BIRS 2012 – Nils Bruin: 6. Workshop Problems – continued

N10. Consider the genus 2 curve

$$X\colon y^2 = -x^6 - x^2 + x + 2$$

We have the following bits of information:

(i) We have the points $P_1 = (1,1) \in X(\mathbb{Q})$ and $P_2 = (1,-1) \in X(\mathbb{Q})$

(ii) $\mathrm{Pic}^0(X/\mathbb{Q}) = \langle G = [P_2 - P_1]\rangle \simeq \mathbb{Z}$.

(iii)

$$\Lambda_3 = \langle 5G\rangle = \langle [Q_1 + Q_2 - 2P_1]\rangle,$$

where

$$Q_1 = (10\sqrt{3} - 44 + O(3^4), \sqrt{3} + 7 + O(3^4))$$
$$Q_2 = (-10\sqrt{3} - 44 + O(3^4), -\sqrt{3} + 7 + O(3^4))$$

(iv) An annihilating 3-adic differential is $\omega = \frac{1}{y}dx + O(3)$.

Together you can use this to determine all rational points on X.

(a) Determine $X(\mathbb{F}_3)$.

(b) Determine the points where ω vanishes modulo 3.

(c) Determine $X(\mathbb{Q})$.

(d) Verify (iii) assuming (ii)

(e) Verify (iv)

N11. A classic curve for explicit Chabauty equations is a curve considered by Poonen, Schaefer and Stoll, arising from considering periodic points under quadratic polynomial maps.

$$X\colon y^2 = x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1$$

(a) Verify that X has good reduction at $p = 3$.

(b) Assume that $[(-3,1) - (0,1)]$ generates Λ_3. Determine an annihilating 3-adic differential. [x is a good uniformizer for $(0,1)$].

(c) Determine $X(\mathbb{F}_3)$.

(d) In order to analyze the points with $x = \infty$, change coordinates to $(z,w) = (\frac{1}{x}, \frac{y}{x^3})$. Verify that modulo 3, the annihilating differential does not have a zero with $z = 0$.

(e) By expanding ω to a little higher precision around $(x,y) = (0,1)$, you can read off that there are at most two rational points that reduce to $(0,1)$ modulo 3.

(f) You can assemble this to a full determination of all rational points on X.

N12. Prove the baby version of Strassman's Lemma: Let $f(z) \sum_{i=0}^{\infty} a_i z^i \in \mathbb{Z}_p[[z]]$ be a power series that converges on $\mathbb{Z}_p$ (i.e., $\lim_{i\to\infty} v_p(a_i) = \infty$). Suppose that $v_p(a_i) > v_p(a_1)$ for all $i = 2,\ldots$. Then $f(z)$ has only one root in $\mathbb{Z}_p$.

Worthwhile questions from previous exercise batches: **N7**, **N3**, except you probably want to do $x^3 - 2y^3 = 5$ instead, in view of $27 - 16 = 11$.