Permanental processes

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Introduction

Let T be an index set and $\{G(x), x \in T\}$ be a mean zero Gaussian process with covariance $u(x,y)$, $x,y \in T$. The process G^2 can be defined by the Laplace transform of its finite joint distributions

$$E\left(\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\alpha_i G^2(x_i)\right)\right) = \frac{1}{|I+\alpha U|^{1/2}} \tag{1}$$

for all $x_1,\ldots,x_n$ in T, where I is the $n \times n$ identity matrix, α is the diagonal matrix with $(\alpha_{i,i} = \alpha_i)$, $\alpha_i \in R_+$ and $U = \{u(x_i,x_j)\}$ is an $n \times n$ matrix, that is symmetric and positive definite. (It is the covariance of $(G(x_1),\ldots,G(x_n))$.)

In 1997, D. Vere-Jones introduced the permanental process $\theta := \{\theta_x, x \in T\}$, which is a real valued positive stochastic process with finite joint distributions that satisfy

$$E\left(\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\alpha_i\theta_{x_i}\right)\right) = \frac{1}{|I + \alpha\Gamma|^{\beta}}, \tag{2}$$

where $\Gamma = \{\Gamma(x_i, x_j)\}_{i,j=1}^{n}$ is an $n \times n$ matrix and $\beta > 0$. (We refer to this as a β–permanental process.)

The generalization here is that Γ need not be symmetric.

For $n = 2$, and $\beta = 1/2$, (2) takes the form

$$E\left(\exp\left(-\frac{1}{2}\left(\alpha_1\theta_x + \alpha_2\theta_y\right)\right)\right)$$
$$= \frac{1}{|I + \alpha\Gamma|^{1/2}} = (1 + \alpha_1\Gamma(x,x) + \alpha_2\Gamma(y,y)$$
$$+\alpha_1\alpha_2\left(\Gamma(x,x)\Gamma(y,y) - \Gamma(x,y)\Gamma(y,x)\right))^{-1/2}.$$

For permanental processes

$$\Gamma(x,x) \geq 0, \qquad \Gamma(x,y)\Gamma(y,x) \geq 0.$$

and

$$\Gamma(x,x)\Gamma(y,y) - \Gamma(x,y)\Gamma(y,x) \geq 0.$$

Therefore, the matrix

$$\begin{bmatrix} \Gamma(x,x) & (\Gamma(x,y)\Gamma(y,x))^{1/2} \\ (\Gamma(x,y)\Gamma(y,x))^{1/2} & \Gamma(y,y) \end{bmatrix}$$

is positive definite, so that we can construct a mean zero Gaussian vector $\{G(x), G(y)\}$ with covariance matrix

$$E\,(G(x)G(y)) = (\Gamma(x,y)\Gamma(y,x))^{1/2}\,. \tag{3}$$

WE MAY HAVE $\Gamma(x,y) \neq \Gamma(y,x)$

Set

$$d(x,y) = \left(E(G(x)-G(y))^2\right)^{1/2}$$
$$= \left(\Gamma(x,x)+\Gamma(y,y)-2\left(\Gamma(x,y)\Gamma(y,x)\right)^{1/2}\right)^{1/2}.$$

Lemma

Suppose that $\theta := \{\theta_x, x \in T\}$ is a 1/2-permanental process with kernel Γ. Then for any pair x, y,

$$\{\theta_x, \theta_y\} \overset{law}{=} \{G^2(x), G^2(y)\} \tag{4}$$

where $\{G(x), G(y)\}$ is a mean zero Gaussian random variable with covariance matrix given by (3).

The function $d(x,y)$ is a metric, or pseudo-metric, on T, (although $(\Gamma(x,y)\Gamma(y,x))^{1/2}$ may not be positive definite.)

Let (T,d) be a separable metric or pseudometric space. Let $B_d(t,u)$ denote the closed ball in (T,d) with radius u and center t. For any probability measure μ on (T,d) we define

$$J_{T,d,\mu}(a)=\sup_{t\in T}\int_0^a\left(\log\frac{1}{\mu(B_d(t,u))}\right)^{1/2}du.$$

A sufficient condition for continuity

Theorem

Let $\theta = \{\theta_x : x \in T\}$ *be a 1/2-permanental process, with kernel* Γ *satisfying* $\sup_{x\in T}\Gamma(x,x) < \infty$*. Let* D *denote the* d *diameter of* T *and assume that* T *is separable for* d*, and that there exists a probability measure* μ *on* $\mathcal{B}(T,d)$ *such that*

$$J_d(D) < \infty.$$

Then there exists a version $\theta' = \{\theta'_x, x \in T\}$ *of* θ *which is bounded almost surely.*

Theorem (continued)

If

$$\lim_{\delta \to 0} J_d(\delta) = 0,$$

there exists a version $\theta' = \{\theta'_x, x \in T\}$ *of* θ *such that*

$$\lim_{\delta \to 0} \sup_{\substack{s,t \in T \\ d(s,t) \le \delta}} |\theta'_s(\omega) - \theta'_t(\omega)| = 0, \qquad a.s.$$

If (3) holds and

$$\lim_{\delta \to 0} \frac{J_d(\delta)}{\delta} = \infty,$$

then

Theorem (continued)

$$\lim_{\delta\to 0} \sup_{\substack{s,t\in T\\ d(s,t)\le\delta}} \frac{|\theta'_s-\theta'_t|}{J_d(d(s,t)/2)} \le 30\left(\sup_{x\in T}\theta'_x\right)^{1/2} \qquad a.s.$$

The proof is immediate. It follows from Lemma 1 that

$$\begin{aligned}\hat{d}(x,y) := \|\theta_x^{1/2}-\theta_y^{1/2}\|_{\psi_2} &= \|\, |G_x|-|G_y|\, \|_{\psi_2} \\ &\le \|\, G_x-G_y\, \|_{\psi_2} = d(x,y).\end{aligned} \tag{5}$$

Also $|\theta_s-\theta_t| \le |\theta_s^{1/2}-\theta_t^{1/2}|\,(\sup_{x\in T}\theta'_x)^{1/2}$

Verre-Jones shows that a sufficient condition for (2) to hold is that all the real non-zero eigenvalues of Γ are positive and that $r\Gamma(I + r\Gamma)^{-1}$ has only non-negative entries for all $r > 0$. Eisenbaum and Kaspi note that this is the case when $\Gamma(x, y)$, $x, y \in T$, is the potential density of a transient Markov process on T. This enables them to find a Dynkin type isomorphism for the local times of Markov processes that are not necessarily symmetric, in which the role of G^2 is taken by the permanental process θ.

Permanental processes associated with Lévy processes

Certain kernels of permanental processes are associated with Lévy processes. Let $X = \{X_t, t \in R_+\}$ be a Lévy process with characteristic function

$$Ee^{i\lambda X_t} = e^{-\psi(\lambda)t}. \tag{6}$$

Assume that X has local times $\{L_t^x, (x,t) \in R \times R_+\}$. Set

$$u_{T_0}(x,y) = E^x\left(L_{T_0}^y\right), \tag{7}$$

where T_0 is the first hitting time of X at zero.

The function $u_{T_0}(x, y)$ is the zero potential of the transient Markov process $\widetilde{X} = \{\widetilde{X}_t\}$, which is X killed at the first time it hits zero, and thus is also the kernel of a permanental process.

Lemma

$$u_{T_0}(x,y) = R(x,y) + H(x,y)$$

and

$$u_{T_0}(y,x) = R(x,y) - H(x,y)$$

where

Lemma (continued)

$$R(x,y) = R(y,x) = \frac{1}{\pi}\int_0^\infty \frac{(1-\cos\lambda x - \cos\lambda y + \cos\lambda(x-y))\,\mathcal{R}e\,\psi(\lambda)}{|\psi(\lambda)|^2}\,d\lambda$$

and

$$H(x,y) = -H(y,x) = \frac{1}{\pi}\int_0^\infty \frac{(\sin\lambda x - \sin\lambda y - \sin\lambda(x-y))\,\mathcal{I}m\,\psi(\lambda)}{|\psi(\lambda)|^2}\,d\lambda.$$

Perm. Proc. are Loop Soups

Following Le Jan, (for symmetric Markov processes), we can identify the peramnental process as an explicit process called a loop soup local time.

Let S a be locally compact set with a countable base. Let $Y = (\Omega, Y_t, P^x, \mathcal{F}_t)$ be a recurrent Markov process with state space S, and jointly measurable transition densities $p_t(x,y)$ with respect to some σ-finite measure m on S. We assume that the 1-potential densities

$$u^1(x,y) = \int_0^\infty e^{-t} p_t(x,y)\,dt \tag{8}$$

are continuous. We do not require that $u^1(x,y)$ is symmetric.

For all $0 < t < \infty$, and $x \in S$, there exists a finite measure $P_t^{x,x}$ on $\mathcal{F}_{t-}$, of total mass $p_t(x,x)$, such that

$$P_t^{x,x}(F) = P^x\left(F\, p_{t-s}(Y_s, x)\right), \tag{9}$$

for all $F \in \mathcal{F}_s$ with $s < t$.

For $\Delta \notin S$, let Ω_Δ denote the set of right continuous paths ω in $S \cup \Delta$ with $\omega_t = \Delta$ for all $t \geq \zeta$. We set $Y_t(\omega) = \omega_t$ and

$$\zeta = \inf\{t > 0 \,|, Y_t = \Delta\}. \tag{10}$$

The killing time ζ is determined by an operator $k_t\omega(s) = \omega(s)$ if $s < t$ and $k_t\omega(s) = \Delta$ if $s \geq t$.

We define a σ-finite measure μ on $(\Omega_\Delta, \mathcal{F})$ by

$$\mu(A) = \int_0^\infty \frac{e^{-t}}{t} \int P_t^{x,x}\left(k_t^{-1}(A)\right)\, dm(x)\, dt, \qquad A \in \mathcal{F}. \tag{11}$$

We refer to μ as the loop measure associated with the Markov process Y. Under certain regularity assumptions on Y, μ is supported on

$$\mathcal{L} = \{Y : Y_{\zeta^-} = Y_0\}. \tag{12}$$

We call μ the loop measure for the Markov process Y.

Let $\mathcal{L}_\alpha$ be a Poisson point process on Ω_Δ with intensity measure $\alpha\mu$. Each realization of the random variable $\mathcal{L}_\alpha$ is countable subset of Ω_Δ. I.e. let

$$N(A) := \#\{\mathcal{L}_\alpha \cap A\}, \qquad A \subseteq \Omega_\Delta. \tag{13}$$

Then for any disjoint measurable subsets $A_1, \ldots, A_n$ of Ω_Δ, the random variables $N(A_1), \ldots, N(A_n)$, are independent, and $N(A)$ is a Poisson random variable with parameter $\alpha\mu(A)$, i.e.

$$P\left(N(A) = k\right) = \frac{(\alpha\mu(A))^k}{k!} e^{-\alpha\mu(A)}. \tag{14}$$

The Poisson point process $\mathcal{L}_\alpha$ is called the 'loop soup' of the Markov process Y.

We define the 'loop soup local time', $\widehat{L}^x$, of Y, by

$$\widehat{L}^x_\alpha = \sum_{\omega \in \mathcal{L}_\alpha} \ell^x(\omega), \tag{15}$$

where $\ell^x(\omega)$ is the local time of the path $\omega \in \Omega_\Delta$.

Theorem

Let $\{\widehat{L}^x_\alpha, x \in S\}$ be the loop soup local time of Y. Then $\{2\,\widehat{L}^x_\alpha, x \in S\}$, is an α-permanental process with kernel $u^1(x,y)$.