Tail bounds and extremal behavior of light-tailed perpetuities

Paweł Hitczenko
Drexel University

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Assuming the first term is negligible and re–numbering (Q_n, M_n)

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And for the almost sure convergence to 0 of

$$R_0 \prod_{j=1}^{n} M_j.$$

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- This basic result has been re–proved and extended by a number of researchers, among others **Goldie** (1991), **Grey** (1994), **Grincievičjus** (1975) ...

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- Such perpetuities are nowadays called Vervaat perpetuities.

Light tails, cont.

- in the special case $\alpha = 1$ the density of Vervaat perpetuity is (up to normalizing constant) the Dickman function $\rho(u)$ appearing in number theory:

$$\rho(u) = \lim_{n \to \infty} \frac{k_n(u)}{n}$$

  where $k_n(u)$ is the number of positive integers $\leq n$ with the largest prime factor no more than $n^{1/u}$, $u \geq 1$.

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- other appearances of Dickman function are discussed in **Hwang and Tsai** (2001) and include the analysis of `Quickselect` algorithm, the degree of the largest irreducible factor in a random polynomial over finite field, and allele frequencies in some biological models.

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for some $\epsilon > 0$, $0 < c, C < \infty$ and for all $\delta \in (0, \epsilon]$

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$$\lim_{x \to \infty} \frac{\ln(P(R \geq x))}{x \ln x} = -\frac{1}{q}.$$

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X is c-decomposable if $\forall c \in [0, 1]\ \exists X_c :\ X \stackrel{d}{=} cX + X_c$, with X and X_c independent on the rhs.

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- In particular, if $Q \equiv q > 0$ and $0 \leq M \leq 1$ then

$$\exp(\frac{2\ln 2}{q} x \ln p_{q/(2x)}) \leq P(R > x) \leq \exp(\frac{1}{4q} x \ln p_{2q/x}).$$

Comments on proof

- techniques for the cases $0 \leq M \leq 1$ and $P(M > 1) > 0$ are completely different.
- techniques previously used for an upper bound in the case $0 \leq M \leq 1$ were generally based on an iteration of the equation $R_n \stackrel{d}{=} M_n R_{n-1} + Q_n$ and they don't seem to work.
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- However, a proof of a lower bound of Goldie–Grübel may be used to yield an upper bound.
- Rough idea for the lower bound: for a small δ, wait for the first time when $M_k \leq 1 - \delta$. Up to that time bound the partial sums forming R_n below by a geometric sum.
- For the upper bound: keep recording consecutive times when $M_k \leq 1 - \delta$, bound above the partial sums by weighted sums of geometric r.v.'s and use exponential bounds for such sums (**Goh, H.** (2008)).

Extremal behavior: heavy tails

We want to analyze the extremal behavior of (R_n) i.e. look at the normalizing constant a_n and b_n so that

$$a_n(\max_{0\le k\le n} R_n - b_n)$$

converges in distribution to a non-degenerate random variable. The theory for i.i.d. sequences (R_n) is completely developed and goes back to **Fisher-Tippett** (1928) and **Gnedenko** (1943) and is presented e.g. in a classic **Leadbetter, Lindren and Rootzén** (1988). The situation is also well understood when (R_n) is a stationary sequence. In our case, if (R_n) converges in distribution to R we can take $R_0 \stackrel{d}{=} R$ and turn (R_n) into a stationary sequence.

Extremal behavior: heavy tails, cont.

de Haan, Resnick, Rootzén, de Vries (1989) showed that

- for $M, Q \geq 0$ under Kesten's conditions for the convergence of (R_n) and $P(M > 1) > 0$ (which implies that $P(R > x) \sim cx^{-\kappa}$) we have

$$\lim_{n\to\infty} P(\frac{R_n^*}{n^{1/\kappa}} \leq x) = \exp(-c\theta x^{-1/\kappa}),$$

  where $R_n^* = \max_{1\leq k\leq n} R_k$.

- That is, there is a convergence to Type II (Frechét) distribution with normalizing constants $a_n = 1/n^{1/\kappa}$ and $b_n = 0$.

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- That is, there is a convergence to Type II (Frechét) distribution with normalizing constants $a_n = 1/n^{1/\kappa}$ and $b_n = 0$.
- $\theta = \kappa \int_1^\infty P(\sup_{j\geq 1} \prod_{i=1}^j M_i \leq \frac{1}{y}) \frac{dy}{y^{\kappa+1}}$, is the *extremal index* of the sequence (R_n).
- the existence of such $\theta \in [0,1]$ (NOT assured in general even for stationary sequences) says that R_n^* behaves like max of $\sim \theta n$ i.i.d. variables with the same marginal distribution.

Extremal behavior: light tails

Theorem (H. (2010)): *Let* $R_n = M_n R_{n-1} + q$ *where* $q > 0$, $0 \le M \le 1$ M *is non-degenerate,* $P(M = 0) = 0$, *and* $\sup\{x : \; P(M > x) > 0\} = 1$. *Then there exist* (a_n), (b_n) *such that*

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- the assumptions on M are needed to exclude trivial cases and the case when R (and hence each R_n) is geometric.
- We may take b_n so that $b_n \ln p_{c/b_n} = -\Theta(\ln n)$ and

$$a_n = \Theta\left(\frac{1}{b_n \ln p_{c/b_n}} f_M(1 - \frac{c}{b_n}) - \ln p_{c/b_n}\right) \overset{*}{\sim} -\Theta(\ln p_{c/b_n}),$$

  where '$\overset{*}{\sim}$' means 'often $\sim$' and c is a constant.
- the extremal index (built-in in a_n, b_n) is $\theta = 1 - P(M = 1)$.

Open problems: tail behavior

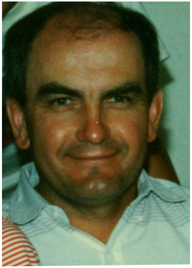
- Get the asymptotics for $P(R > x)$ when $R \stackrel{d}{=} MR + q$, $0 \leq M \leq 1$, $q > 0$.
  Knowing the tail behavior would give the asymptotics of the normalizing constants a_n, b_n in the limit theorem for the extremes.
- Get rid of the assumption $Q \equiv q$ and/or $|Q| \leq q$.
  Without that some of the basic cases when we know the tail behavior are not covered, e.g. the α–stable distributions:

$$R \stackrel{d}{=} 2^{-1/\alpha}(R + R') \stackrel{d}{=} MR + Q; \;\; M = 2^{-1/\alpha}, Q \stackrel{d}{=} 2^{-1/\alpha} R$$

or

$$M \stackrel{d}{=} \beta(\alpha_1, \alpha_2), Q \stackrel{d}{=} \Gamma(\alpha_2, \gamma) \Longrightarrow R \stackrel{d}{=} \Gamma(\alpha_1 + \alpha_2, \gamma).$$

Thank you :)



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