Some Lasso procedure for multivariate counting processes and its particular link with some exponential inequalities for martingales

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(Conditional) Intensity

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An intensity is a predictable process wrt a filtration that defines "past". If it exists, $\int_0^t \lambda(x)dx$ is the compensator of N_t, ie

$$M_t = N_t - \int_0^t \lambda(x)dx$$

is a (local) martingale.

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- Hawkes ...

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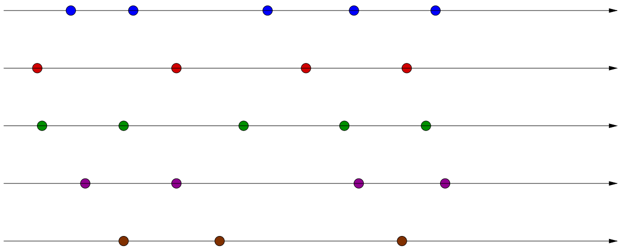

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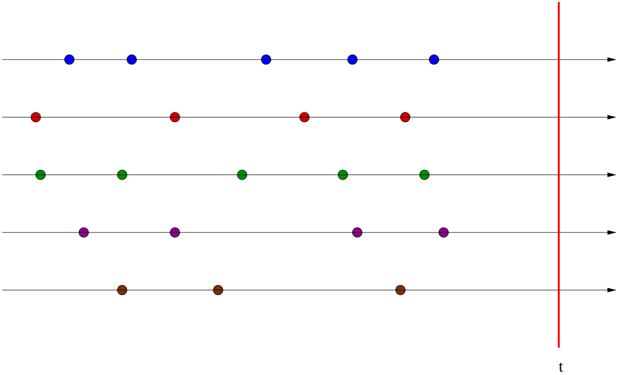

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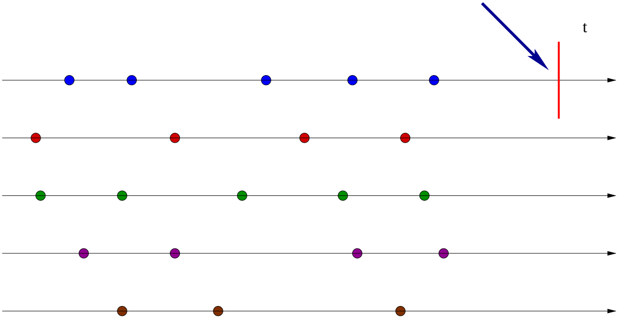

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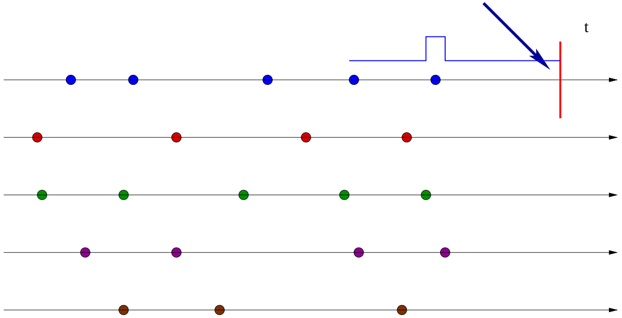

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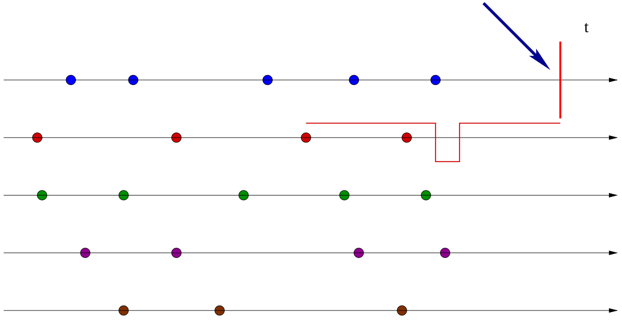

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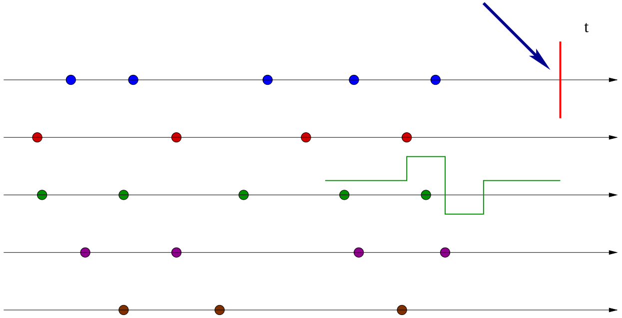

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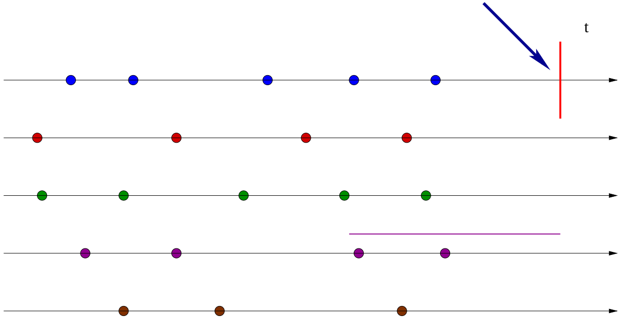

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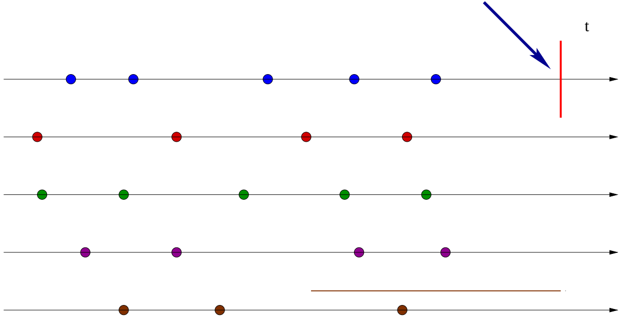

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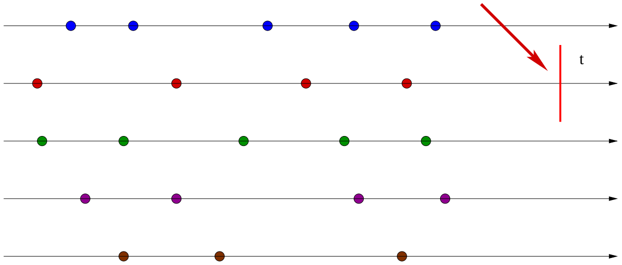

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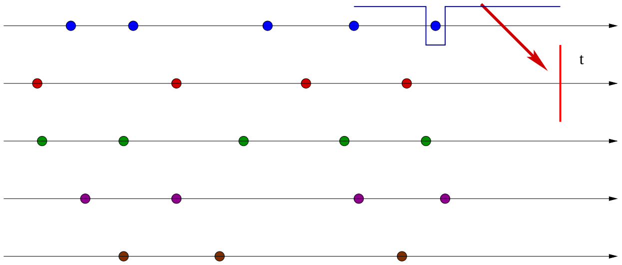

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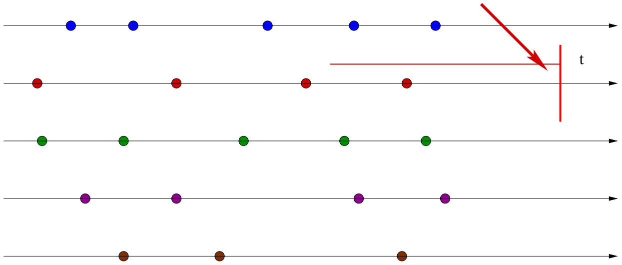

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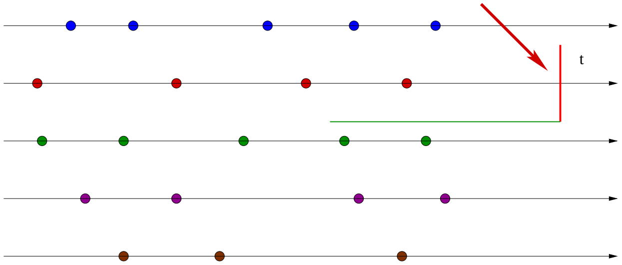

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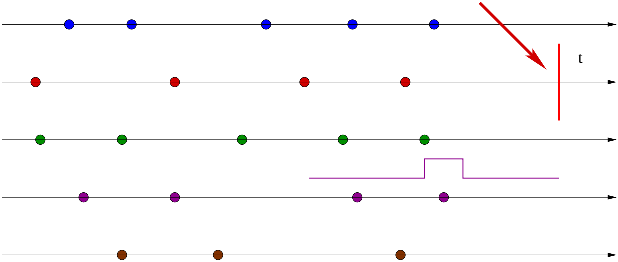

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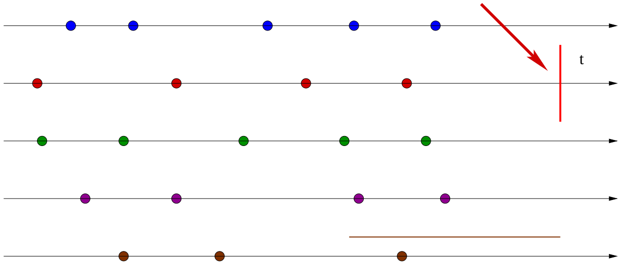

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Link with graphical model of local independence (see Didelez (2008)). Estimating the interaction functions and finding out which one is zero gives a picture of the synergy between the different processes (neurons, elements)

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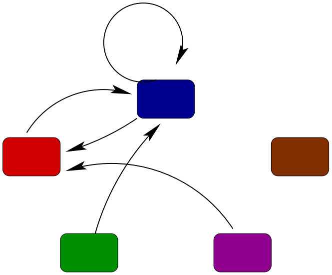

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We want to estimate $s = \left((\nu_r, (h_\ell^{(r)})_{\ell=1,\dots,M})_{r=1,\dots,M}\right)$ in

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Intensity candidate per mark

$\psi_f^{(r)}(t) = \mu_r + \sum_\ell \int_{-\infty}^t g_\ell^{(r)}(t-u)dN_u^{(\ell)}$.

Lasso estimate

- $\Phi = (\phi_\lambda)_{\lambda\in\Lambda}$ = dictionary in $\mathcal{H}$(Orthonormal family ...) and $f = \sum_{\lambda\in\Lambda} a_\lambda\phi_\lambda$. (Hope : decomposition of s sparse)

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$$\gamma(f) = \sum_{m=1}^{M} \left(-2 \int_0^T \Psi_f^{(m)}(t) dN_t^{(m)} + \int_0^T [\Psi_f^{(m)}(t)]^2 dt \right).$$

to minimize in order to find a good estimate.

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since $\gamma(f) \simeq -2 \sum_m \int \Psi_f^{(m)}(t) \Psi_s^{(m)}(t) dt + \sum_m \int [\Psi_f^{(m)}(t)]^2 dt$
minimal when $\Psi_f^{(m)} = \Psi_s^{(m)} \rightsquigarrow f = s$

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$\gamma(f) = -2a'b + a'Ga$ with

$$b_{\lambda_1} = \sum_{m=1}^{M} \int_0^T \Psi^{(m,\lambda_1)} dN_t^{(m)}, \quad G_{\lambda_1,\lambda_2} = \sum_{m=1}^{M} \int_0^T \Psi^{(m,\lambda_1)} \Psi^{(m,\lambda_2)} dt.$$

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Quadratic form (norm ?)

$$\|f\|^2_{T,M} = \sum_{m=1}^{M} \int_0^T [\Psi_f^{(m)}(t)]^2 dt.$$

An analytical result

Theorem

Let $c > 0$. *If*

1. $\inf_{x \in \mathbb{R}_*^{|\Lambda|}} \frac{x' G x}{\|x\|_{\ell^2}^2} \geq c$,
2. $\forall \lambda \in \Lambda, \quad |b_\lambda - \bar{b}_\lambda| \leq d_\lambda$, *where* $\bar{b}_\lambda = \sum_{m=1}^{M} \int_0^T \Psi^{(m,\lambda)}(t) \Psi_s^{(m)}(t) dt$,

then, there exists an absolute constant C *such that*

$$\|\hat{s} - s\|_{T,M}^2 \leq C \inf_{a \in \mathbb{R}^{|\Lambda|}} \left\{ \left\| s - \sum_{\lambda \in \Lambda} a_\lambda \phi_\lambda \right\|_{T,M}^2 + c^{-1} \sum_{\lambda \in S(a)} (d_\lambda)^2 \right\},$$

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Oracle inequality (see also Tsybakov (et al.), Bertin, Le Pennec, Rivoirard (2011))

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2. $\forall \lambda \in \Lambda, \quad |\sum_{m=1}^{M} \int_0^T \Psi^{(m,\lambda)}(t)(dN_t^{(m)} - \Psi_s^{(m)}(t)dt)| \leq d_\lambda,$
   Choice of d_λ crucial to have a full data-driven procedure

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  $\mathbb{P}\left(M_\tau \geq \sqrt{2\rho x} + Bx/3 \text{ and } \int_0^\tau H_t^2 \lambda(t)dt \leq \rho \text{ and } \sup_{t\leq\tau} |H_t| \leq B\right)$
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- (Dzhaparidze and van Zanten (2001), Barlow, Jacka, Yor (1986), de la Peña (1999) and Bercu and Touati (2008)) If symetric (or heavy on the left)
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One satisfying exponential inequality

Theorem

Let $B > 0$ and $v > w > 0$. For every $x > 0$ and $\mu > 0$ such that $\mu > \phi(\mu)$, define

$$\hat{V}_t^{\mu} = \frac{\mu}{\mu - \phi(\mu)} \int_0^t H_s^2 dN_s + \frac{B^2 x}{\mu - \phi(\mu)},$$

where $\phi(u) = \exp(u) - 1 - u$. Then for any almost surely finite stopping time τ and any $\varepsilon > 0$

$$\mathbb{P}\left(M_\tau \geq \sqrt{2(1+\varepsilon)\hat{V}_\tau^{\mu} x} + \frac{Bx}{3} \text{ and } w \leq \hat{V}_\tau^{\mu} \leq v \text{ and } \sup_{t \in [0,\tau]} |H_t| \leq B \right)$$

$$\leq 2 \frac{\log(v/w)}{\log(1+\varepsilon)} e^{-x}.$$

inspired by Lipster and Spokoiny (2000)