The Gurariĭ space

The Gurariĭ space $\mathcal{G}$ is obtained setting the input data at

- The ordinal ω.
- For each $n < \omega$ the Banach space $X_n = \mathrm{PO}_n$ and X_1 is any separable Banach space.
- For each $n < \omega$, $\mathcal{F}_n$ is a countable family of finite-dimensional Banach spaces that is dense, in the Banach-Mazur distance, in the space of all finite-dimensional Banach spaces.
- For each $n < \omega$, $\mathfrak{J}_n$ denotes a countable and dense family of almost isometric embeddings between elements of $\mathcal{F}_n$.
- For each $n < +\infty$, $\mathfrak{L}_{n+1}$ is the family of all isometric embeddings of elements of $\mathcal{F}_n$ into X_n.

Gurariĭ introduces in [*Spaces of universal placement, isotropic spaces and a problem of Mazur on rotations of Banach spaces (Russian).* Sibirsk. Mat. Ž. 7 (1966), 1002–1013] the notions of spaces of universal and almost-universal disposition for a given class $\mathfrak{M}$ as follows.

Definition 0.1. Let $\mathfrak{M}$ be a class of Banach spaces. A Banach space U is said to be of almost universal disposition for the class $\mathfrak{M}$ if, given $A, B \in \mathfrak{M}$, isometric embeddings $u : A \to U$ and $\imath : A \to B$, and $\varepsilon > 0$, there is a $(1+\varepsilon)$-isometric embedding $u' : B \longrightarrow U$ such that $u = u'\imath$.

- Gurariĭ shows that there exists a separable Banach space of almost-universal disposition for the class $\mathfrak{F}$ of finite dimensional spaces.
- Gevorkyan shows in [*The universality of spaces of almost universal displacement*, (Russian) Funkcional. Anal. i Priložen 8 (1974), no. 2, 72] that a space of almost-universal disposition for finite dimensional spaces must contain isometric copies of all separable spaces.
- Lusky shows in [*The Gurariĭ spaces are unique* . Arch. Math. 27 (1976) 627-635.] that two separable spaces of almost-universal disposition for finite-dimensional Banach spaces are isometric. Therefore, there exists a unique separable space of almost-universal disposition for finite dimensional Banach spaces, that we will call the Gurariĭ space and denote by $\mathcal{G}$.
- Pełczyński and Wojtaszczyk [*Banach spaces with finite dimensional expansions of identity and universal bases of finite dimensional spaces,* Studia Mathematica, XL (1971) 91–108] show that there is a separable Lindenstrauss space $\mathcal{PW}$ in which every separable Lindenstrauss embedds almost isometrically as a 1-complemented subspace.
- Wojtaszczyk shows in [*Some remarks on the Gurarij space*, Studia Mathematica, XLI (1972), 207–210] that $\mathcal{PW}$ can be constructed as a space of almost universal disposition for finite dimensional spaces.

The Kubis space

The Kubis space [*Fraïssé sequences - a category-theoretic approach to universal homogeneous structures*, arXiv:0711.1683v1, 2007] is obtained setting the input data at

- The ordinal ω_1.
- For each countable α $X_\alpha = \mathrm{PO}_\alpha$ and X_1 is any separable Banach space .
- For each countable α, $\mathcal{F}_\alpha$ is the family of all separable Banach spaces.
- For each countable α, $\mathfrak{J}_\alpha$ denotes the family of all isometric embeddings between elements of $\mathcal{F}_\alpha$.
- For each countable α, $\mathfrak{L}_{\alpha+1}$ is the family of all isometric embeddings of elements of $\mathcal{F}_\alpha$ into X_α.

The resulting space X_{ω_1} is, under CH, the only space (up to isometries) space having density character $\aleph_1$ which is of universal disposition for separable spaces.

- Gurariĭ conjectured the existence of spaces of universal disposition for the classes $\mathfrak{F}$ of finite dimensional spaces and $\mathfrak{S}$ of separable spaces: see the footnote to Theorem 5.

  Definition 0.2. A Banach space U is of universal disposition for the class $\mathfrak{M}$ if, given $A, B \in \mathfrak{M}$ and isometric embeddings $u : A \to U$ and $\imath : A \to B$, there is an isometric embedding $u' : B \to U$ such that $u = u'\imath$.

- Kubis (paper above) constructed the Fraïssé limit in the category of separable Banach spaces and isometric embeddings.
- In [A. Avilés, F. Cabello, J.M.F. Castillo, M. González and Y. Moreno, *Banach spaces of universal disposition* J. Functional Anal. 261 (2011) 2347-2361] it was shown that this means that the space is of universal disposition for the class of separable spaces
- Under CH, there is only one one space (up to isometries) of universal disposition for separable spaces and density character $\aleph_1$. We called it $\mathcal{K}$.
- Do there exist other spaces of universal disposition apart from the push-out generated?
- Under CH, $\mathcal{K} = \mathcal{G}_{\mathfrak{U}}$ (since the ultrapower of a space of almost-universal disposition is of universal disposition; I'm cheating!).

Spaces of universal disposition

(1) A Banach space of universal disposition for separable spaces must contain an isometric copy of each Banach space of density $\aleph_1$ or less.

(2) There are spaces of universal disposition for finite-dimensional spaces that are not of universal disposition for separable spaces.

Actually a space of universal disposition for separable spaces cannot contain complemented copies of c_0; while there are spaces of universal disposition for finite-dimensional spaces on which every copy of c_0 is complemented.

(3) This suggests that quite plausibly there is –even under CH– a continuum of mutually non-isomorphic spaces of universal disposition for finite-dimensional spaces. Such is the case –outside CH, of course– for separable spaces.

(4) (Transitivity affairs) A Banach space in which every isometry between 1- dimensional subspaces can be extended to an isometry of the space have been called transitive. Gurariĭ had essentially observed that separable spaces of universal disposition for finite-dimensional spaces are transitive (so they cannot exist). This suggests the problem of whether spaces of universal disposition for finite-dimensional spaces are transitive. It can be proved that push-out generated spaces are even "ultratransitive" (in the denomination of Kalton) or f-homogeneous (in that of Kubis): isometries between finite dimensional subspaces can be extended to an isometry of the space.

(5) (Automorphic affairs) A Banach space X of universal disposition for separable spaces and density character $\aleph_1$ enjoys the property that any isomorphism between two separable subspaces of X can be extended to an automorphism of X.

The automorphic space problem

Definition 0.3. A Banach space is said to be *automorphic* if every isomorphism between two subspaces such that the corresponding quotients have the same density character can be extended to an automorphism of the whole space.

The motivation for such definition is in the Lindenstrauss-Rosenthal theorem [*Automorphisms in c_0, ℓ_1 and m.* Israel J. Math. 9 (1969), 227–239] asserting that c_0 is automorphic, in the extension ($c_0(\Gamma)$ is automorphic) presented by Y. Moreno and A. Plichko [*On automorphic Banach spaces.* Israel J. Math. 169 (2009) 29-45].

Automorphic space problem of Lindenstrauss and Rosenthal:
Does there exist an automorphic space different from $c_0(I)$ or $\ell_2(I)$?

- A. Aviles and Y. Moreno, [*Automorphisms in spaces of continuous functions on Valdivia compacta*, Topology Appl. 155 (2008) 2027–2030] showed that most $C(K)$ spaces are not automorphic.
- Y. Moreno and A. Plichko (paper above) showed that most Banach spaces are not automorphic.
- J.M.F. Castillo and A. Plichko [*Banach spaces in various positions.* J. Funct. Anal. 259 (2010) 2098-2138] showed that an automorphic space must be either an $\mathcal{L}_\infty$ space or a B-convex near-Hilbert weak-type 2 space.
- Can the methods above produce an automorphic $\mathcal{L}_\infty$? They can produce, for each cardinal β a Banach space automorphic for all subspaces with density character at most β.

Hereditarily Indecomposable automorphic spaces (do they exist?)

We had three HI candidates to be automorphic: the Argyros-Haydon space, Tarbard space and Ferenczi space.

- (Ferenczi) His own space is not automorphic (it is not near-Hilbert).
- (Ferenczi-Moreno; also Johnson) The Argyros-Haydon space is not automorphic.

Proposition 0.4. *If there exists an HI $\mathcal{L}_\infty$-space with the property that every operator $Y \to X$ from a subspace $i : Y \to X$ has the form $\lambda i + K$ with K compact, it would be automorphic.*

or

Proposition 0.5. *If there exists a reflexive HI space X with the property that for some $K > 0$ and every finite dimensional subspace F of X every norm one operator $F \to X$ can be extended to an operator $X \to X$ with norm at most K then X would be automorphic.*