0$ such that for any $v_1, \dots v_n$ in E we have

$$\left[\sum_{j=1}^{n} \|Tv_j\|^p\right]^{1/p} \leq C \sup_{\phi \in B_{E^*}} \left[\sum_{j=1}^{n} |\phi(v_j)|^p\right]^{1/p}.$$

The *p-summing norm of T* is

$$\pi_p(T) := \inf C.$$

The space of p-summing operators from E to F is denoted

$$\Pi_p(E, F).$$

Chevet-Saphar norms

Theorem (Saphar 1970)

$$\left(E \otimes_{d_p} F\right)^* = \Pi_{p'}(E, F^*).$$

Where

Definition (Chevet 1969, Saphar 1965,1970)

For $1 \leq p \leq \infty$ and $w \in E \otimes F$, define p' by $1/p + 1/p' = 1$ and

$$\|w\|_{d_p} := \inf\left\{ \sup_{\phi \in B_{E^*}} \left[\sum_{j=1}^{n} |\phi(u_j)|^{p'}\right]^{1/p'} \cdot \left[\sum_{j=1}^{n} \|v_j\|^p\right]^{1/p} : w = \sum_{j=1}^{n} u_j \otimes v_j \right\}.$$

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Definition (Pietsch, 1966)

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Lipschitz p-summing operators

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$$\left[\sum_{j=1}^{n} d(Tx_j, Tx'_j)^p\right]^{1/p} \leq C \sup_{\phi \in B_{E^*}} \left[\sum_{j=1}^{n} |\phi(v_j)|^p\right]^{1/p}$$

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The Lipschitz p-summing norm of T is

$$\pi_p^L(T) := \inf C.$$

Duality for Lipschitz p-summing operators

Theorem (C 2011)

$$\left(X \boxtimes_{d_p} F\right)^* = \Pi^L_{p'}(X, F^*).$$

Where Π^L_p denotes the Lipschitz p-summing operators of [Farmer/Johnson 2009] and

Definition (C 2011)

For an E-valued molecule m on a metric space X,

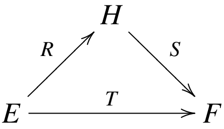
$$\|m\|_{d_p} = \inf\left\{ \left(\sum_j \lambda_j^p \|v_j\|^p \right)^{1/p} \sup_{f \in B_{X^\#}} \left(\lambda_j^{-p'} |f(x_j) - f(x_j')|^{p'} \right)^{1/p'} : m = \sum_j v_j m_{x_j x_j'}, \lambda_j > 0 \right\}.$$

Linear factorization through Hilbert space

Define for a linear map $T : E \to F$

$$\gamma_2(T) := \inf \{ \|R\| \cdot \|S\| \}$$

where



and H is a Hilbert space.

$\Gamma_2(E, F)$ will denote the space of all operators admitting such a factorization.

Duality for $\Gamma_2(E,F)$

Theorem

$(E \otimes_{w_2} F)^* = \Gamma_2(E, F^*)$

Where for $w \in E \otimes F$

$$\|u\|_{w_2} = \inf\left\{\left(\sum_{j=1}^{n}\|u_j\|^2\right)^{1/2}\left(\sum_{i=1}^{n}\|v_i\|^2\right)^{1/2} : \right.$$
$$\left. u = \sum_{ij} a_{ij} u_j \otimes v_i,\ \|(a_{ij}) : \ell_2^n \to \ell_2^n\| \leq 1\right\}$$

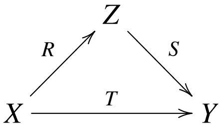
and the identification is given again via trace duality.

Lipschitz factorization through subsets of Hilbert space

Define for a Lipschitz map $T : X \to Y$

$$\gamma_2^{\mathrm{Lip}}(T) := \inf \left\{ \mathrm{Lip}(R) \cdot \mathrm{Lip}(S) \right\}$$

where



and Z is a subset of a Hilbert space.

Duality for Γ_2^{Lip}

The norm on molecules that gives the duality for Γ_2^{Lip} is

$$\|m\|_{w_2} = \inf\left\{ \left(\sum_{i=1}^{n} \|v_i\|^2 \right)^{1/2} \left(\sum_{j=1}^{m} d(x,x_j')^2 \right)^{1/2} : \right.$$

$$\left. m = \sum_{i=1}^{n} v_i m_{y_i y_i'},\ m_{y_i y_i'} = \sum_{j=1}^{m} a_{ij} m_{x_j x_j'},\ \|(a_{ij}) : \ell_2^m \to \ell_2^n\| \le 1 \right\}$$

Tensoring with the identity

Representation theorems

Operator ideals satisfying certain technical properties can be characterized by theorems of the following form:

Representation theorem

A linear operator $T : E \to F$ belongs to the operator ideal $\mathfrak{A}$ if and only if for every Banach space G, the map

$$T \otimes id_G : E \otimes_\alpha G \to F \otimes_\beta G$$

is continuous.

Here, α and β are certain tensor norms.

Example

A linear operator $T : E \to F$ is p-summing if and only if for every Banach space G the map

$$T \otimes id_G : E \otimes_{d_{p'}} G \to F \otimes_\pi G$$

is continuous.

Moreover, in this case

$$\pi_p(T) = \inf_G \|T \otimes id_G\|$$

A nonlinear version

Theorem (C, 2011)

TFAE:

(a) $T : X \to Y$ *is Lipschitz* p*-summing.*

(b) *For every Banach space* E *(or only* $E = Y^{\#}$*),*

$$\left\| T \boxtimes id_E : X \boxtimes_{d'_p} E \to Y \boxtimes_{\pi} E \right\| < \infty$$

(q,p)-mixing operators

Theorem

Let $T : E \to F$ be a linear map, $1 \leq p \leq q \leq \infty$. TFAE:

(a) $\exists\, C > 0$ *such that for every* $S : F \to G$,

$$\pi_p(S \circ T) \leq C\pi_q(S).$$

(b) *For every Banach space G (or only $G = \ell_{q'}$),*

$$\left\| T \otimes id_G : E \otimes_{d_{p'}} G \to F \otimes_{d_{q'}} G \right\| < \infty$$

Similarly

Theorem (C, 2011)

Let $T : X \to Y$ be a Lipschitz map, $1 \leq p \leq q \leq \infty$. TFAE:

(a) $\exists\, C > 0$ *such that for every* $S : Y \to Z$,

$$\pi_p^L(S \circ T) \leq C\pi_q^L(S).$$

(b) *For every Banach space E (or only $E = \ell_{q'}$),*

$$\left\| T \boxtimes id_E : X \boxtimes_{d_{p'}} E \to Y \boxtimes_{d_{q'}} E \right\| < \infty$$