Convectively unstable fronts in high Lewis number combustion model and some other examples

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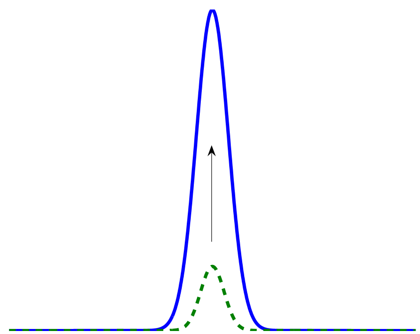
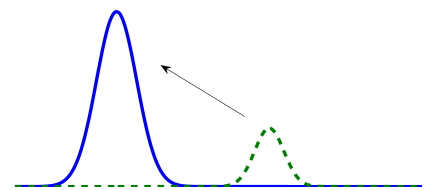
Convective instability

- **Instability:** **not only growth but also propagation of perturbations**

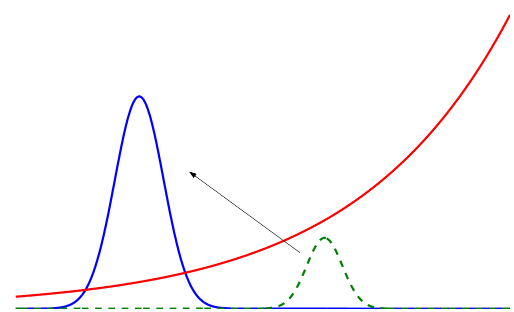


absolute



convective

Exponential weights:



Far Field Spiral Breakup

SPIRALS: **aggregation of Dictyostelium amoebae**

catalytic oxidation of CO on platinum

Belousov-Zhabotinski chemical reaction,...

BZ reaction:

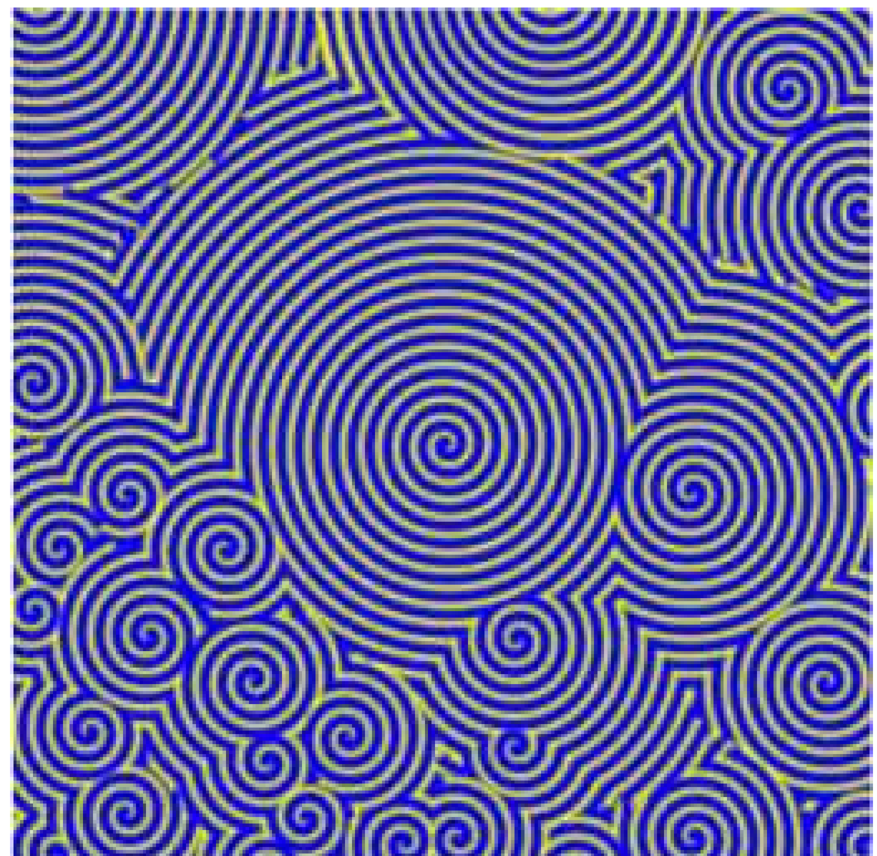
oxidation of malonic acid by

bromade in the acidic solution,

catalyzed by a metal ion.

Laboratory experiments, 1996:

[Belmonte, Flesselles, Ouyang]

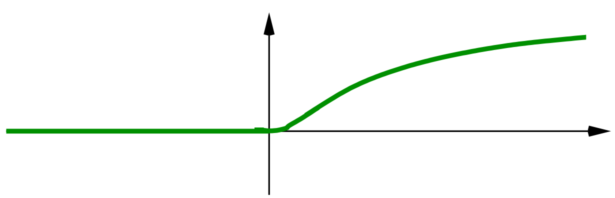


Combustion model

$$\partial_t u_1 = \partial_{xx} u_1 + u_2 \Omega(u_1) \qquad x \in \mathbb{R}, \quad t > 0$$

$$\partial_t u_2 = \frac{1}{\mathrm{Le}} \partial_{xx} u_2 - \beta u_2 \Omega(u_1)$$

$\Omega(u_1) = e^{-1/u_1}$ **for** $u_1 > 0$ **and** 0 **for** $u_1 \leq 0$



u_1 **- temperature,** u_2 **- concentration**

β **- exothermicity**

$\varepsilon = 1/\mathrm{Le}$, Le **- Lewis number**

$\varepsilon > 0$: **gaseous combustion - fuels with a gaseous phase;**

burning of synthetic polymers, cedar wood

$\varepsilon \ll 1$: **fuels with a liquid phase or high density liquid fuels;**

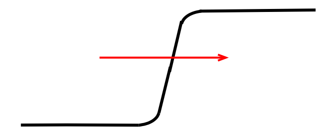
burning of toxic wastes

$\varepsilon = 0$: **solid fuels - gasless combustion, no liquid phase;**

arises in synthesis of ceramic and metallic alloys (thermites)

Combustion Fronts

$$u_1(x,t) = u_1(\xi), u_2(x,t) = u_2(\xi), \quad \xi = x - ct$$



Rest states: $(u_1, u_2) = (1/\beta, 0)$ **completely burned state** $-\infty$

$(u_1, u_2) = (0, 1)$ **unburned state** ∞

$$u_1'' + cu_1' = -u_2 \Omega(u_1)$$

$$\varepsilon u_2'' + cu_2' = \beta u_2 \Omega(u_1)$$

- **no standing fronts** $c \neq 0$**; fix the direction of propagation** $c > 0$
- **there exists a conserved quantity:** $\beta u_1' + \beta c u_1 + \varepsilon u_2' + c u_2 = c$
- **solutions are monotone**
- **seek fronts with exponential decay to the equilibria**

[Varas, Vega: 2002]

Existence and uniqueness of fronts $0 \leq \varepsilon \ll 1$

$0 < \varepsilon < 1$ **Leray-Schauder Degree Theory**

[Berestycki, Nicolaenko, Scheurer: 1985]

$0 < \varepsilon \ll 1$ **Geometric Singular Perturbation Theory**

- **Melnikov integral calculations**

  [Balasuriya, Gottwald, Hornibrook, Lafortune: 2007]

- **Direct check of transversality** [G., Jones]

$\varepsilon \geq 0$ **Numerical observations**

[Balasuriya, Gottwald, Hornibrook, Lafortune: 2007]

[Weber, Mercer, Sidhu, Gray: 1997]

$\varepsilon = 0$ **Phase plane analysis** [Billingham: 2000] [Varas, Vega: 2002]

Stability

- $\varepsilon = 1$ [Roquejoffre, Terman, 1993]
- relation between cases $\varepsilon = 0$ and $\varepsilon \ll 1$ [G., Jones, 2009]
- $\varepsilon \ll 1$ numerical calculation of the spectrum

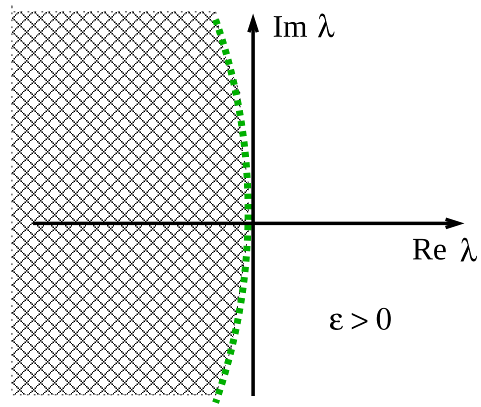
  [Weber, Mercer, Sidhu, Gray, 1997]

  [G., Humpherys, Lytle]
- There exists a parameter regime when the instability is convective

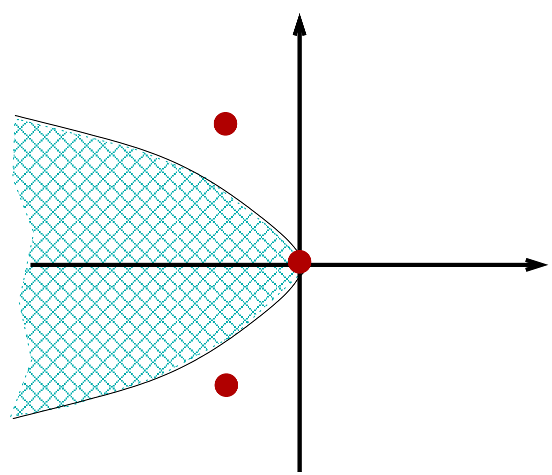
  [G., 2009]

Stability Analysis: Spectrum

Essential spectrum

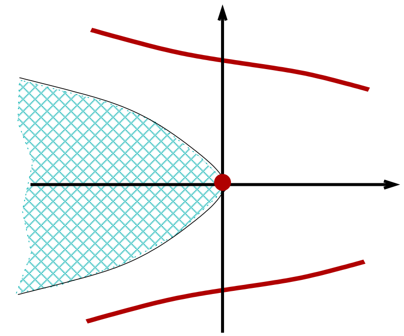


Discrete spectrum; numerics, $0 < \varepsilon \ll 1$ There exists β^* such that for any fixed $\beta < \beta^*$ [Weber et al: 1997]



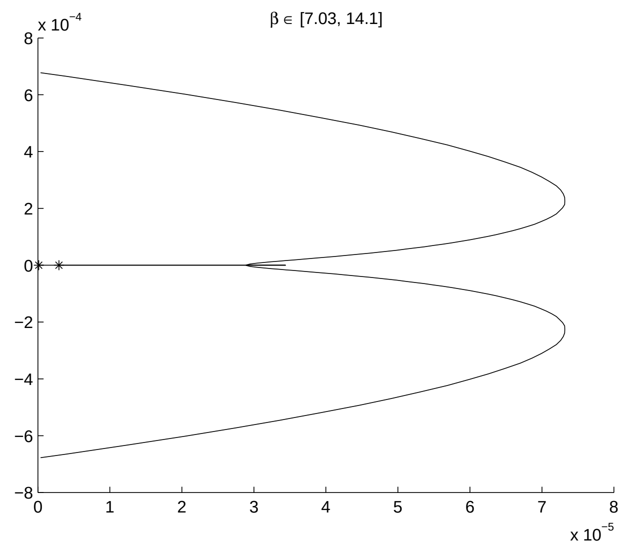
Instability, $\beta > \beta^*$

[Weber et al: 1997]



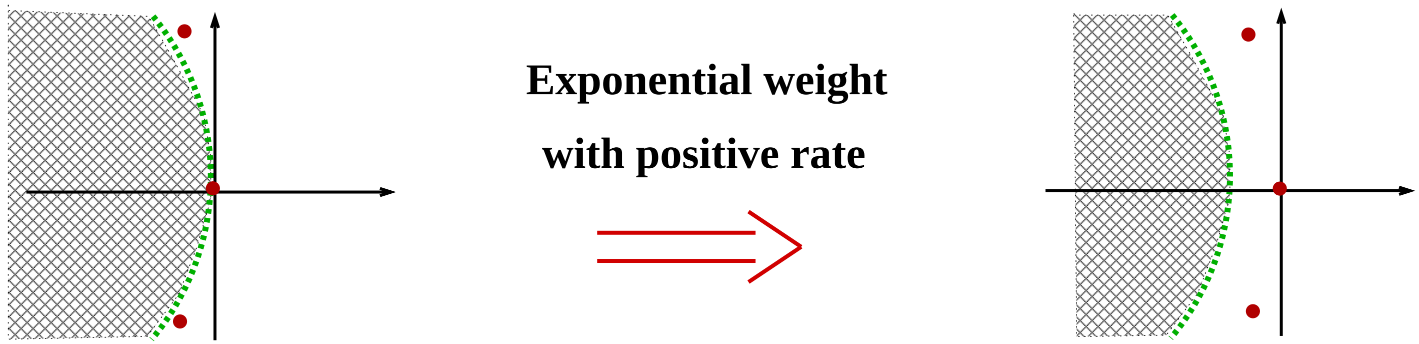
Energy-like estimates $\Longrightarrow$ $\mathrm{Re}\,\lambda \le C\beta^2 e^{-\beta}, \quad \beta \ge 2$

[G., Humpherys, Lytle]



Stability analysis, $\beta < \beta^*$

Use weights to stabilize the front on the linear level



The issue: In terms of $(w_1, w_2) = e^{\alpha\xi}(u_1, u_2)$, N is not good

$$e^{\alpha\xi} u_2 e^{-1/u_1} = w_2 e^{-e^{\alpha\xi}/w_1}$$

Solution: Consider a system for u_1, u_2, w_1 and w_2 with N written as

$$\begin{aligned} e^{\alpha\xi} u_2 e^{-1/u_1} &= w_2 e^{-1/u_1} \\ \| e^{\alpha\xi} u_2 e^{-1/u_1} \|_{H^1} &\leq M \| e^{-1/u_1} \|_{C^1} \| w_2 \|_{H^1} \end{aligned}$$

[Pego, Weinstein, 1994] - Hamiltonian context

Nonlinear stability results

Theorem. Let initial conditions be $U^0 = H + V^0$. There exist α^*, δ^*, q^* $\nu^* > 0$ for which the following is true:

For every δ^*, there exists ρ_* such that if

$$\|V^0\|_{H^1} + \|V^0 e^{\alpha\xi}\|_{H^1} \leq \rho^* \quad \text{then}$$

- a unique global solution U exists and can be decomposed as

$$U(\xi, t) = H(\xi - q(t)) + V(\xi, t) \quad \text{for some } q(t),\ q(0) = 0$$

- is nonlinearly stable in the exponentially weighted space

  $|q(t) - q_*| + \|V(\xi, t)e^{\alpha\xi}\|_{H^1} \leq \delta^* e^{-\nu^* t}$

- Moreover, there exists a constant $K > 0$ such that for $V = (v_1, v_2)$
  - $\|v_1(\xi, t)\|_{H^1} \leq K\delta^*, \qquad \|v_2(\xi, t)\|_{H^1} \leq K\delta^* e^{-\frac{\nu^*}{2}t}$
  - under more assumptions on V^0, W^0: $\|v_1(\xi, t)\|_{L^1} \leq K\delta^* t^{1/2}$

Strategy

- **Seek solution as** $Y(\xi, t) = H(\xi - q(t)) + \tilde{Y}(\xi, t)$
- **Obtain equations for** Y **in the weighted and unweighted norms**
- **Eliminate the translational direction**
- **Prove local existence, uniqueness and continuous dependence on initial data as long as the solution in not large**
- **Obtain a-priori estimates that show that the solution in the unweighted norm is always small**
- **Use semigroup estimates, a-priori estimates and exponential decay of the front to the rest states in the equation for the weighted variable**

System

$$V = (v_1, v_2), W = (w_1, w_2) = e^{\alpha\xi} V,$$

$$\begin{aligned} \partial_t V &= \mathcal{L}(\partial_\xi)V + \mathcal{R}(\xi)V + \Delta\mathcal{R}(\xi, q(t))V + N(V)v_1 + \dot{q}(t)H'(\xi - q(t)) \\ \partial_t W &= \Lambda_\alpha W + \Delta\mathcal{R}(\xi, q(t))W + N(V)w_1 + \dot{q}(t)e^{\alpha\xi}H'(\xi - q(t)) \end{aligned}$$

where

$$\mathcal{L}(\partial_\xi) = \begin{pmatrix} \partial_{\xi\xi} + c\partial_\xi & 0 \\ 0 & \mathrm{Le}^{-1}\partial_{\xi\xi} + c\partial_\xi \end{pmatrix}, \quad \Lambda_\alpha = \mathcal{L}(\partial_\xi - \alpha) + \mathcal{R}(\xi)$$

$$\mathcal{R}(\xi) = \begin{pmatrix} h_2(\xi)\Omega_{h_1}(h_1(\xi)) & \Omega(h_1(\xi)) \\ -\beta h_2(\xi)\Omega_{h_1}(h_1(\xi)) & -\beta\Omega(h_1(\xi)) \end{pmatrix}$$

$$\Delta\mathcal{R}(\xi, q(t)) = \mathcal{R}(\xi - q(t)) - \mathcal{R}(\xi)$$

System, projections

$\mathcal{P}^{\mathrm{c}}_{\alpha} W = 0$. Apply $\mathcal{P}^{\mathrm{c}}_{\alpha}$ to

$$\begin{aligned} \partial_t W &= \Lambda_\alpha W + \Delta\mathcal{R}(\xi, q(t))W \\ &\quad + N(V)w_1 + \dot{q}(t)e^{\alpha\xi}H'(\xi - q(t)), \end{aligned}$$

to obtain

$$\begin{aligned} \partial_t W &= \mathcal{P}^{\mathrm{s}}_{\alpha}\Lambda_\alpha W + \mathcal{P}^{\mathrm{s}}_{\alpha}\Delta\mathcal{R}(\xi, q(t))W \\ &\quad + \mathcal{P}^{\mathrm{s}}_{\alpha}[N(V)w_1 + \dot{q}(t)e^{\alpha\xi}H'(\xi - q(t))], \\ \dot{q}(t) &= -\left[\mathcal{P}^{\mathrm{c}}_{\alpha}e^{\alpha\xi}H'(\xi - q(t))\right]^{-1}\mathcal{P}^{\mathrm{c}}_{\alpha}\left[\Delta\mathcal{R}(\xi, q(t))W + N(V)w_1\right], \\ \partial_t V &= \mathcal{L}(\partial_\xi)V + \mathcal{R}(\xi)V + \Delta\mathcal{R}(\xi, q(t))V + N(V)v_1 + \dot{q}(t)H'(\xi - q(t)) \\ q(0) &= 0 \end{aligned}$$

A-priori estimates

As long as $|q(t)| + \|V\|_{H^1} < A^*$

$$|q'(t)| + \|V(\xi,t)e^{\alpha\xi}\|_{H^1} \le C\|W^0\|_{H^1} e^{-\nu^* t}$$

$$\begin{aligned}
\partial_t v_2 &= \mathrm{Le}^{-1}\partial_{\xi\xi} v_2 + c\partial_\xi v_2 - \beta\Omega(h_1(\xi - q(t))v_2 + \ldots. \\
&= \mathrm{Le}^{-1}\partial_{\xi\xi} v_2 + c\partial_\xi v_2 - \beta e^{-\beta\xi} v_2 + \beta(e^{-\beta\xi} - \Omega(h_1(\xi - q(t)))v_2 + \ldots. \\
&= \mathrm{Le}^{-1}\partial_{\xi\xi} v_2 + c\partial_\xi v_2 - \beta e^{-\beta\xi} v_2 + \beta(e^{-\beta\xi} - \Omega(h_1(\xi - q(t)))e^{-\alpha\xi} w_2 + \ldots
\end{aligned}$$

$$\Longrightarrow \|v_2(\xi,t)\|_{H^1} \le K_1\|W^0\|_{H^1} e^{-\frac{\nu^*}{2}t}$$

$$\Longrightarrow \|v_1(\xi,t)\|_{H^1} \le K_1\|W^0\|_{H^1}$$

Exothermic-endothermic systems

$$\partial_t z_1 = \partial_{xx} z_1 + z_2 f_2(z_1) - \sigma z_3 f_3(z_1)$$

$$\partial_t z_2 = d_2 \partial_{xx} z_2 - z_2 f_2(z_1)$$

$$\partial_t z_3 = d_3 \partial_{xx} z_3 - \tau z_3 f_3(z_1)$$

z_1 - temperature

z_2 - quantity of an exothermic reactant

z_3 - quantity of an endothermic reactant

d_2, $d_3 \geq 0$, $0 < \sigma < \tau$

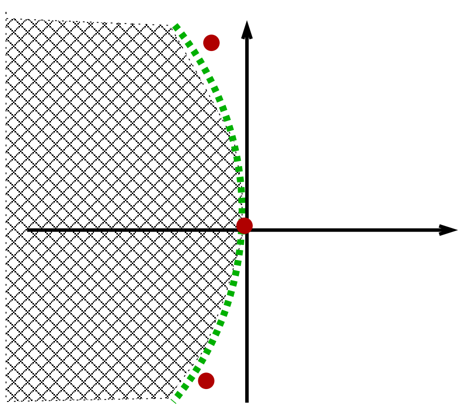
$f_i(z_1) = a_i e^{-\frac{b_i}{z_1}}$ for $z_1 > 0$, for some a_i and $b_i > 0$; 0 for $z_1 \leq 0$

References

Simon, Kalliadasis, Merkin, and Scott, [2003], [2004], [2004]

There exists a parameter regime such that

- **traveling waves exist**
- **traveling fronts converge to their rest states at exp. rates**
- **the zero eigenvalue of the linearization is simple**
- **no other eigenvalues in the closed right half plane**
- **essential spectrum touches the imaginary axis**



Autocatalysis

Quadratic Autocatalysis

$$\begin{aligned} \partial_t u_1 &= \partial_{xx} u_1 + u_2 u_1^2 \qquad x \in \mathbb{R}, \quad t > 0 \\ \partial_t u_2 &= \delta \partial_{xx} u_2 - u_2 u_1^2 \end{aligned}$$

Autocatalytic Reactions:

$$\begin{aligned} \partial_t u_1 &= \partial_{xx} u_1 + u_2 u_1^m \qquad x \in \mathbb{R}, \quad t > 0 \\ \partial_t u_2 &= \delta \partial_{xx} u_2 - u_2 u_1^m \end{aligned}$$

u_1 **- autocatalyst,** u_2 **- reactant,** δ**-the ratio of diffusivities**

Common features

- **Conserved quantity**
- **Special structure of the nonlinearity:** $N(u_1, u_2) = u_2 \Omega(u_1, u_2)$, $N(u_1, 0) = 0$
- **exponentially decaying fronts**
- **instability caused by only essential spectrum**
- **transport of perturbations,** $c \neq 0$**, on the linear level, towards the equilibrium behind the front**
- **in the linearization about the equilibrium behind the front, equation for** u_2 **are decoupled from the equations for** u_1**.**
- **in terms of** u_2**, the equilibrium behind the front is stable**

General Case

The proof works

- for any number of reactants
- more general class of nonlinearities
- not only for fronts, but also for pulses

Consider a reaction-diffusion system

$$Y_t = DY_{xx} + R(Y),$$

- $Y \in \mathbb{R}^n$, $x \in \mathbb{R}$, $t \geq 0$
- $D =$ diag$(d_1, \ldots, d_n)$, $d_i \geq 0$
- $R(Y)$ is C^3

Assumptions

H1. **Assume there is a front or a pulse solution $Y_*(\xi)$, $\xi = x - ct$, $c > 0$**

$$\lim_{\xi \to -\infty} Y_*(\xi) = Y_-, \quad \lim_{\xi \to \infty} Y_*(\xi) = Y_+$$

Moreover, there exist numbers $K > 0$ and $\omega_- < 0 < \omega_+$ such that

$$\begin{aligned} \|Y_*(\xi) - Y_-\| &\leq Ke^{-\omega_- \xi} \text{ for } \xi \leq 0 \\ \|Y_*(\xi) - Y_+\| &\leq Ke^{-\omega_+ \xi} \text{ for } \xi \geq 0 \end{aligned}$$

H2. $Y = (U, V)$, $U \in \mathbb{R}^{n_1}$, $V \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$

$$R(U, V) = \begin{pmatrix} A_1 U + \tilde{R}_1(U, V)V \\ \tilde{R}_2(U, V)V \end{pmatrix}$$

A_1 **is an** $n_1 \times n_1$ **matrix**

$\tilde{R}_1$ **is** $n_1 \times n_2$ **matrix-valued function,** $\tilde{R}_2$ **is** $n_2 \times n_2$ **matrix-valued function**

Assumptions on the Spectrum in the Weighted Norm

H3. **There exists α such that for**

$$\tilde{Y}_t = D\tilde{Y}_{\xi\xi} + c\tilde{Y}_\xi + DR(Y_*)\tilde{Y} = L\tilde{Y}$$

1. $\sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}_{\mathrm{ess}}(\mathcal{L}_\alpha)\} < 0$

2. $\mathrm{Sp}(\mathcal{L}_\alpha \cup \{\lambda : \mathrm{Re}\,\lambda \geq 0\}$ **is a simple eigenvalue 0.**

The nonlinear term in the weighted space is no longer locally Lipschitz

Recall: $Y = (U, V)$, $U \in \mathbb{R}^{n_1}$, $V \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$

$$U_t = D_1 U_{\xi\xi} + cU_\xi + R_1(U, V),$$

$$V_t = D_2 V_{\xi\xi} + cV_\xi + R_2(U, V).$$

$Y_*(\xi) = (U_*(\xi), V_*(\xi))$

Assumptions on the Spectrum in the Unweighted Norm

$$L^{(1)} = D_1 \partial_{\xi\xi} + c\partial_\xi + D_U R_1(Y_-) = D_1 \partial_{\xi\xi} + c\partial_\xi + A_1$$

$$L^{(2)} = D_2 \partial_{\xi\xi} + c\partial_\xi + D_V R_2(Y_-)$$

H4. • **The operator $L^{(1)}$ generates a bounded semigroup**

• **The operator $L^{(2)}$ satisfies** $\sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}(\mathcal{L}_0^{(2)})\} < 0$

$$L \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix} = \begin{pmatrix} L^{(1)} & D_V R_1(Y_-) \\ 0 & L^{(2)} \end{pmatrix} \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix} + (DR(Y_*) - DR(Y_-)) \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix}$$

Then

$$L\tilde{Y} = L^- \tilde{Y} + (DR(Y_*) - DR(Y_-))\tilde{Y}$$

$$L\tilde{Y} = L^- \tilde{Y} + (DR(Y_*) - DR(Y_-))e^{-\beta\xi}(\tilde{Y} e^{\beta\xi})$$

2-d case

$$\begin{aligned} \partial_t u_1 &= \partial_{xx} u_1 + \partial_{yy} u_1 + u_2 \Omega(u_1) \qquad (x,y) \in \mathbb{R} \times \mathbb{R}, \quad t > 0 \\ \partial_t u_2 &= \delta \partial_{xx} u_2 + \delta \partial_{yy} u_2 - \beta u_2 \Omega(u_1) \end{aligned}$$

Planar wave $\Phi = \Phi(z)$, $z = x - ct$

$\Phi(z) \to (1,0)$ **as** $z \to -\infty$, $\Phi(z) \to (0,1)$ **as** $z \to +\infty$

$$\begin{aligned} \partial_t u_1 &= \partial_{zz} u_1 + \partial_{yy} u_1 + c\partial_z u_1 + u_2 \Omega(u_1) \\ \partial_t u_2 &= \delta \partial_{zz} u_2 + \delta \partial_{yy} u_2 + c\partial_z u_2 - \beta u_2 \Omega(u_1) \end{aligned}$$

Linearization about the wave:

$$\begin{aligned} \partial_t \tilde{u}_1 &= \partial_{zz} \tilde{u}_1 + \partial_{yy} \tilde{u}_1 + c\partial_z \tilde{u}_1 + \phi_2 \Omega'(\phi_1) \tilde{u}_1 + \Omega(\phi_1) \tilde{u}_2 \\ \partial_t \tilde{u}_2 &= \delta \partial_{zz} \tilde{u}_2 + \delta \partial_{yy} \tilde{u}_2 + c\partial_z \tilde{u}_2 - \beta \phi_2 \Omega'(\phi_1) \tilde{u}_1 - \beta \Omega(\phi_1) \tilde{u}_2 \end{aligned}$$

Continuous spectrum, $\mathbb{R} \times \mathbb{R}$

Linearization about $(1,0)$:

$$\begin{aligned} \partial_t \tilde{u}_1 &= \partial_{zz}\tilde{u}_1 + \partial_{yy}\tilde{u}_1 + c\partial_z \tilde{u}_1 + \Omega(1)\tilde{u}_2 \\ \partial_t \tilde{u}_2 &= \delta\partial_{zz}\tilde{u}_2 + \delta\partial_{yy}\tilde{u}_2 + c\partial_z \tilde{u}_2 - \beta\Omega(1)\tilde{u}_2 \end{aligned}$$

at $(1,0)$: $e^{ikz}e^{i\mu y}e^{\lambda t}$ $\Longrightarrow$ $\lambda = -k^2 - \mu^2 + cik$, $k, \mu \in \mathbb{R}$

Exp. weight $e^{\alpha z}$ $\Longrightarrow$ $\lambda = -k^2 - \mu^2 + (c - 2\alpha)ik + \alpha^2 - c\alpha$

$\Phi'(z)$ is an eigen-function corresponding to 0 eigenvalue

$e^{i\mu y}e^{\lambda t}\Phi'(z)$ $\Longrightarrow$ $\lambda = -\mu^2$, $\mu \in \mathbb{R}$

Exponential weight $e^{\alpha z}$ does not work: $e^{i\mu y}e^{\lambda t}\Phi'(z)e^{\alpha z}$ $\Longrightarrow$ $\lambda = -\mu^2$

Reduction to the stability of 1-d front, $\mathbb{R} \times \mathbb{R}$

Kapitula, 1997

$u_t = \alpha \Delta u + f(u)$, $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$

$\phi(z)$, $z = x_1 - ct$, **is an exp. decaying to its rest states front**

$L_1 = \partial_{zz} + c\partial_z + Df(\phi)$

$\sigma(L_1) \subset \{\lambda : \mathrm{Re}\ \Lambda \leq -\gamma\} \cup \{0\}$

$\Longrightarrow$ **the front is stable at algebraic rates**

Situation to consider:

- $\sigma(L_1) \subset \{\lambda : \mathrm{Re}\ \lambda \leq 0\}$
- **The 1-d front is nonlinearly stable in an exp. weight**

Issue: the diffusion matrix is not αI

Special classes of perturbations

Simplest case:

Perturbations constant in y-direction:

$\Longrightarrow$ **the pde system reduces to the 1-d system**

$\Longrightarrow$ **the front is stable in an exponential weighted norm**

$\Longrightarrow$ **the front is convectively unstable**

Look at perturbations periodic in y: $e^{imy}\tilde{U}(z,t)$

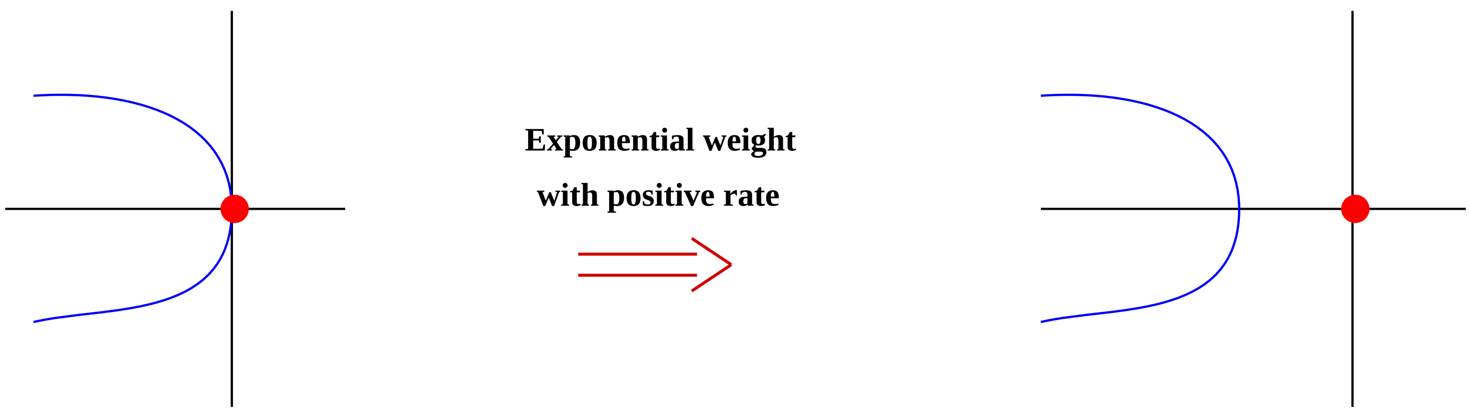
Spectrum, $\mathbb{R} \times \mathbb{S}$

Essential spectrum: $\Longrightarrow$ $\lambda = -k^2 - m^2 + cik$, $k \in \mathbb{R}$, $m \in \mathbb{N}$

Exp. weight $e^{\alpha z}$ $\Longrightarrow$ $\lambda = -k^2 - m^2 + (c - 2\alpha)ik + \alpha^2 - c\alpha$

$\Phi'(z)$ **is an eigen-function corresponding to** 0 **eigenvalue**

$e^{imy}e^{\lambda t}\Phi'(z)$ $\Longrightarrow$ $\lambda = -m^2$, $m \in \mathbb{N}$.



Evans function for discrete spectrum:

- **analysis [Terman, 1990]**
- **numerical calculation [Balmforth, Craster, Malham, 1998], [Ledoux, Malham, Niesen, Thummler, 2009],**