3 mflds, Slopes, foliations Banff.

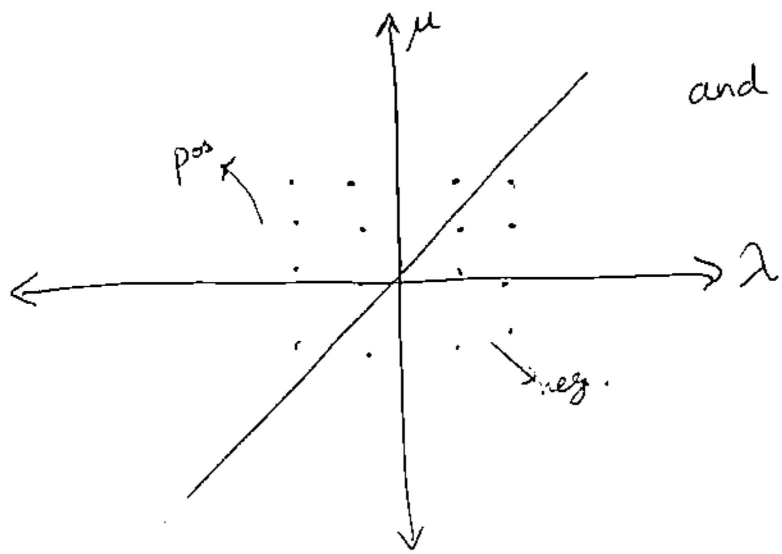
Joint w Liam/Steve. (Goal: foliations $\Leftrightarrow$ $\pi_1(M)$ LO for graph M)

Let M be a 3-mfld (closed, connected, irreducible) with torus boundary.

So BRW $\Rightarrow$ $\pi_1(M)$ is LO.

If we choose a basis $\{\mu, \lambda\}$ of the subgroup
$$\pi_1(\partial M) \cong \pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$$
then every LO of $\pi_1(M)$ 'detects' a slope on the boundary:

$$LO(\pi_1(M)) \xrightarrow{\text{restrict}} LO(\pi_1(T))$$



and get a line defining a slope.
$[\alpha] \in \mathbb{P}H_1(T; \mathbb{R})$.

For 3-mfld theorists, a slope is usually $[\alpha] \in \mathbb{P}H_1(\partial M; \mathbb{Z})$, but we use $\mathbb{R}$ coefficients to allow for lines of irrational slope.

Every rational slope ~~α~~ can be written

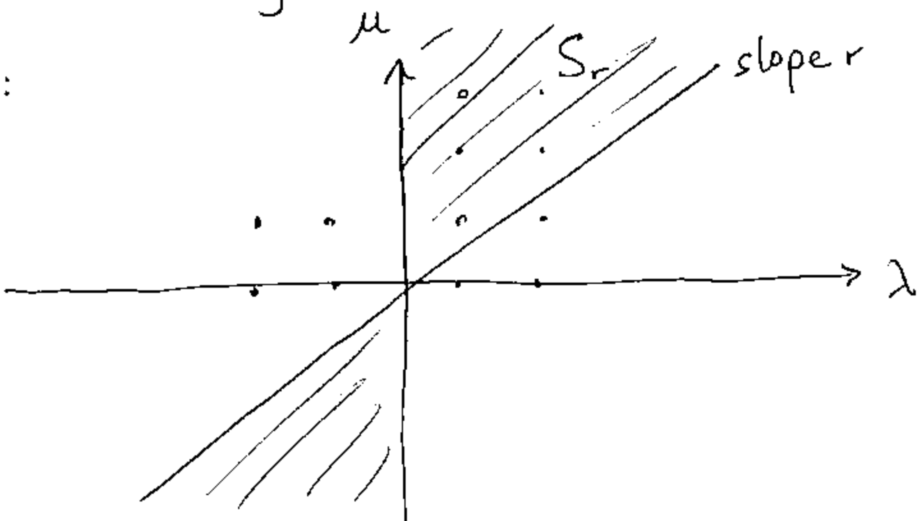
$$[\alpha] = \pm\left(p[\mu] + q[\lambda]\right)$$

so $[\alpha]$ can be associated to $r = p/q \in \mathbb{Q}$.

Detected slopes are related to r-decay

Prop [Watson-C]
A knot K is r-decayed if and only if no left-ordering of $\pi_1(M)$ detects a slope $s > r$.

Proof:



r-decay $\Rightarrow$ all elements of ~~S_r~~ of $S_r \cap \pi_1(\partial M)$ have same sign in every LO.

LO detecting $s > r$ would contradict this.

Related to orderability of $\pi_1(M(\alpha))$

Prop: If $\pi_1(M(\alpha))$ is LO, then α is detected by a LO.

Proof: Build a LO of $\pi_1(M)$ using

$$1 \longrightarrow \langle\langle \alpha \rangle\rangle \longrightarrow \pi_1(M) \longrightarrow \pi_1(M(\alpha)) \longrightarrow 1.$$

and check it restricts to $\pi_1(\partial M)$ to give slope $[\alpha]$.

NOT EVERY DETECTED SLOPE ARISES THIS WAY, i.e. not all detected slopes correspond to LO fillings.

Consider

$M = S^3 \setminus$  , so $\pi_1(M) \cong B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$.

with $\pi_1(\partial M) = \langle \underset{\mu}{\underset{\parallel}{\sigma_1}}, \underset{\lambda}{\underset{\parallel}{\sigma_1^{-6}\Delta^2}} \rangle$ (here $\Delta = \sigma_1\sigma_2\sigma_1$).

From Boyer/Rolfsen/Wiest/Watson/Gordon,

- $\pi_1(M(r))$ is LO for $r < 1$ (so slopes < 1 are detected)
- $\pi_1(M(r))$ is <u>not</u> LO for $r \geq 1$ (may or may not be detected slopes).

However, we can consider a slope as a point in the image of the map:

$$\mathrm{LO}(\pi_1(M)) \xrightarrow{\text{restrict}} \underset{\mathbb{Z}\times\mathbb{Z}}{\underset{\cong}{\mathrm{LO}(\pi_1(\partial M))}} \longrightarrow \underset{\text{"slopes"}}{S^1}$$

Then $\mathrm{LO}(\pi_1(M))$ compact
$\Rightarrow$ set of detected slopes is compact + closed

$\Rightarrow$ Since slopes $r < 1$ are detected in $\pi_1(M) = B_3$, the slope $r = 1$ is detected, but not LO.

I.e. $\exists$ a LO of B_3 with $(\sigma_1^{-5}\Delta^2)^k < \Delta^2 \ \forall k$, but it doesn't come from

$$1 \longrightarrow \langle\langle \sigma_1^{-5}\Delta^2 \rangle\rangle \longrightarrow B_3 \longrightarrow B_3 / \langle\langle \sigma_1^{-5}\Delta^2 \rangle\rangle \longrightarrow 1.$$

So, for <u>rational</u> slopes $[\alpha] \in \mathbb{P}H_1(\overset{\overset{T}{\shortparallel}}{\partial M}; \mathbb{R})$

- We say $[\alpha]$ is <u>strongly</u> detected by a LO if $\pi_1(M(\alpha))$ is LO,

otherwise $[\alpha]$ is just detected.

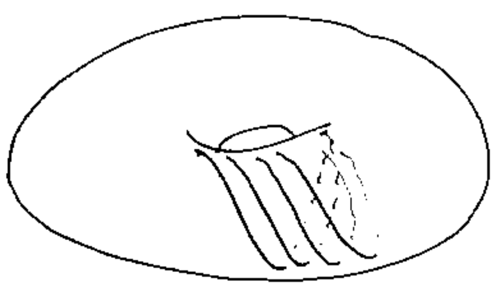
(cooriantable, taut).

Slopes associated to foliations (transverse to ∂M).

Suppose M has $\partial M = T$, and foliation $\mathcal{F}$ meeting ∂M transversely.

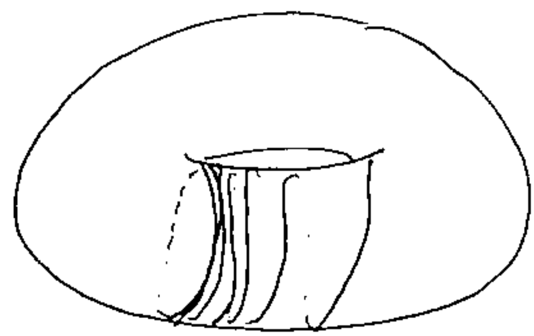
A foliation $\mathcal{F}$ on M 'detects' a rational slope on T if some leaf of $\mathcal{F}|_T$ is a closed curve of that slope.

For example:



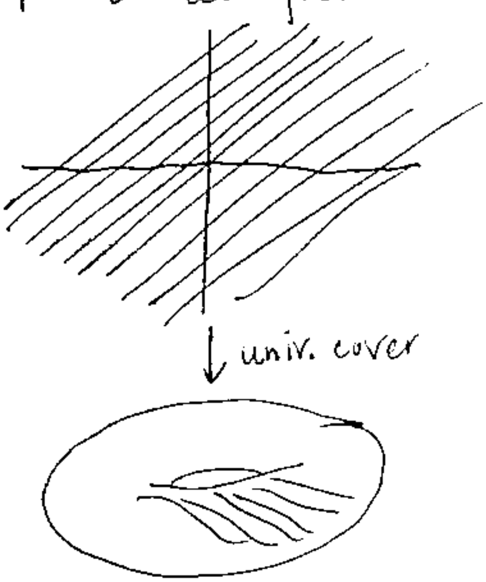
parallel curves

or



some leaves spiral into closed leaves.

In the special case on the left, the foliation on T comes from



parallel lines of slope $[\alpha]$.

parallel curves of slope $[\alpha]$.

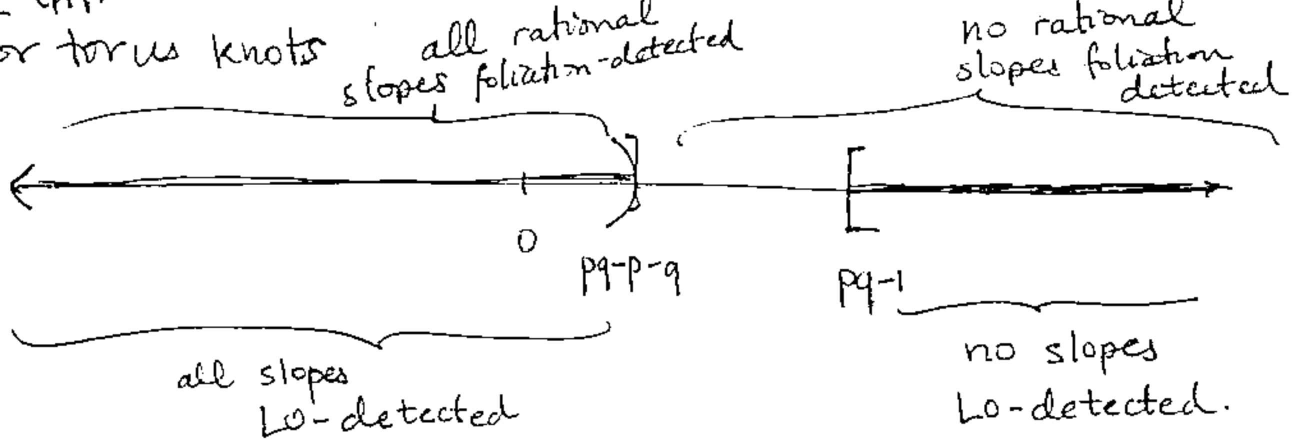
$\mathcal{F}$ strongly detects the slope $[\alpha]$ if the induced foliation on ∂M is parallel curves of slope $[\alpha]$.

- Strongly detected slopes are related to foliations in $\pi_1(M(\alpha))$.

Many results concerning strongly detected slopes:
- Eisenbud Hirsch Naumann Jankins Naimi (SF manifolds)
- Gabai, Roberts, + lots more.

From known foliations / r-decay results, it looks like these notions of ~~slop~~ "detected slopes" should correspond.

For (p,q) torus knots



So we want:

slopes detected by LO's $\longleftrightarrow$ slopes detected by foliations?

We expect this if

$\pi_1(M(\alpha))$ LO $\longleftrightarrow$ $M(\alpha)$ admits coorientable taut foliation.

Restrict to M a SF manifold over $D^2(a_1,\dots,a_n)$ $(\partial M = T)$.

$\pi_1(M) = \langle x_1,\dots,x_n, h \mid x_1^{a_1} = x_2^{a_2} = \dots = x_n^{a_n} = h \rangle$,
and
$\pi_1(\partial M) = \langle x_1 \cdot\cdot x_n, h\rangle \cong \mathbb{Z}\times\mathbb{Z}$.

LO $\Longrightarrow$ foliation (coorientable, horizontal).

Given a LO of $\pi_1(M)$, create the dynamical realization $\rho: \pi_1(M) \longrightarrow \mathrm{Homeo}_+(\mathbb{R})$.

<u>Lemma</u>: h is cofinal in every LO
$\Rightarrow \rho(h)$ has no fixed points
$\Rightarrow \rho(h)$ can be conjugated to $sh(1)$.

So can assume $\rho: \pi_1(M) \longrightarrow \widetilde{\mathrm{Homeo}_+}(S^1)$
$= \{ f \in \mathrm{Homeo}_+(\mathbb{R}) \mid f(x+1) = f(x)+1\}$.

Now build a foliation on M:
If $X \longrightarrow D^2(a_1,\dots,a_n)$ univ. cover, then $\pi_1(M)$ acts on X by $\varphi: \pi_1(M) \longrightarrow \pi_1(D^2(a_1,\dots,a_n))$.

$\pi_1(M)$ acts on $X\times\mathbb{R}$ freely discontinuously by:

$\gamma \in \pi_1(M)$, $\quad (x,t) \in X\times\mathbb{R}$

$$\gamma\cdot(x,t) = (\varphi(\gamma)(x), \rho(\gamma)(t)).$$

Then $M' = X\times\mathbb{R}/\pi_1(M) \simeq M$ inherits a foliation from the copies of X

Coorientable horizontal foliation $\Longrightarrow$ LO.

The foliation is $\mathbb{R}$-covered.

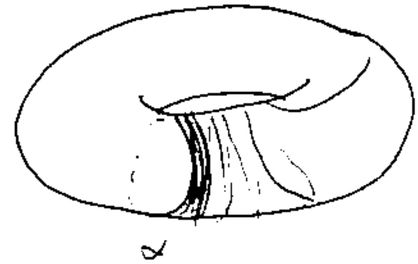
<u>Prop</u>: [Boyer-C] For M as above (SF over D^2)

These constructions establish a correspondence:

LO's detecting slope $[\alpha]$ $\longleftrightarrow$ foliations (coor, horiz) detecting $[\alpha]$

In particular, strongly detected slopes are associated with strongly detected slopes.

detected slope $[\alpha]$ with $\pi_1(M(\alpha))$ non-LO $\longleftrightarrow$ horiz, coor. foliation with



<u>Theorem</u>: [EHN, Naimi, Jankins] + some work

M SF over $D^2(a_1, \ldots, a_n)$ with torus boundary.

The set of detected slopes $T(M)$ is a closed interval with rational endpts, or a single slope.

Every rational ~~endp~~ slope in the interior is strongly detected. The endpts are not.

Example : Gluing two SF pieces.

Take M_1, M_2 with $f: \partial M_1 \to \partial M_2$

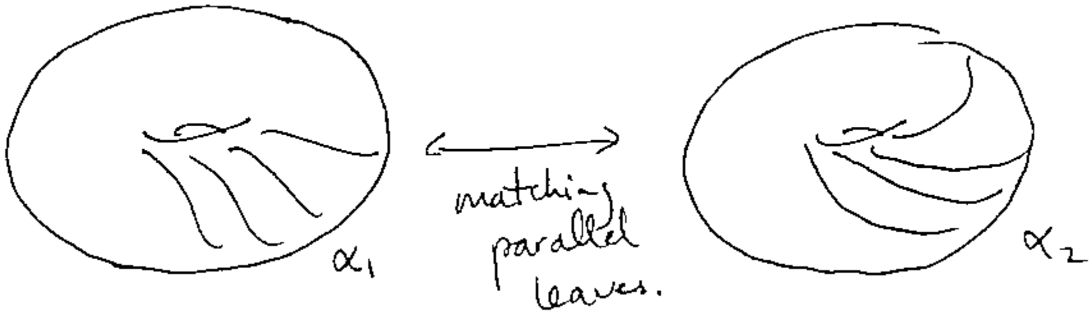
Boyer-Gordon give necessary and suff. conditions for $\pi_1(M_1) *_f \pi_1(M_2)$ to be LO.

We find $\pi_1(M_1 \cup_f M_2)$ is LO if and only if $\exists$ LO detected slopes $[\alpha_1] \in \mathbb{P}H_1(T; \mathbb{R})$ ($T = \partial M_1$) and $[\alpha_2] \in \mathbb{P}H_1(\partial M_2; \mathbb{R})$ s.t. $f([\alpha_1]) = [\alpha_2]$.

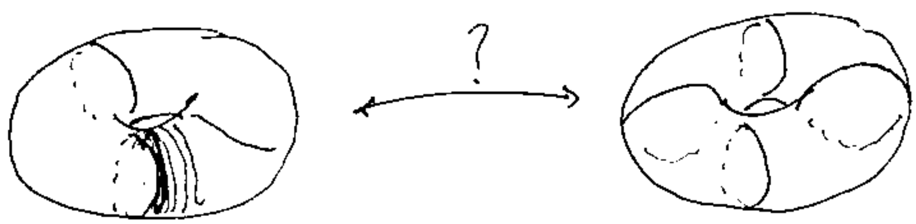
Does this hold for foliations as well? (Yes)
coort, horizon. (at least 2 pieces)

If there exists foliation-detected slopes $[\alpha_i] \in \mathbb{P}H_1(\partial M_i; \mathbb{Z})$ such that $f([\alpha_1]) = [\alpha_2]$ then $M_1 \cup_f M_2$ admits a coorient, horizontal foliation.

Clear if α_1 and α_2 are strongly detected.



but not clear if



So, e.g gluing two such pieces:

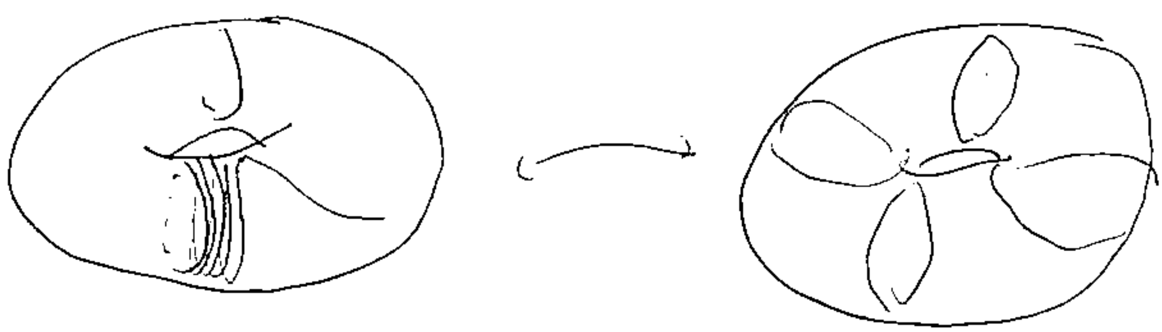
We can determine exactly when $M_1 \cup_T M_2$ is LO using Bludov-Glass.
(detected slope glued to detected slope).

Does this reflect the gluing behaviour of foliations in M_1 and M_2?

Yes, ~~as to~~

If $T(M_1) \cap T(M_2)$ has nonempty interior, it works since we're gluing strongly detected slopes.

If $T(M_1) \cap T(M_2)$ doesn't have interior, we need more work, because we're gluing:



need to have same number of closed leaves.

So for graph manifolds w 2 pieces

$\pi_1(M)$ LO iff M admits coorientable horiz. foliation.

What about ~~$\geq$~~ more pieces?

Here, a problem arises:

If $\exists$ multiple boundary comp. then $\exists$

$$1 \to K \longrightarrow \pi_1(M) \longrightarrow \mathbb{Z} \longrightarrow 1$$
$$h \longmapsto 1.$$

So we can LO $\pi_1(M)$ ~~with~~ detecting the fibre slope h.

$\iff$ foliations with vertical leaves?