Scaling laws for slanted snaking

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Outline

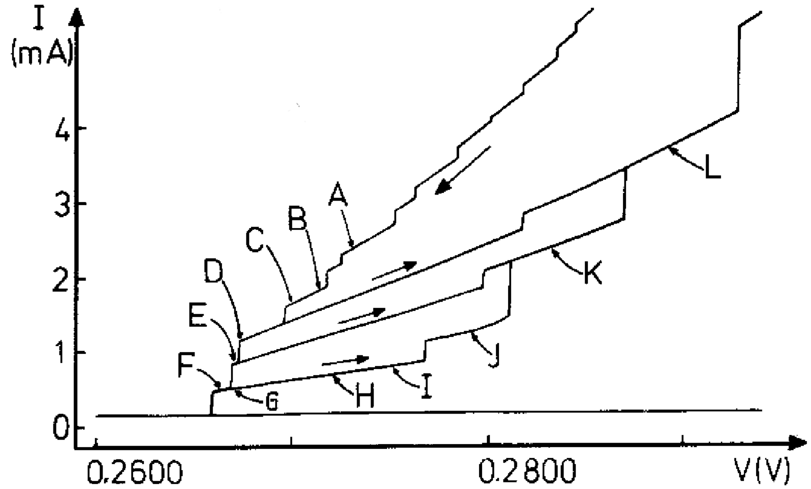
- Physical examples
  - Dielectric gas discharge / nonlinear optics
  - Vertically shaken granular media / viscoelastic fluid
  - Magnetoconvection
- Toy model: Swift–Hohenberg + nonlinear diffusion eqn
  - ‘Slanted snaking’
  - Reduction to a nonlocal Ginzburg–Landau equation
  - Scaling laws
- Construction of fully nonlinear solutions - to try to understand scaling laws
  - 1. Patching
  - 2. via the ‘Variational Approximation’
- 2D

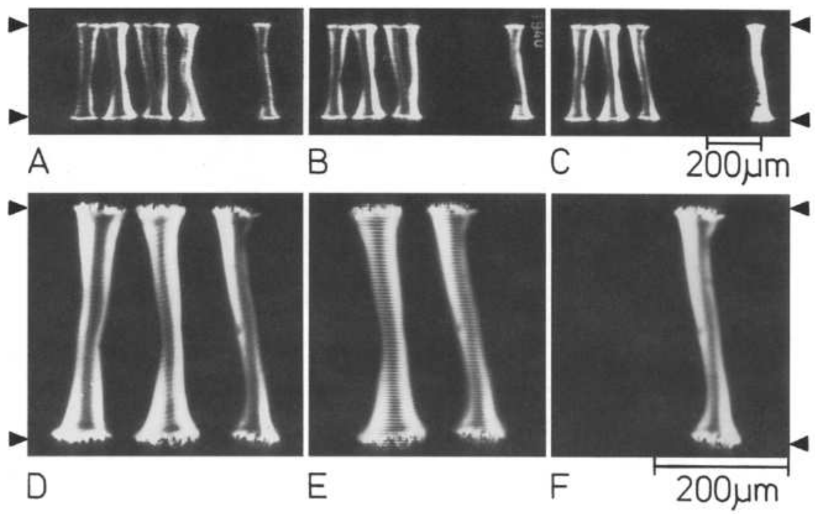
Catherine Penington

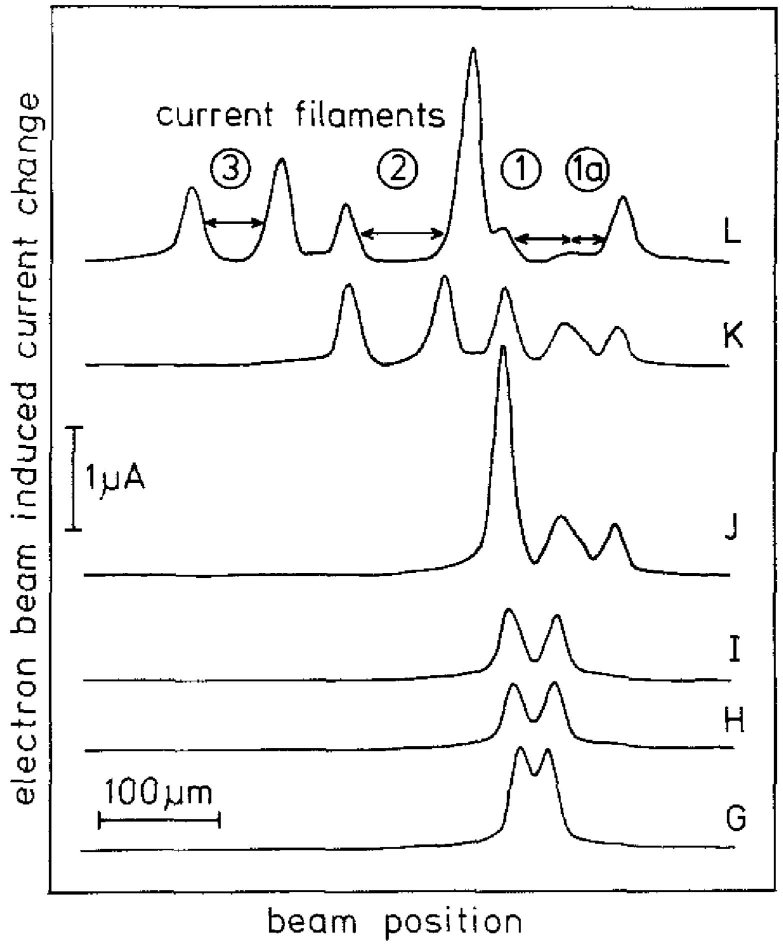
J.H.P. Dawes, The emergence of a coherent structure for coherent structures: localized states in nonlinear systems. *Phil. Trans. Roy. Soc.* **368**, 3519–3534 (2010)

Slanted snaking - examples

Semiconductor filaments:

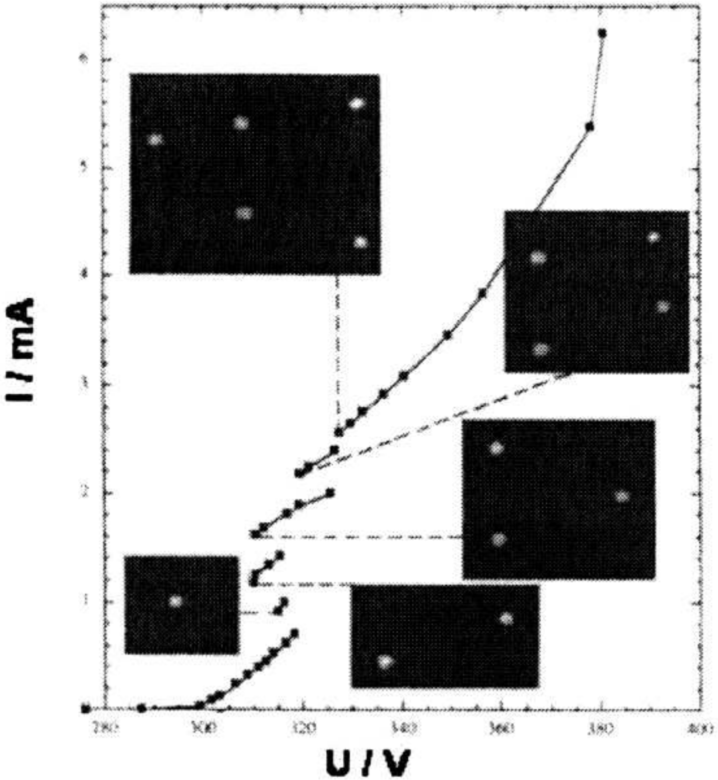




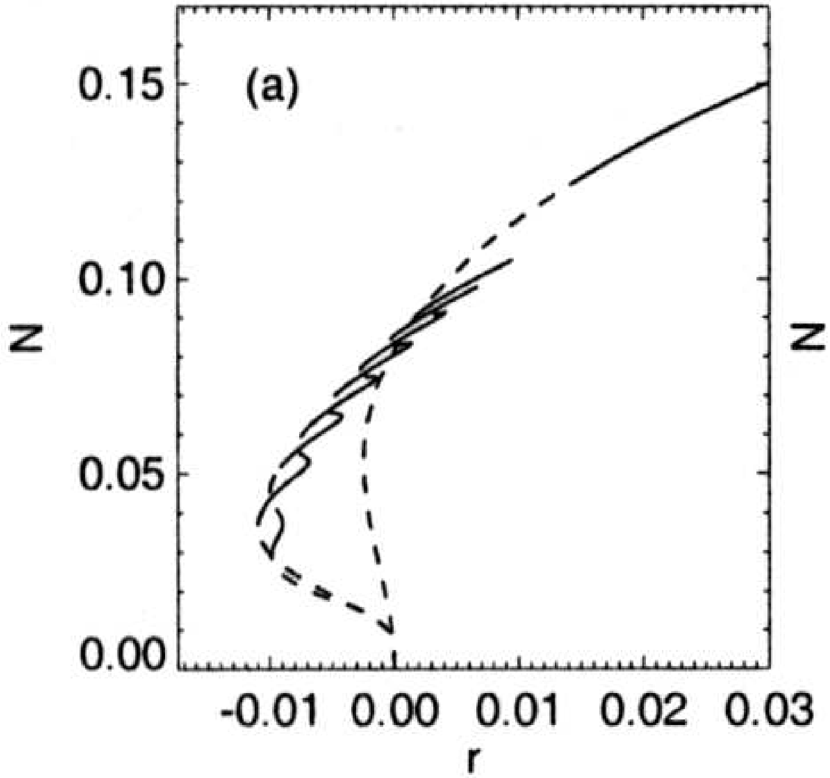


K.M. Mayer, J. Parisi and R.P. Huebner, *Z. Phys.* B **71**, 171-178 (1988)

Slanted snaking – examples



Dielectric gas discharge
H.-G. Purwins *et al*



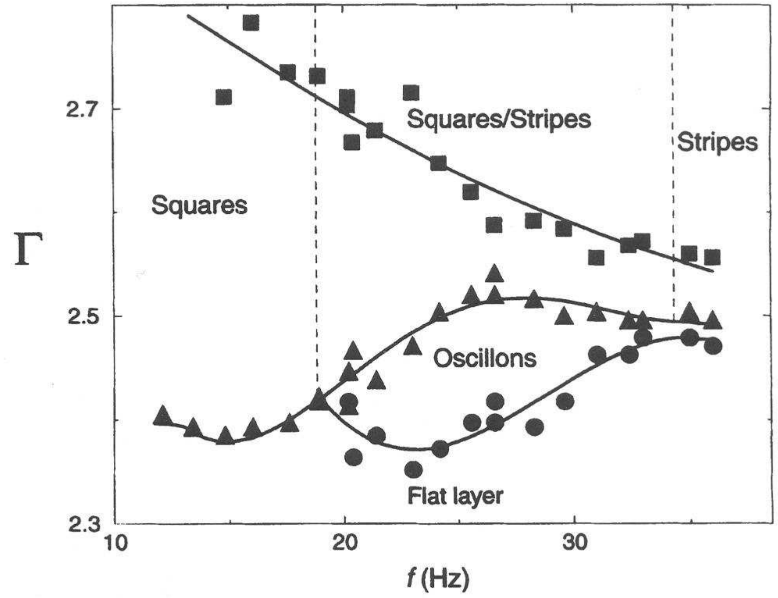
Model equation
W.J. Firth *et al*

$$u_t = [r - (1 + \partial_x^2)^2]u + b_2 u^2 - u^3 - \gamma \langle u^2 \rangle u$$

E. Ammelt. PhD Thesis, University of Münster (1995)
W.J. Firth, L. Columbo, A.J. Scroggie, *Proposed resolution of theory–experiment discrepancy in homoclinic snaking*. *Phys. Rev. Lett.* **99**, 104503 (2007)

Granular Faraday Experiment

Granular 'oscillons'

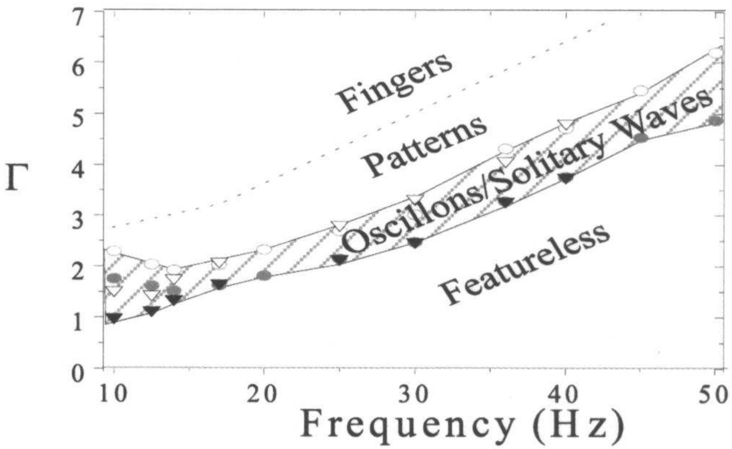


... are observed below periodic patterns.

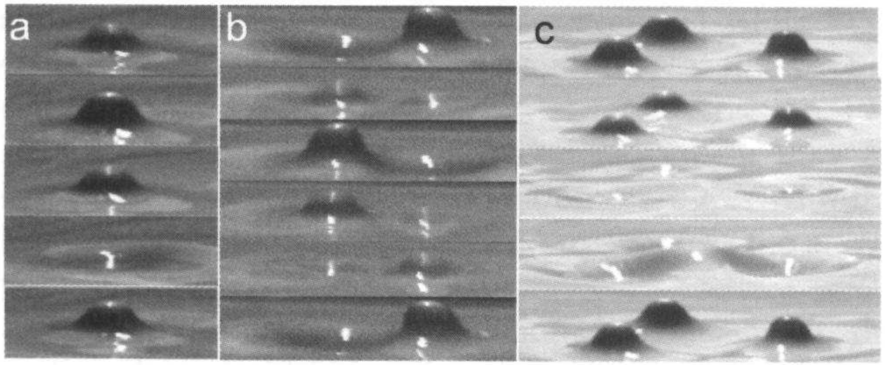
P.B. Umbanhowar, F. Melo and H.L. Swinney, *Nature* **382**, 793 (1996)
J.H.P. Dawes & S. Lilley, *SIAM J. Appl. Dyn. Syst.* **9**, 238–260 (2010)

Viscoelastic Faraday Experiment

- Regime diagram:



- Side views (over 2 driving periods)

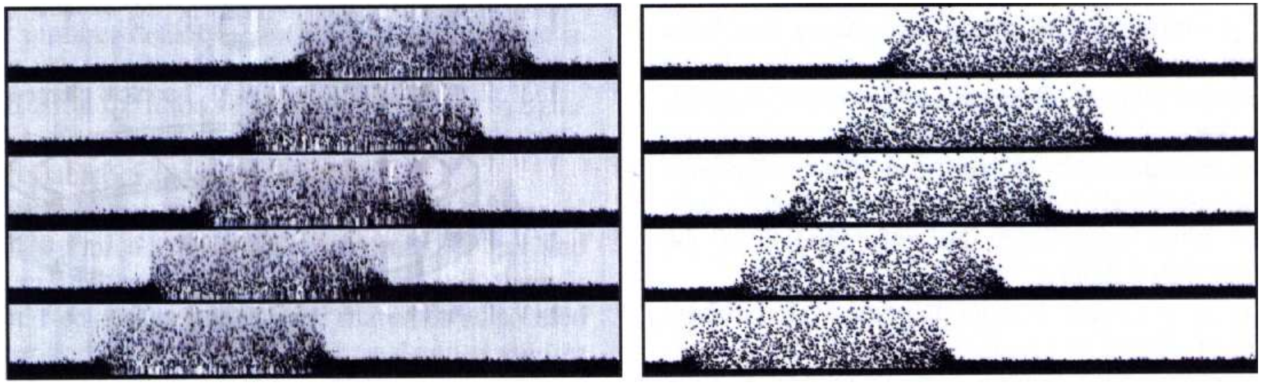


oscillon pair triad

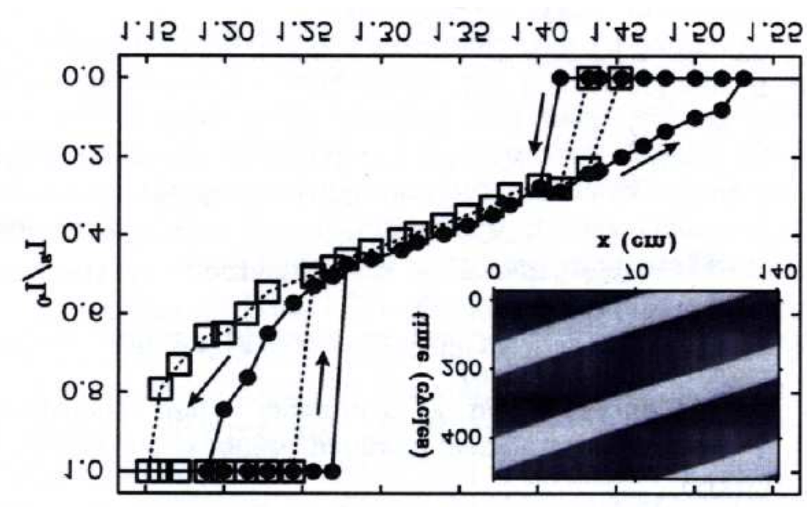
Oscillons are again observed BELOW hysteresis region for patterns

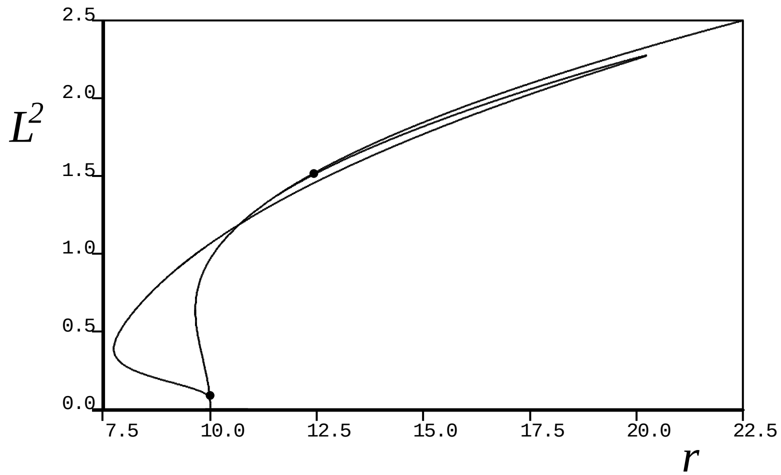
O. Lioubashevski, H. Arbell and J. Fineberg, *Phys. Rev. Lett.* **76**, 3959–3962 (1996)
O. Lioubashevski, Y. Hamiel, A. Agnon, Z. Reches and J. Fineberg, *Phys. Rev. Lett.* **83**, 3190–3193 (1999)

Thin layer Granular Faraday



Left: Experiment; Right: molecular dynamics simulation





A. Götzendorfer, J. Kreft, C.A. Kruelle and I. Rehberg, Sublimation of a vibrated granular monolayer: coexistence of gas and solid *Phys. Rev. Lett.* **95**, 135704 (2005)

Magnetoconvection

Numerical simulations: Rayleigh–Bénard convection with a vertical magnetic field

$R = 10^5$, $Q = 1600$

$\sigma = 0.1$, $\zeta = 0.2$

stress-free, T fixed (lower)

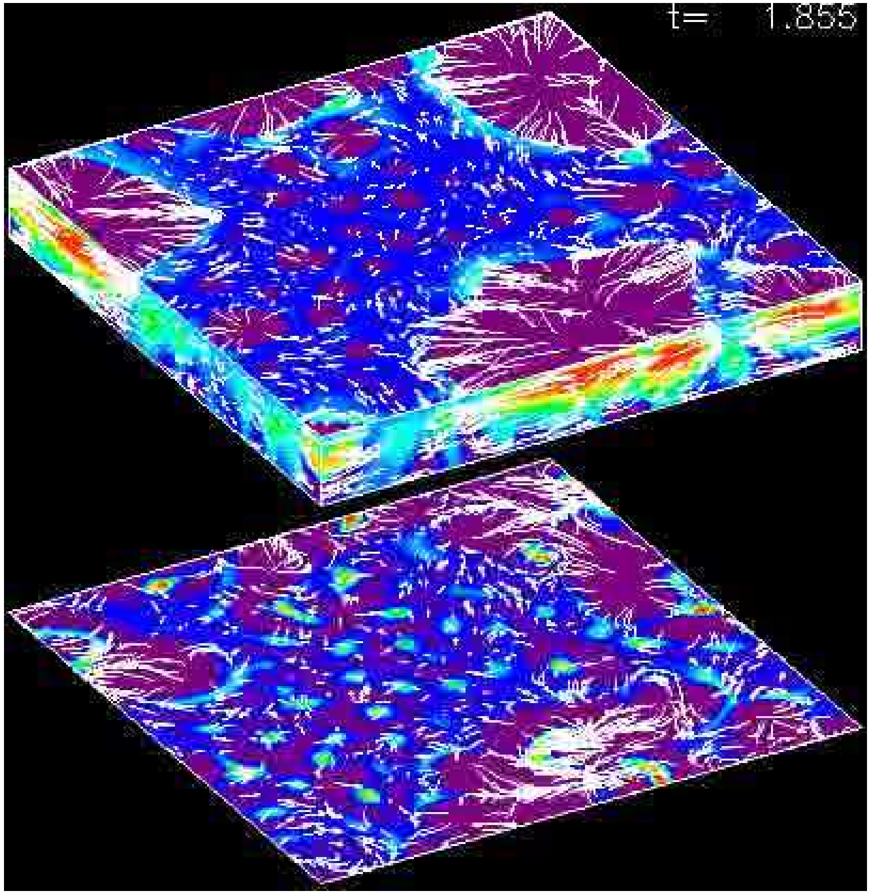
radiative b.c. (upper)

$8 \times 8 \times 1$ stratified layer

density contrast approx 11.

blue = strong field

purple = weak field

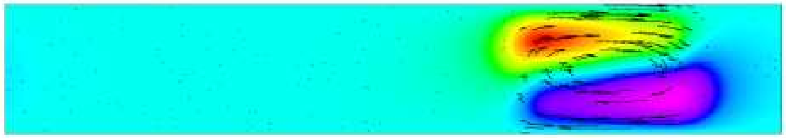


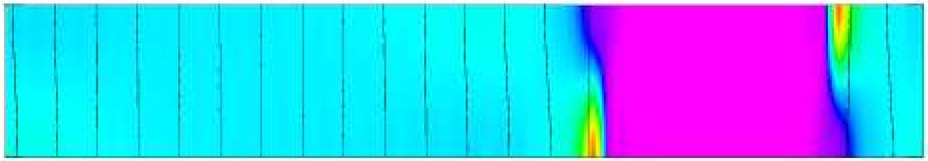
A.M. Rucklidge, N.O. Weiss, D.P. Brownjohn, P.C. Matthews & M.R.E. Proctor, *J. Fluid Mech.* **419**, 283–323 (2000)

Localised magnetoconvection

$R = 20\,000$, $Q = 14\,000$, $\zeta = 0.1$, $\sigma = 1.0$, $L = 6.0$

Temperature (deviation) & velocity: $|\mathbf{B}|^2$:





Subcritical finite-amplitude magnetoconvection noted by several previous authors:

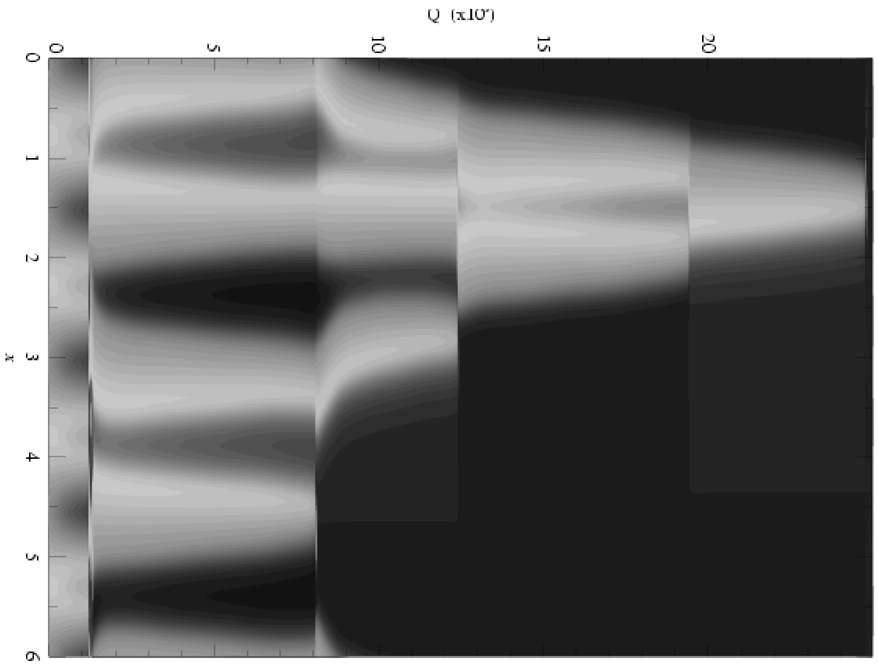
- N.O. Weiss *Proc. Roy. Soc. Lond.* (1966) – flux expulsion
- F.H. Busse, *J. Fluid Mech.* **71** 193–206 (1975):

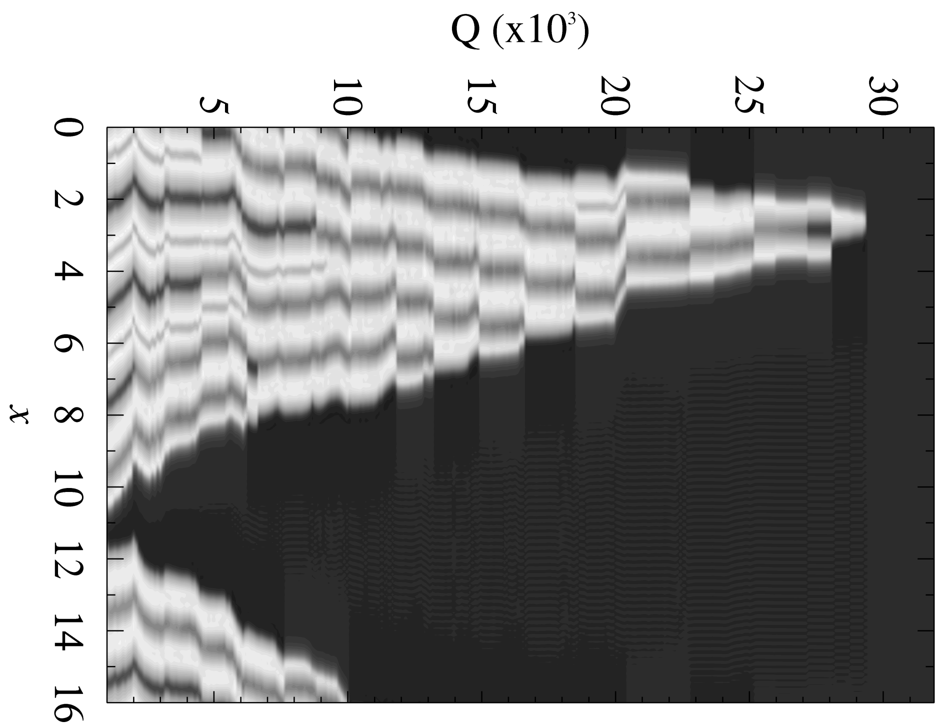
  "...thus finite amplitude onset of steady convection becomes possible at Rayleigh numbers considerably below the values predicted by linear theory."

- ... and recent work by
  D. Lo Jacono, A. Bergeon and E. Knobloch. Preprint. (2011)

Localised magnetoconvection

- Strongly nonlinear localised states (‘convectons’) persist for strong fields.
- $\frac{dT}{dz}|_{\rm top}$ for increasing Q at fixed R:





S.M. Blanchflower, *Phys. Lett.* A **261**, 74–81 (1999); PhD thesis, University of Cambridge (1999)

J.H.P. Dawes, Localised convection cells in the presence of a vertical magnetic field. *J. Fluid Mech.* **570**, 385–406 (2007)

Magnetoconvection

Boussinesq equations for 2D thermal convection, vertical magnetic field:

$$
\begin{aligned}
\partial_t \omega + J[\psi, \omega] &= -\sigma R \partial_x \theta - \sigma \zeta Q (J[A, \nabla^2 A] + \partial_z \nabla^2 A) + \sigma \nabla^2 \omega \\
\partial_t \theta + J[\psi, \theta] &= \nabla^2 \theta + \partial_x \psi \\
\partial_t A + J[\psi, A] &= \partial_z \psi + \zeta \nabla^2 A
\end{aligned}
$$

- Jacobian: $J[f, g] \equiv \partial_x f \partial_z g - \partial_z f \partial_x g$
- $\theta(x, z, t)$ – perturbation to the conduction profile $T = 1 - z$
- $\psi(x, z, t)$ – streamfunction.
- $\mathbf{u} = \nabla \times (\psi(x, z, t)\hat{\mathbf{y}})$, so $\omega = -\nabla^2 \psi$
- Magnetic field $\mathbf{B} = \mathbf{B}_0 + \nabla \times (A(x, z, t)\hat{\mathbf{y}}) = (-\partial_z A, 0, 1 + \partial_x A)$
- Nondimensionalised so that $\mathbf{B}_0 = (0, 0, 1)$

Magnetoconvection

Take analytically simple boundary conditions:

- Stress–free + fixed temperature + vertical field
  $\psi = \omega = \theta = \partial_z A = 0$ on $z = 0, 1$
- Periodic in horizontal direction: $0 \le x \le L$

Four dimensionless parameters:

- thermal Prandtl number: $\sigma = \nu/\kappa$
- magnetic Prandtl number: $\zeta = \eta/\kappa$
- Rayleigh number: $R = \frac{\hat{\alpha} g \Delta T d^3}{\kappa \nu}$
- Chandrasekhar number: $Q = \frac{|\mathbf{B_0}|^2 d^2}{\mu_0 \rho_0 \nu \eta}$

S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*. OUP (1961)
M.R.E. Proctor and N.O. Weiss, *Rep. Prog. Phys.* **45**, 1317–1379 (1982)

Weakly nonlinear theory

- Introduce long length and time scales $X = \varepsilon x$, $T = \varepsilon^2 t$.
- Matthews and Cox pointed out the need to include a large-scale mode $A_0(X,T)$ for the magnetic field:

$$\begin{aligned}
\psi &= \varepsilon a(X,T)\mathrm{e}^{\mathrm{i}kx}\sin\pi z + c.c. + O(\varepsilon^2) \\
\theta &= \varepsilon c_1 a(X,T)\mathrm{e}^{\mathrm{i}kx}\sin\pi z + c.c. + O(\varepsilon^2) \\
A &= \varepsilon c_2 a(X,T)\mathrm{e}^{\mathrm{i}kx}\cos\pi z + \varepsilon c_2 A_0(X,T) + c.c. + O(\varepsilon^2)
\end{aligned}$$

- At $O(\varepsilon^3)$ and $O(\varepsilon^4)$ we derive amplitude equations:

$$\begin{aligned}
a_T &= \mu a + a_{XX} - a|a|^2 - aA_{0X} \\
A_{0T} &= \zeta A_{0XX} + \pi(|a|^2)_X
\end{aligned}$$

- Coupling terms represent suppression by the magnetic field and flux expulsion by the fluid flow

Weakly nonlinear theory

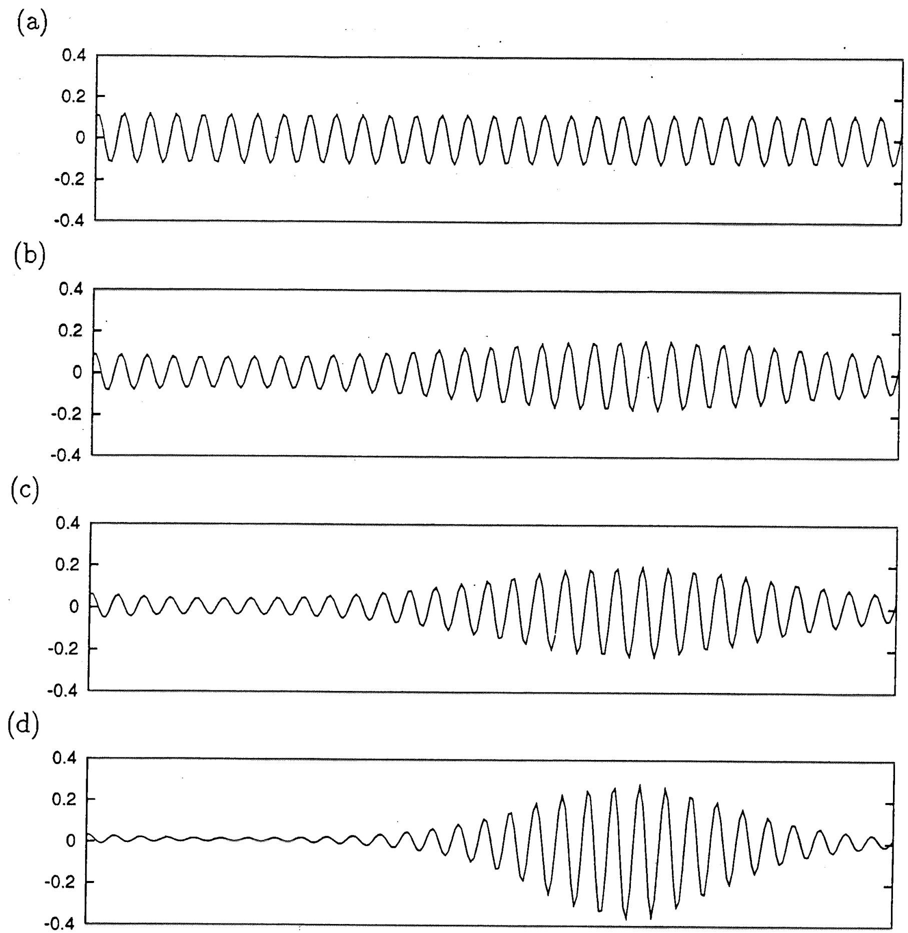
Constant-amplitude, steady rolls:

$a = \text{const}, \; A_{0X} = 0$

are unstable (at onset) to modulational disturbances if:

$\zeta^2 k^4 (\pi^2 + k^2) < \pi^2 (2k^2 - \pi^2)(k^2 + 3\pi^2)$

i.e. if ζ is sufficiently small, for fixed Q.



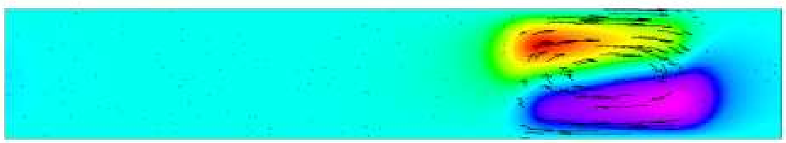
P.C. Matthews and S.M. Cox, *Nonlinearity* **13**, 1293–1320 (2000)

Weakly nonlinear theory - restrictions

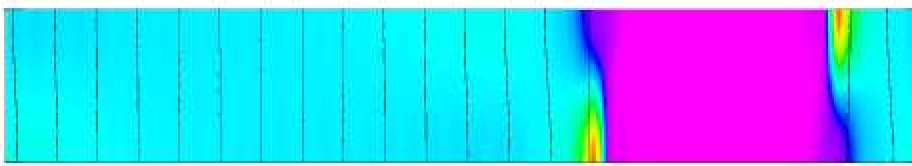
- Convection amplitude is assumed small
- Deviation from uniform field strength is assumed small

Solutions look much more ‘fully nonlinear’

Temperature (deviation) & velocity:



$|\mathbf{B}|^2$:



- Can we employ a different asymptotic limit to investigate more strongly nonlinear solutions?

Magnetoconvection: model problem

$$w_t = [r - (1 + \partial_x^2)^2]w - w^3 - QB^2 w \qquad (1)$$

$$B_t = \zeta B_{xx} + \tfrac{c}{\zeta}(w^2 B)_{xx} \qquad (2)$$

Symmetries:

- $w \to -w$ (Boussinesq problem)
- $B \to -B$ (direction of magnetic field)

Parameters:

- r - reduced Rayleigh number $r = R/R_c$
- Q - Chandrasekhar number $\propto |B_0|^2$
- $\langle B \rangle = 1$ after nondimensionalising
- ζ - magnetic/thermal diffusivity ratio $\zeta = \eta/\kappa$

Remark: Traditional weakly nonlinear analysis would be

$$w = \varepsilon w_1 + \cdots, \qquad B = 1 + \varepsilon^2 B_2 + \cdots, \qquad X = \varepsilon x, \qquad T = \varepsilon^2 t$$

P.C. Matthews & S.M. Cox *Nonlinearity* **13**, 1293–1320 (2000)

Magnetoconvection: model problem

Set $\partial_t \equiv 0$. Integrate (2) twice:

$$\zeta P = B\left(\zeta + \frac{cw^2}{\zeta}\right)$$

where P is a constant of integration.

Re-arrange and integrate over the domain $[0, L]$: $\left\langle \frac{P}{1+cw^2/\zeta^2} \right\rangle = \langle B \rangle \stackrel{\text{def}}{=} 1$

Hence

$$\frac{1}{P} = \left\langle \frac{1}{1+cw^2/\zeta^2} \right\rangle$$

So $P[w]$ measures the higher concentration of the large-scale mode in the region *outside* the localised pattern. Substituting, we obtain

$$0 = [r - (1+\partial_x^2)^2]w - w^3 - \frac{QP^2 w}{(1+cw^2/\zeta^2)^2}$$

Nonlocal Ginzburg–Landau eqn

$$0 = [r - (1+\partial_{xx}^2)^2]w - w^3 - \frac{QP^2 w}{(1+cw^2/\zeta^2)^2}$$

- Suppose $\zeta \ll 1$
- Introduce the long scales $X = \zeta x$, $T = \zeta^2 t$.
- Rescale: $Q = \zeta^2 q$ and $r = \zeta^2 \mu$.
- Expand: $w(x,t) = \zeta A(X,T)\sin x + O(\zeta^2)$, assuming $A(X,T)$ real.

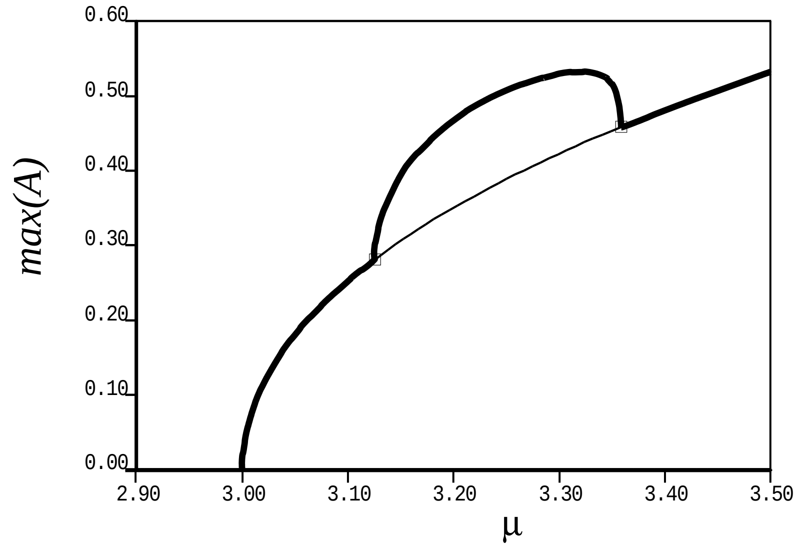
Interpret spatial average as over both $x \in [0, 2\pi]$ and X:

$$\frac{1}{P} = \left\langle \left\langle \frac{1}{1+A^2\sin^2 x} \right\rangle_x \right\rangle_X = \left\langle \frac{1}{\sqrt{1+A^2}} \right\rangle_X$$

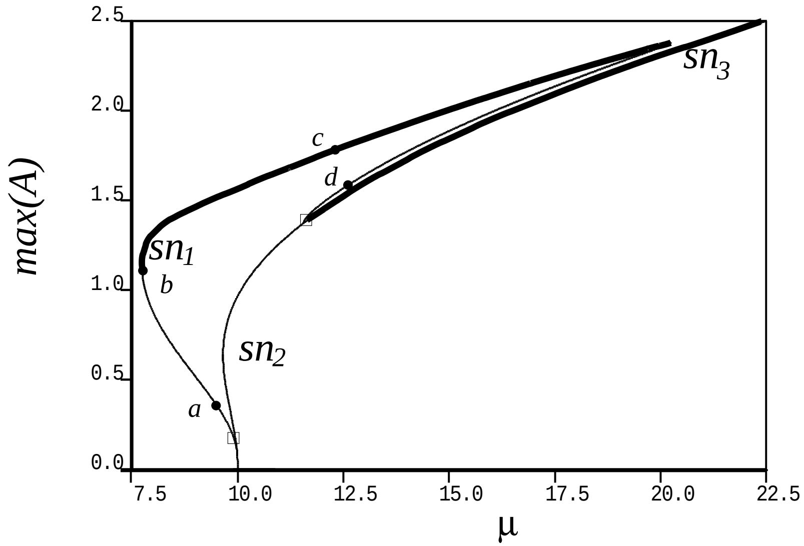
Extract solvability condition by multiplying by $\sin x$ and integrating over x:

$$0 = \mu A + 4A_{XX} - \frac{3}{4}A^3 - \frac{qP^2 A}{(1+cA^2)^{3/2}}$$

Nonlocal Ginzburg–Landau reduction



$q = 3$



$q = 10$

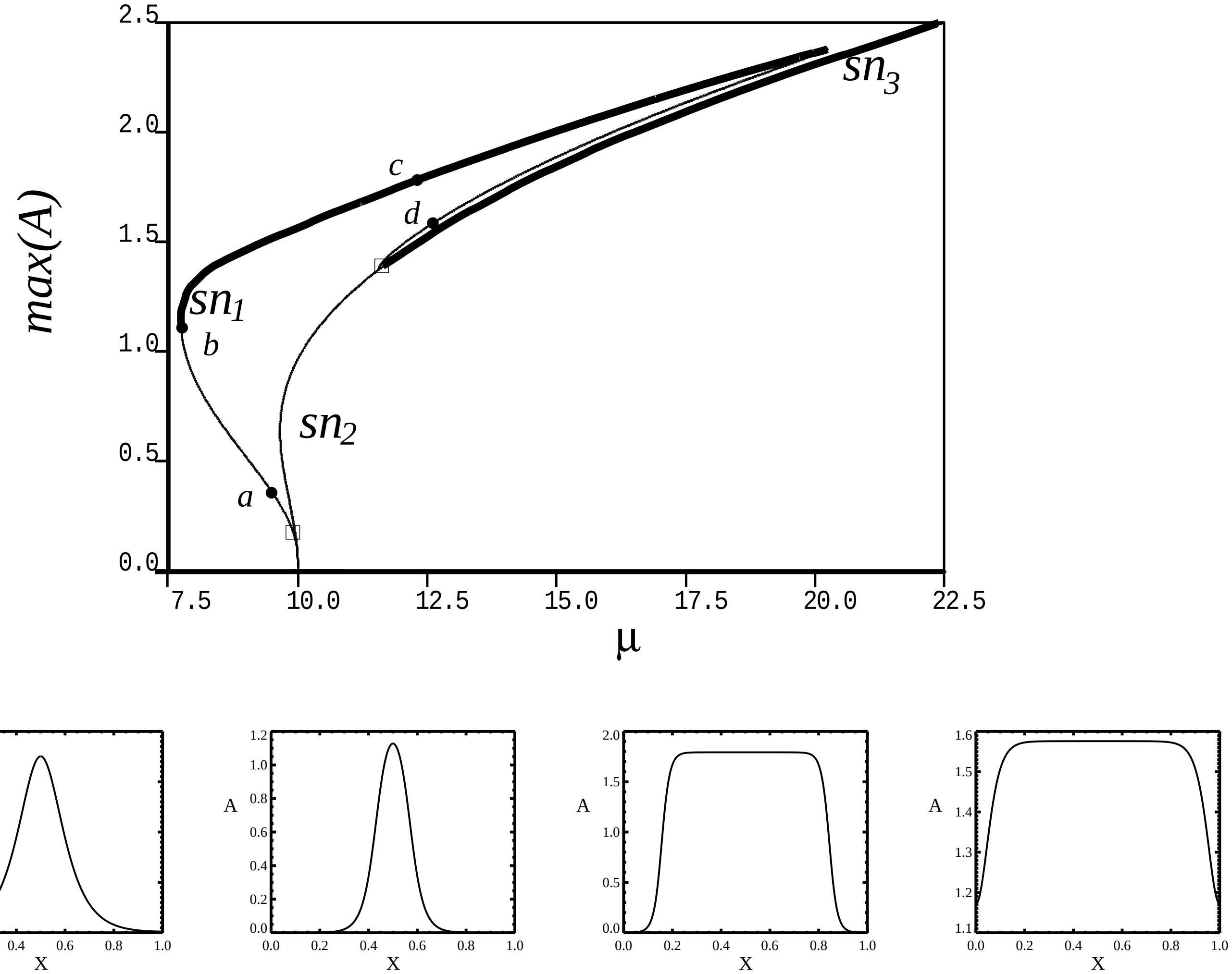
For large q:

- localised branch has saddle-node far below uniform branch
- and second saddle-node bifurcation before it rejoins uniform branch

$c = 0.25$, $\varepsilon L = 10\pi$.

Nonlocal Ginzburg–Landau reduction

$q = 10$, $c = 0.25$, $\varepsilon L = 10\pi$.



(a) (b) (c) (d)

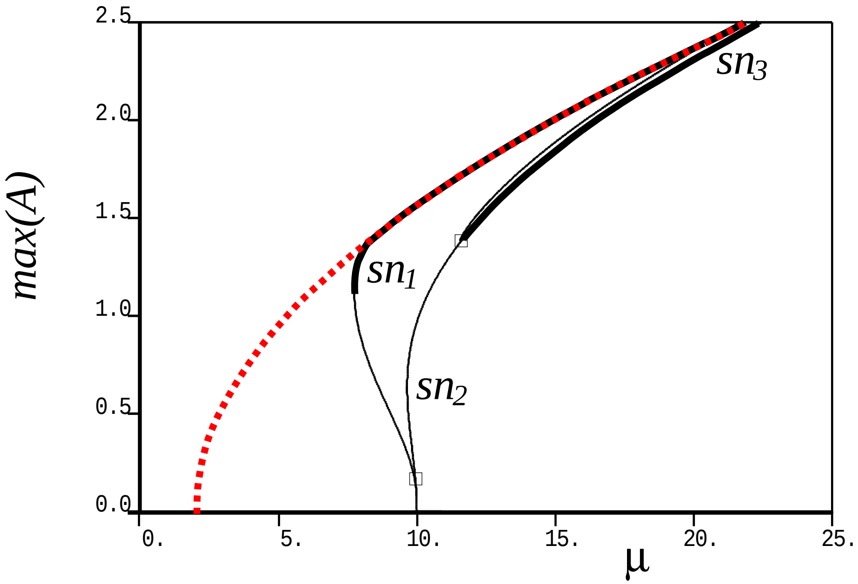
Maxwell point → 'Maxwell curve'

Nonlocal Ginzburg–Landau equation (3) has a first integral:

$$E = \frac{\mu}{2}A^2 + 2A_X^2 - \frac{3}{16}A^4 + \frac{qP^2}{c}\frac{1}{\sqrt{1+cA^2}}$$

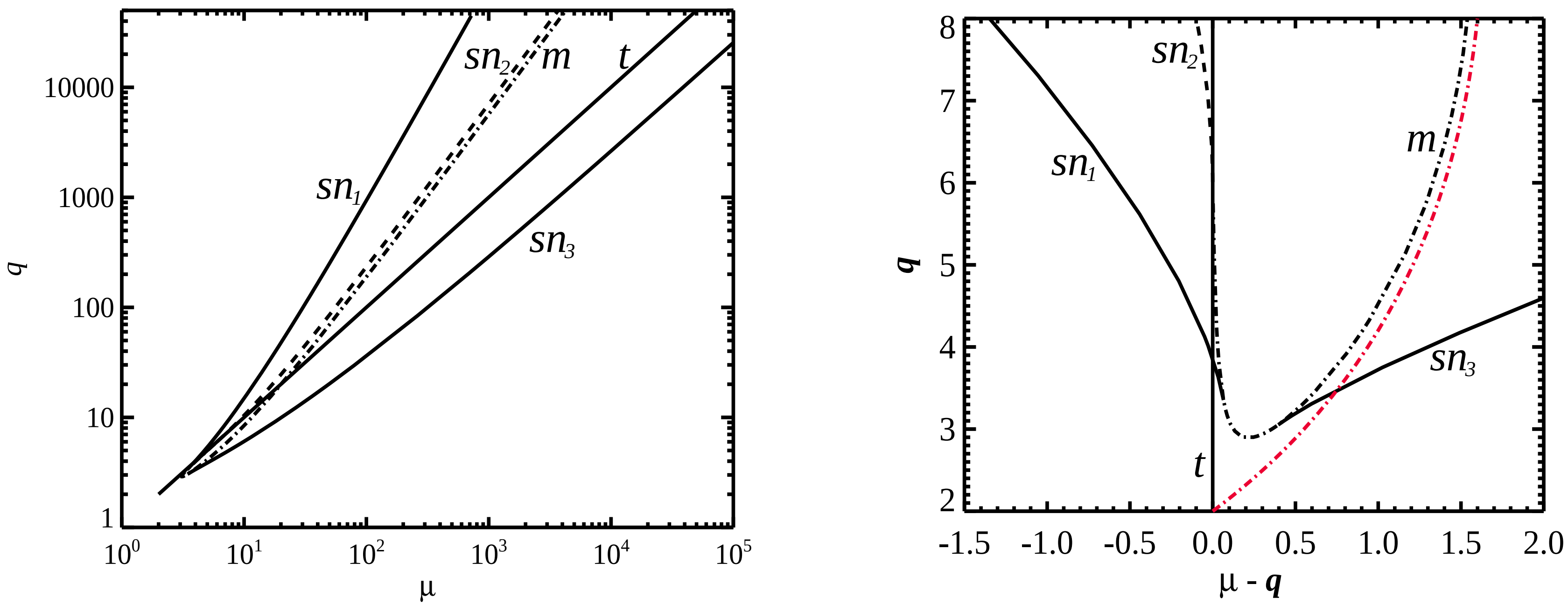
Condition $E|_{A=0} = E|_{A=A_0}$, assuming that nontrivial state occupies a fraction ℓ_c/L of the domain, yields an analytic prediction for the 'Maxwell curve':

$$144c^2A_0^6 + (207 - 384c\mu)cA_0^4 + (72 - 432c\mu + 256c^2\mu^2)A_0^2 + 96\mu(2c\mu - 1) = 0$$



Nonlocal Ginzburg–Landau reduction

Bifurcation curves in the (μ, q) plane:

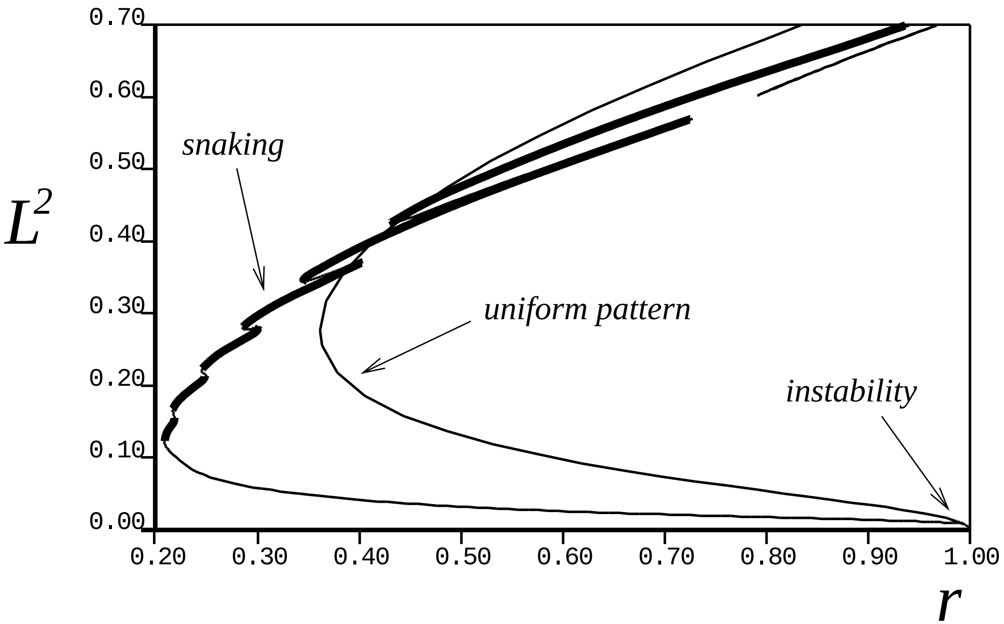


- t – bifurcation from trivial state, at $\mu = q$.
- Solid lines: sn_1, sn_3 – saddle-nodes on the localised branch. Modulated states exist between sn_1 and sn_3. Scalings: $q \approx 0.0927(\mu + 3.55)^{1.987}$, $q \approx 0.298(\mu + 27.9)^{0.986}$. Different!

Return to (w, B) equations

$$w_t = [r - (1 + \partial_{xx}^2)^2]w - w^3 - QB^2 w \qquad (1)$$

$$B_t = \zeta B_{xx} + \frac{c}{\zeta}(w^2 B)_{xx} \qquad (2)$$

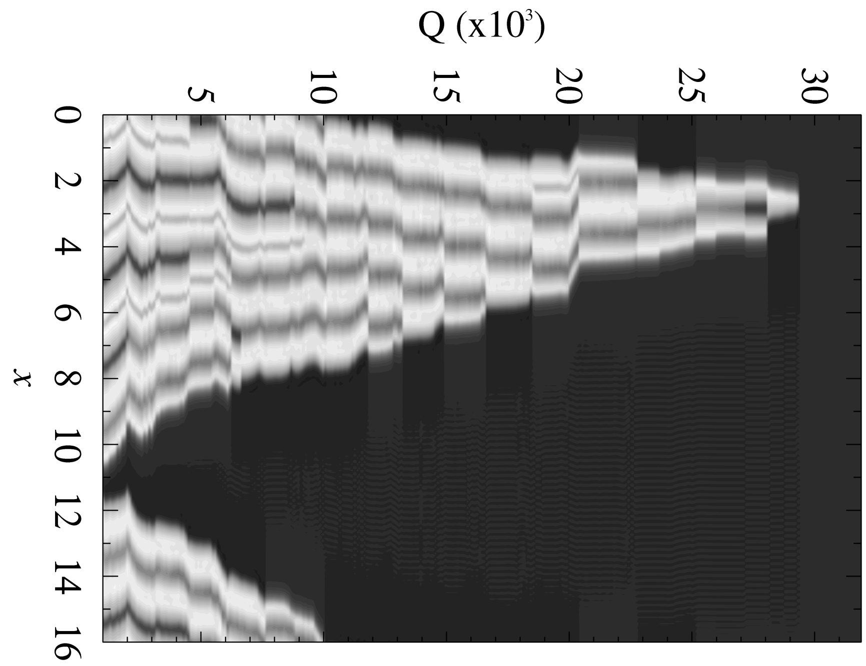


Slanted snaking

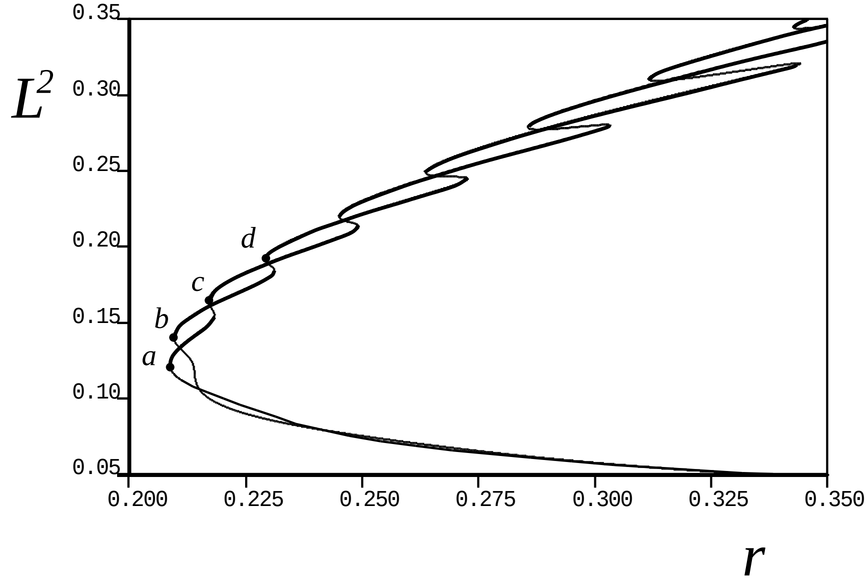
J.H.P. Dawes, Localised pattern formation with a large-scale mode: slanted snaking. *SIAM J. App. Dyn. Syst.* **7**, 186–206 (2008)

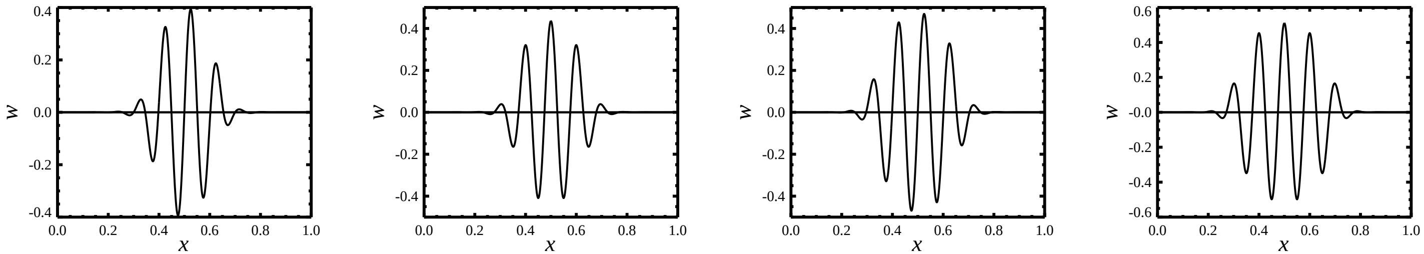
Slanted snaking - details

Full magnetoconvection equations:



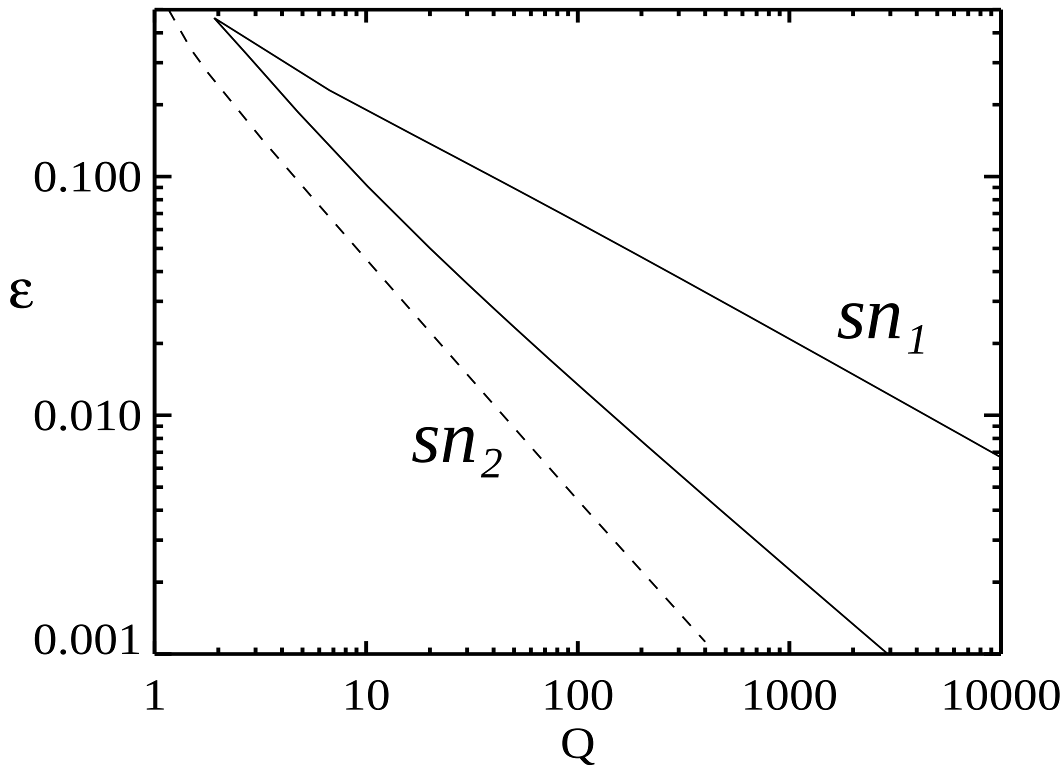
Toy model:





(a) (b) (c) (d)

Scaling laws for (w, B) equations



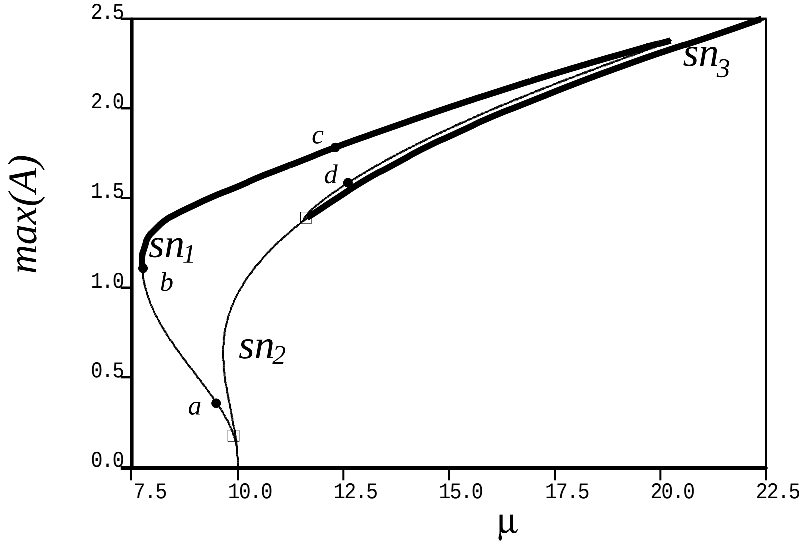
Solid lines contain the most subcritical part of snake. Dashed line = limit of subcriticality of periodic pattern.

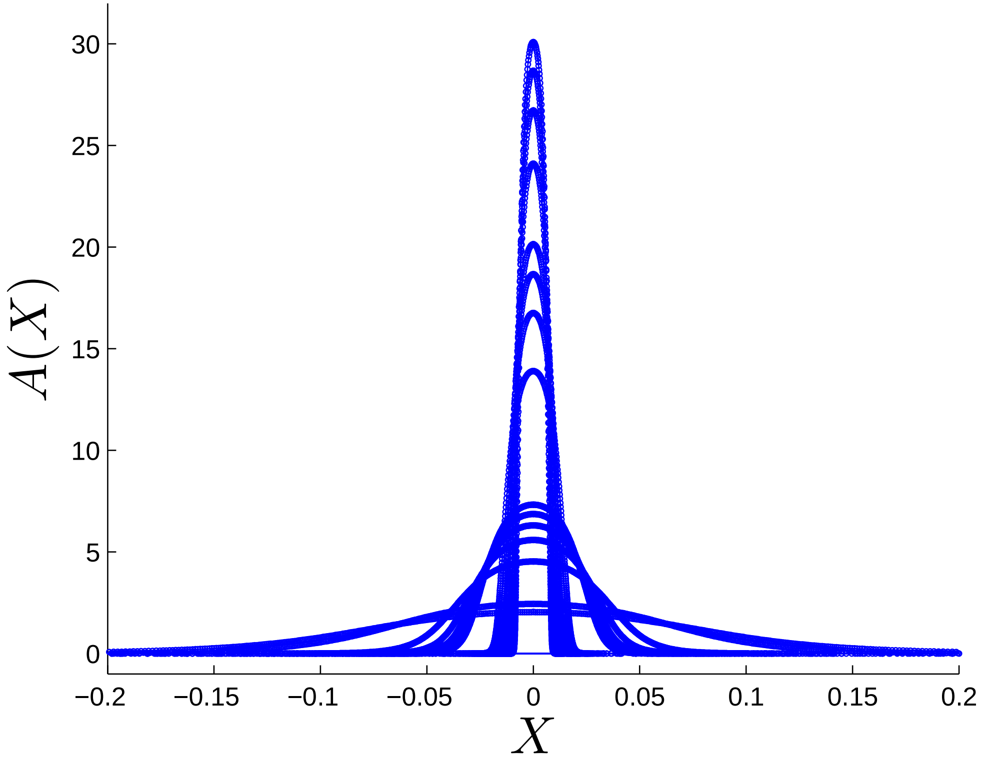
- For sn_1 (lower limit of snake): $\varepsilon \sim Q^{-1/2}$ which agrees with nonlocal GL equation.
- Next twist above sn_1 scales as $\varepsilon \sim Q^{-3/4}$ - this scaling is not obvious.

Fully nonlinear solutions

Nonlocal G-L equation:

$$0 = \mu A + 4A_{XX} - \frac{3}{4}A^3 - \frac{qP^2 A}{(1+cA^2)^{3/2}} \qquad \text{where} \qquad \frac{1}{P} = \frac{1}{L}\int_0^L \frac{1}{\sqrt{1+cA^2}}\, dX$$



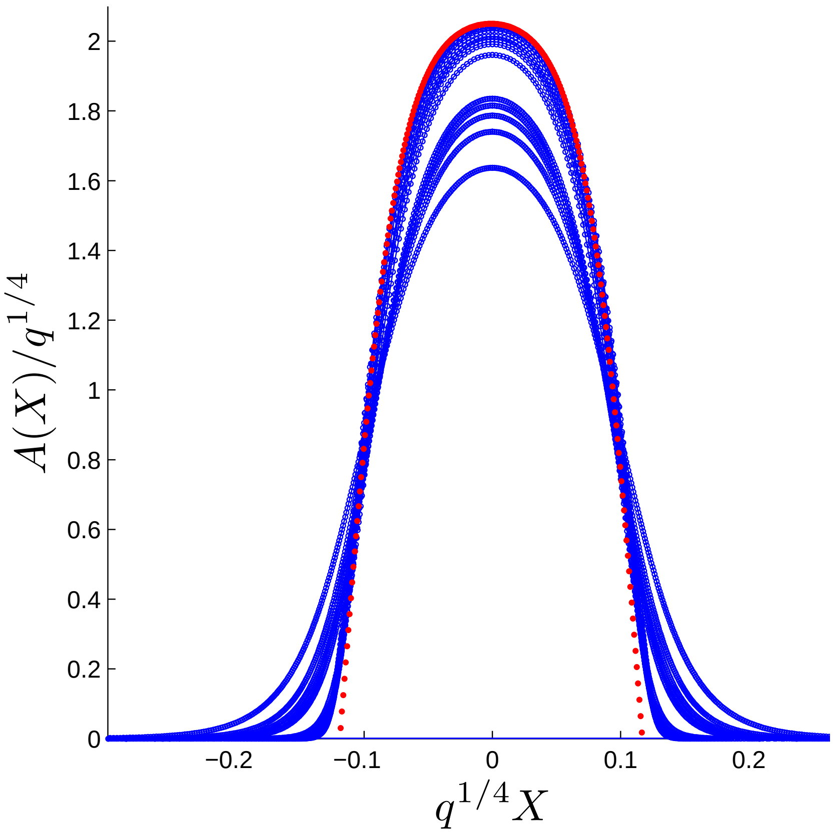


sn_1 - minimum $\mu_{sn}(q)$ at which localised states exist.

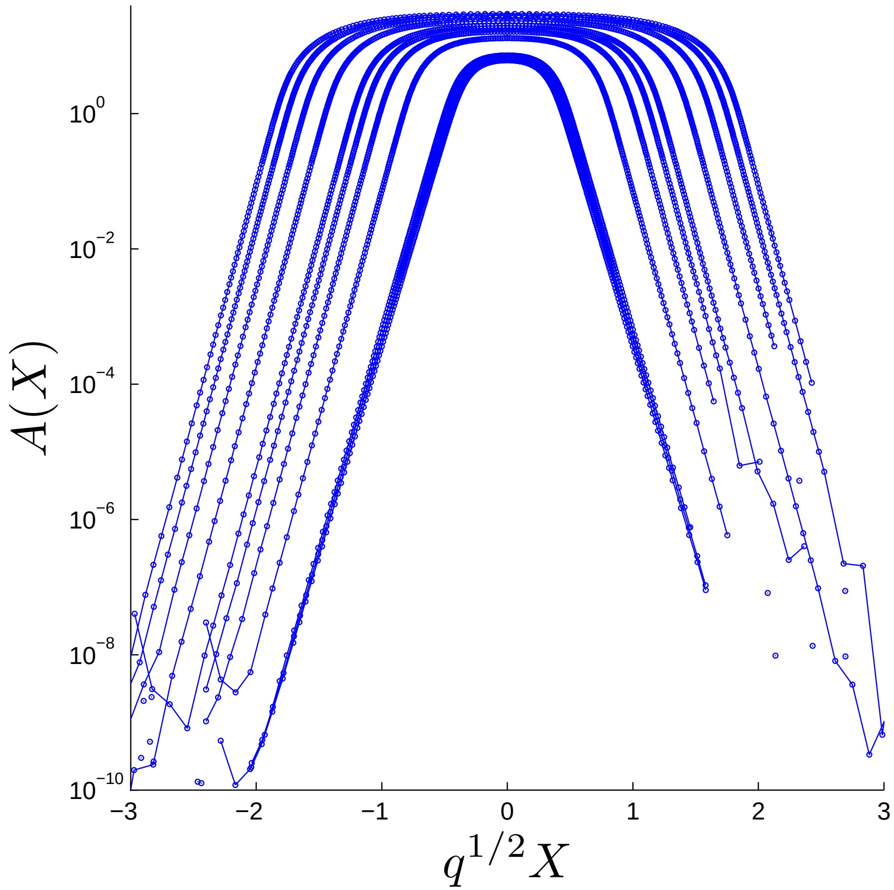
Profiles along sn_1: $c = 0.25$, $L = 10\pi$.

Suggestive rescalings

Centre:



Tails:



Parameters: $c = 0.25$, $L = 10\pi$.

Asymptotic regimes

$$0 = \mu A + 4A_{XX} - \frac{3}{4}A^3 - \frac{qP^2A}{(1+cA^2)^{3/2}}$$

- Consider the general rescaling $A(X) = q^\alpha B(\xi) \qquad \xi = q^\beta X$.
- Four regimes:

1. $\alpha < 0$ $\Rightarrow 4q^{2\beta}B_{\xi\xi} \sim qP^2B$ and $\beta = 1/2$. Linear
2. $\alpha = 0$ $\Rightarrow 4B_{\xi\xi} \sim \frac{P^2B}{(1+cB^2)^{3/2}}$ and $\beta = 1/2$. Difficult
3. $\alpha = \beta = 1/5$ $\Rightarrow 4B_{\xi\xi} \sim \frac{3}{4}B^3 \sim \frac{P^2}{c^{3/2}B^2}$. Difficult
4. $\alpha > 1/5$ $\Rightarrow \alpha = \beta$ and so $4B_{\xi\xi} \sim \frac{3}{4}B^3$. Large amplitude

- Focus on regimes 1 and 4: ‘outer’ and ‘inner’.

Patching

Construct even-symmetric solutions in $-L/2 \le X \le L/2$:

- Outer solution $A_{out}(X)$ in $X^* < |X| < L/2$ – regime 1.
- Inner solution $A_{in}(X)$ in $-X^* < X < X^*$ – regime 4.

Outer solution: $0 = \mu A + 4A_{XX} - qP^2 A$

$$\begin{aligned} A_{out}(X) &= \tilde{A}_1 \cosh((X - L/2)\sqrt{qP^2 - \mu}/2) \\ \Rightarrow A_{out}(X) &\approx \frac{A_1}{2} \exp\left(-X\sqrt{qP^2 - \mu}/2\right) \end{aligned}$$

1 unknown constant: A_1.

Inner solution: let $\lambda = q^{-2\alpha}\mu$, $\xi = q^{-\alpha}X$, $B(\xi) = q^{-\alpha}A(X)$ for some $\alpha > 1/5$. Then

$$0 = \lambda B + 4B_{\xi\xi} - \frac{3}{4}B^3$$

Patching

Inner solution:

$$0 = \lambda B + 4B_{\xi\xi} - \frac{3}{4}B^3$$

has the solution

$$B(\xi) = B_0 \,\mathrm{sn}\,(\eta \,|\, m)$$

where

$$\eta := \xi \left(\frac{\lambda}{4} - \frac{3B_0^2}{32}\right)^{1/2} + K(m) \qquad m := \frac{3B_0^2/32}{\lambda/4 - 3B_0^2/32}$$

and

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{(1 - m\sin^2\theta)^{1/2}}.$$

$K(m)$ is a quarter-period of sn, i.e. $\mathrm{sn}(\eta + 4K|m) = \mathrm{sn}(\eta|m)$.

Patching

- In $0 < X < X^*$: $A_{in}(X) = A_0 \,\mathrm{sn}\left(\left(\frac{\mu}{4} - \frac{3A_0^2}{32} \right)^{1/2} X + K(m) \middle| m \right)$
- In $X^* < X < L/2$: $A_{out}(X) = \frac{A_1}{2} \exp\left(-X\sqrt{qP^2 - \mu}/2 \right)$
- There are 2 unknown constants: A_0 and A_1, plus the patch point X^*.
- Requires 3 equations:

$$\begin{aligned} a &= A_{in}(X^*) \\ a &= A_{out}(X^*) \\ A'_{in}(X^*) &= A'_{out}(X^*) \end{aligned}$$

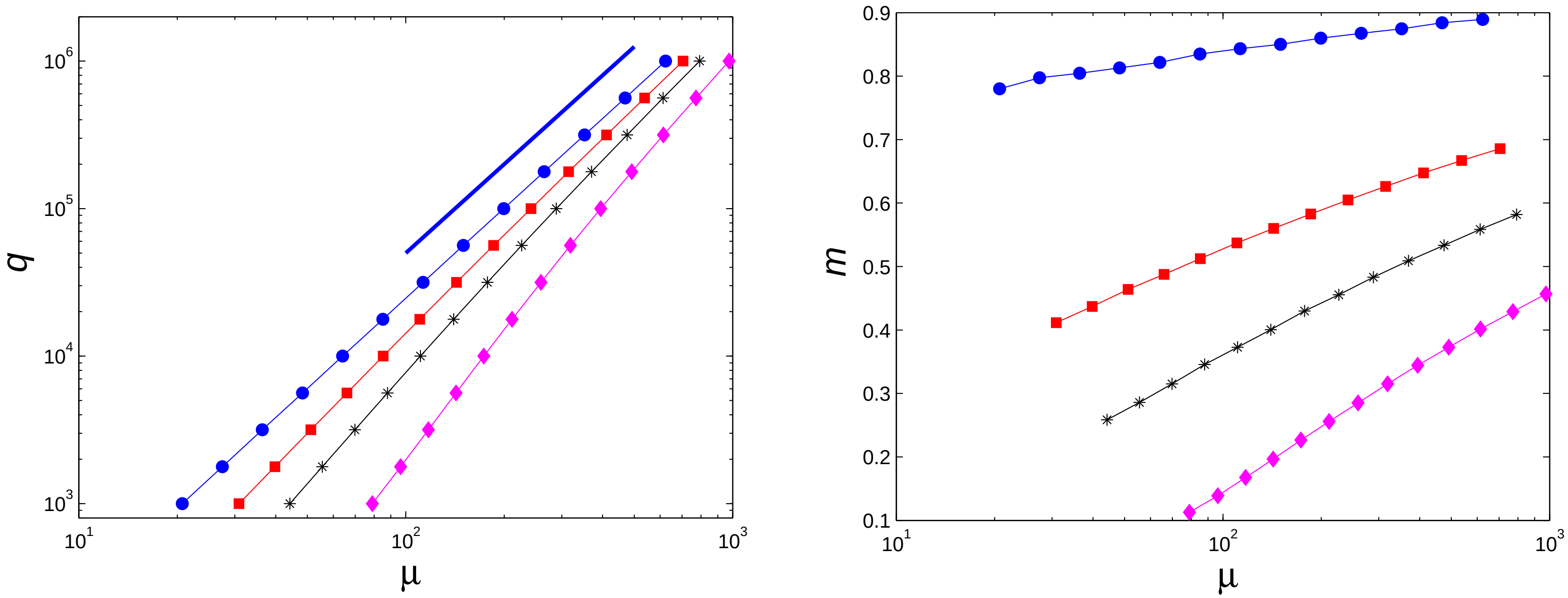
where we fix the constant $a = 0.1$ (fit parameter);

- In addition we have to solve for P:

$$\frac{1}{P} = \frac{2}{L}\left(\int_0^{X^*} \frac{1}{\sqrt{1 + cA_{in}(X)^2}}\, dX + \int_{X^*}^{L/2} \frac{1}{\sqrt{1 + cA_{out}(X)^2}}\, dX \right)$$

Patching - results

- Locations of saddle-node bifurcations $\mu_{sn}(q)$:



Domain sizes: $L = 10\pi$, dots; $L = \pi$, squares; $L = \pi/2$, asterisks; $L = \pi/4$, diamonds.

Remarks:

- Blue line indicates $\mu \sim q^{1/2}$ scaling
- At fixed L, values of $0 < m < 1$ tend (slowly) to 1 as $q \to \infty$.

Derivation of $\mu_{sn} \sim q^{1/2}$

- As $q \to \infty$, μ_{sn} increases, hence so does the constant $A_0 (= A(X = 0)$.
- So $\mathrm{sn}(\cdot|m)$ must tend to zero, and so can be approximated by Taylor series.
- The patching conditions can be combined into the form

$$\frac{A'_{in}(X^*)}{A_{in}(X^*)} = \frac{A'_{out}(X^*)}{A_{out}(X^*)},$$

which is useful since the $\exp(\cdot)$ factors on the RHS cancel, leaving

$$\frac{(\mu/4 - 3A_0^2/32)^{1/2}}{K - (\mu/4 - 3A_0^2/32)^{1/2} X^*} = -q^{1/2}\frac{P}{2}.$$

- Further simplification of the denominator of the LHS leads to

$$a \sim \left(\frac{4\lambda}{3}\right)^{1/2} q^{\alpha/2} \frac{2}{Pq^{1/2}} \left(\frac{\lambda}{8} q^{\alpha}\right)^{1/2}.$$

- which implies $\alpha = 1/2$.

Variational Approximation (VA)

Idea:

- For equations whose steady solutions extremise a Lagrangian

$$\mathcal{L} = \int_0^\infty F(w, w_x, w_{xx}, \ldots)\, dx$$

choose a parameterised family for $w(x)$, say $w(x) = f(x; a_1, \ldots, a_k)$.

- Then compute $\mathcal{L}$ by direct integration to give an 'effective Lagrangian' restricted to the family f:

$$\mathcal{L}_{\text{eff}}(a_1, \ldots, a_k) = \int_0^\infty F(f, f_x, f_{xx}, \ldots)\, dx$$

- Then, functions that extremise $\mathcal{L}$ can be approximately found by extremising $\mathcal{L}_{\text{eff}}$ with respect to the parameters $a_1, \ldots, a_k$.

- So we are left with the simpler problem of solving the k (nonlinear algebraic) equations

$$\frac{\partial \mathcal{L}_{\text{eff}}}{\partial a_1} = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial a_2} = \cdots = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial a_k} = 0.$$

H. Susanto & P.C. Matthews *PRE* **83** 035201(R) (2011); Chong, Pelinovsky, Malomed ...

VA for modulation equation

The nonlocal Ginzburg–Landau equation

$$0 = \mu A + 4A_{XX} - \frac{3}{4}A^3 - \frac{qP^2 A}{(1+cA^2)^{3/2}}$$

has the (surprisingly simple) Lagrangian

$$\mathcal{L} = \int_0^L \left(-\frac{\mu}{2}A^2 + 2(A_X)^2 + \frac{3}{16}A^4 \right) dX + \frac{qL}{c}P$$

where, as before,

$$\frac{1}{P} = \frac{1}{L}\int_0^L \frac{1}{\sqrt{1+cA^2}}\, dX$$

Choose a *very* simple family of solutions to extremise over – step functions:

$$A(X) = \begin{cases} a \text{ in } 0 < X < \ell \\ 0 \text{ in } \ell < X < L \end{cases}$$

2 parameters: a, ℓ.

VA calculations

We obtain:

$$\mathcal{L}_{\text{eff}} \quad = \quad -\frac{\mu}{2}a^2\ell + \frac{3a^4}{16}\ell + \frac{qL^2}{c}\frac{\sqrt{1+ca^2}}{\ell + (L-\ell)\sqrt{1+ca^2}}$$

- Now compute $\partial\mathcal{L}_{\text{eff}}/\partial a$ and $\partial\mathcal{L}_{\text{eff}}/\partial\ell$, and solve.
- ... at least, solve in the limit of large amplitude, $a \gg 1$.

The limit of large a:

- First compute P:

$$P = \frac{L\sqrt{c}a}{\ell + (L-\ell)\sqrt{c}a}$$

- Two cases:
  - if $L - \ell = O(1)$ then $P \sim L/(L-\ell) = O(1)$
  - if $L - \ell = u/a \ll 1$ where $u \sim 1$ then $P \sim aL\sqrt{c}/(\ell + u\sqrt{c}) = O(a) \gg 1$

VA calculations

Case 1: $a \gg 1$, $L - \ell = O(1)$.

- We find that $\mu = O(a^2)$, so, expanding in powers of a, we obtain

$$\mu = \frac{3a^2}{4} + \frac{3}{16\sqrt{c}}a + O(1)$$

and so

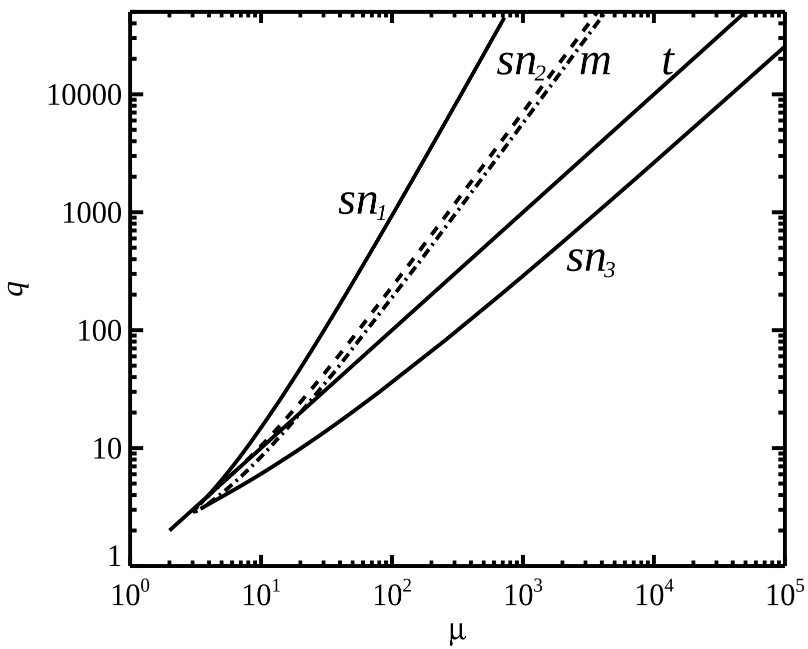
$$q = \left(\frac{L-\ell}{L}\right)^2 \frac{3c}{16}a^4 + \cdots \sim \left(\frac{L-\ell}{L}\right)^2 \frac{c}{3}\mu^2$$

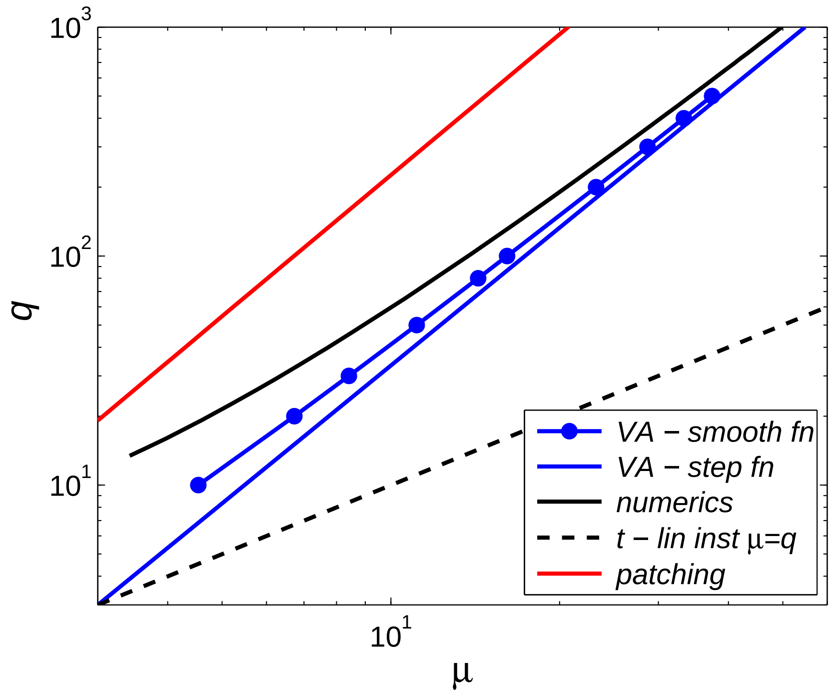
- Lower limit (i.e. saddle-node) of this is then at $\ell = 0$.
- This prediction for the location of sn_1 is broadly in agreement with numerics in the case $c = 0.25$:

$$\begin{aligned} \text{numerics}: &\qquad q \sim 0.0927\mu^{1.987} \\ \text{theory}: &\qquad q \sim 0.0833\mu^2 \end{aligned}$$

VA results for sn_1

$c = 0.25$, $L = 10\pi$:





- Recall that patching method has a free parameter.
- VA using a step function performs as well as using a smooth ansatz in the form

$$A(X) \quad = \quad \frac{a}{\sqrt{1 + \exp(b(|X| - \ell))}}$$

VA calculations

Case 2: $a \gg 1$**,** $L - \ell = \frac{u}{a} \ll 1$**.**

- As in case 1, $\mu = O(a^2)$, so, expanding in powers of a, we obtain

$$\mu = \frac{3a^2}{4} + \frac{3}{16\sqrt{c}}a + O(1)$$

but now

$$q = \left(\frac{L + u\sqrt{c}}{L}\right)^2 \frac{3}{16}a^2 + \cdots \sim \left(\frac{L + u\sqrt{c}}{L}\right)^2 \frac{\mu}{4}$$

- Upper limit (i.e. saddle-node) of this is then at $u = 0$, and is independent of c (at leading order).
- This prediction for the location of sn_3 is broadly in agreement with numerics in the case $c = 0.25$:

$$\text{numerics}: \quad q \sim 0.298\mu^{0.986}$$

$$\text{theory}: \quad q \sim 0.25\mu$$

Axisymmetric solutions

Steady, axisymmetric solutions to the system

$$\begin{aligned} w_t &= rw - (1+\nabla^2)^2 w - w^3 - QB^2 w \\ B_t &= \varepsilon \nabla^2 B + \frac{c}{\varepsilon}\nabla^2 (w^2 B) \end{aligned}$$

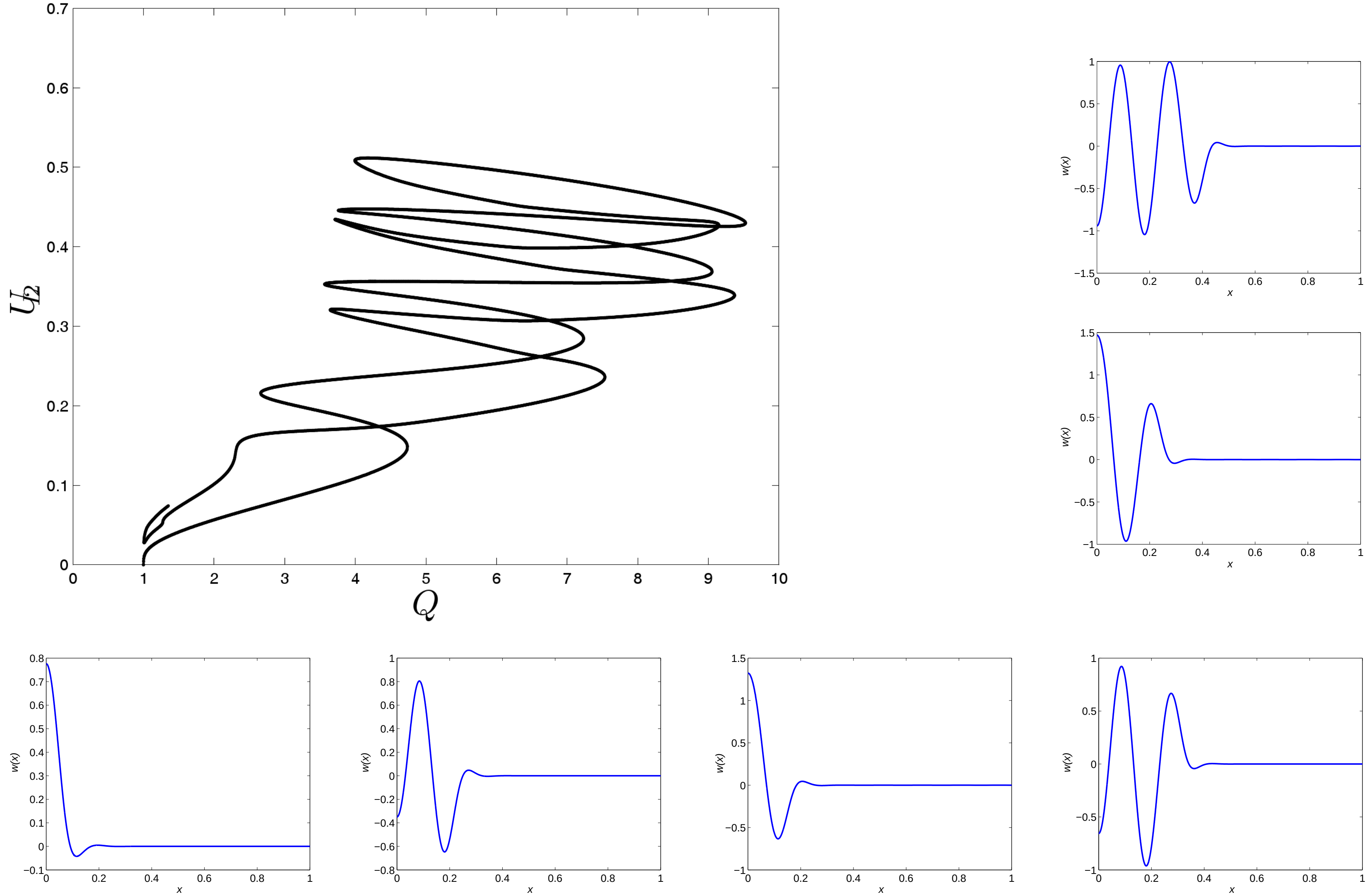
in $\mathbb{R}^n$ satisfy the nonlocal ODE

$$\begin{aligned} w_{rrrr} &= (\mu-1)w - 2w_{rr} - \frac{2(n-1)}{r} w_r + \frac{(n-1)(n-3)}{r^3} w_r \\ &\quad - \frac{(n-1)(n-3)}{r^2} w_{rr} - \frac{2(n-1)}{r} w_{rrr} - w^3 - \frac{QP^2 w}{(1+cw^2/\varepsilon^2)^2} \end{aligned}$$

which contains the integral contribution

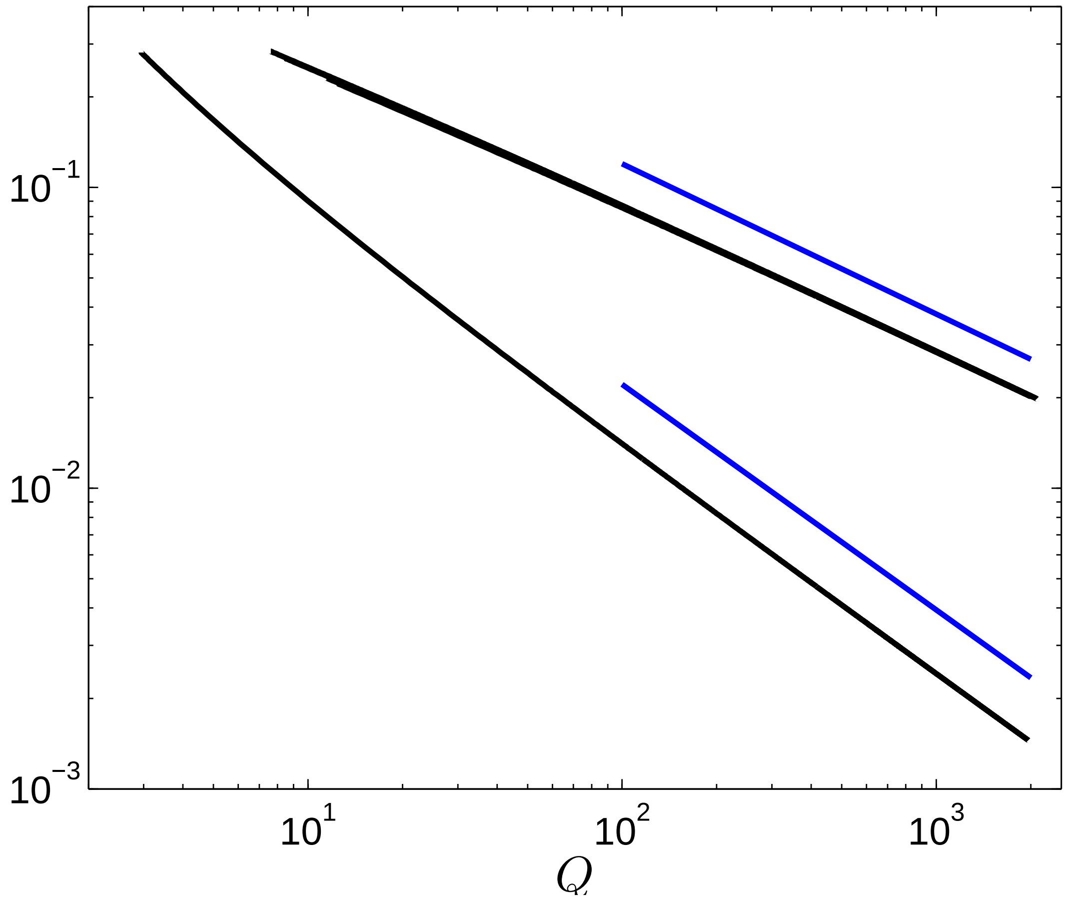
$$P^{-1} = \langle (1+cw^2/\varepsilon^2)^{-1} \rangle \qquad \text{where} \qquad \langle f \rangle := \frac{n}{L^n}\int_0^L f(r) r^{n-1}\, dr$$

2D – Spot AB



2D – Spot AB

Existence region (limits of snaking curve) opens out with same scalings as in 1D:



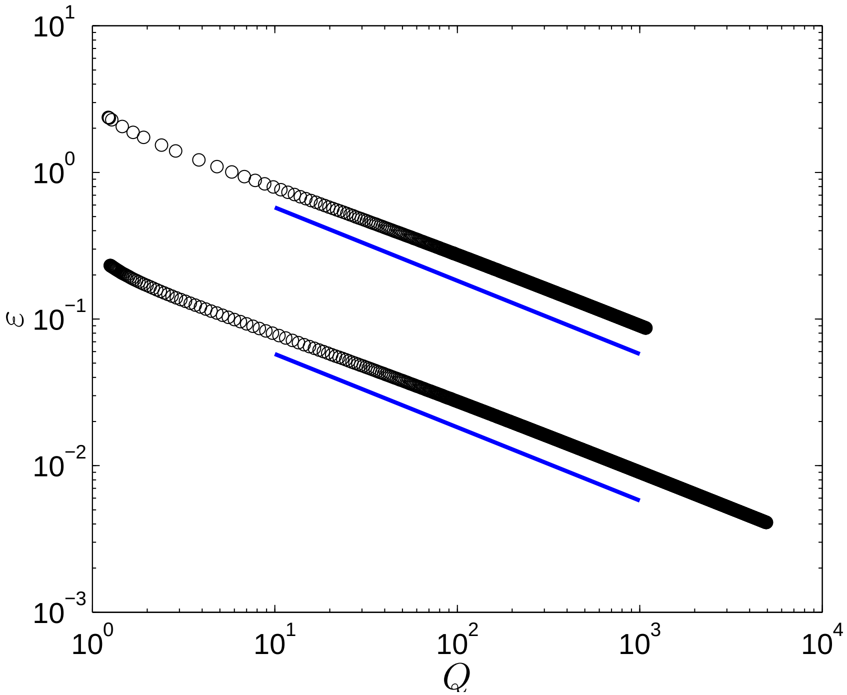
- Upper line: $\varepsilon \sim Q^{-1/2}$; lower line: $\varepsilon \sim Q^{-3/4}$.

2D - varying c

- The scaling law for sn_1 from the 1D nonlocal GL equation predicts

$$\varepsilon \sim \left(\frac{cr^2}{3}\right)^{1/2} Q^{-1/2}$$

- This appears to hold in 2D as well, for both the Q and c dependencies.
- $c = 10$ (upper) and $c = 0.1$ (lower).
- $r = 1$ (i.e. not 'large')



Summary

- Nonlocal terms arise naturally from conservation laws.
- Such terms strongly distort standard snaking into slanted snaking.
- This distortion means localised states exist over a wider region of parameter space - and perhaps are physically more robust
- Reduction to nonlocal GL equation and construction of approximate solutions helps reveal the origin of scaling laws, and hence prediction of (wider) parameter regime over which localised states exist.
- Region of existence is reduced, but not in fact by much, as domain size L decreases.
- Parameter c affects prefactors but not exponents in scaling laws.
- ... and the 1D scaling laws appear to carry over into 2D (for spots).