Heat kernel and mixing time convergence for sequences of simple random walks on graphs

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David Croydon (University of Warwick)

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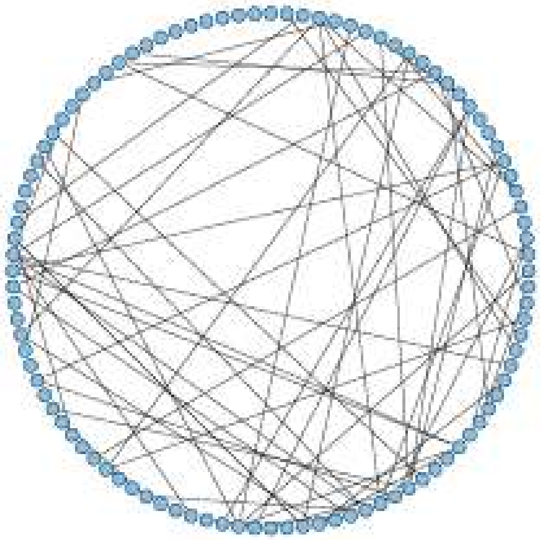
B. M. Hambly (Oxford) and T. Kumagai (Kyoto)

CRITICAL ERDŐS-RÉNYI RANDOM GRAPH

$G(N, p)$ is obtained via bond percolation with parameter p on the complete graph with N vertices. We concentrate on critical window: $p = N^{-1} + \lambda N^{-4/3}$. e.g. $N = 100$, $p = 0.01$:



All components have:
- size $\Theta(N^{2/3})$ and surplus $\Theta(1)$ [Erdős-Rényi], [Aldous],
- diameter $\Theta(N^{1/3})$ [Nachmias, Peres].

Moreover, asymptotic structure of components is known [Addario-Berry, Broutin, Goldschmidt].

COMPONENT MIXING TIMES

For a component $\mathcal{C}$, let $(X_t^{\mathcal{C}})_{t \geq 0}$ be the corresponding discrete-time simple random walk.

The invariant probability measure for $X^{\mathcal{C}}$ is given by

$$\pi^{\mathcal{C}}(\{x\}) \propto \mathrm{deg}(x).$$

The mixing time of $X^{\mathcal{C}}$ is given by

$$t_{\mathrm{mix}}(\mathcal{C}) := \inf \left\{ t \geq 0 : \sup_{x \in \mathcal{C}} \left\| \mathrm{P}_x^{\mathcal{C}}(X_t^{\mathcal{C}} = \cdot) - \pi^{\mathcal{C}}(\cdot) \right\|_{TV} \leq 1/8 \right\}.$$

The mixing times of critical random graph components are $\Theta(N)$ in probability [Nachmias, Peres].

CONVERGENCE OF MIXING TIMES

Suppose $t_{\text{mix}}(\mathcal{C}_1)$ is the mixing time of the largest component of $G(N,p)$ in the critical window, can we prove that

$$N^{-1}t_{\text{mix}}(\mathcal{C}_1)$$

converges in distribution?

Plan:

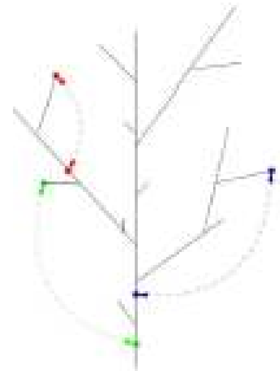
- Recall metric space scaling limit $\mathcal{M}_1$.
- Construction of diffusion on $\mathcal{M}_1$.
- Random walk scaling limit result.
- Convergence of mixing times.
- Other examples of mixing time convergence.

CRITICAL RANDOM GRAPH SCALING LIMIT
[Addario-Berry, Broutin, Goldschmidt] .....

The random metric space scaling limit $\mathcal{M}_1$ of the largest component of the critical random graph is defined by:

1. Choosing a random compact real tree $\tilde{\mathcal{T}}$.

2. Gluing a random, but finite, number of pairs of points together.



Picture produced by Christina Goldschmidt.

We will let $\phi : \tilde{\mathcal{T}} \to \mathcal{M}_1$ be the natural quotient map induced by the gluing of pairs of vertices of $\tilde{\mathcal{T}}$.

BROWNIAN MOTION ON REAL TREES

Let $(\mathcal{T}, d_{\mathcal{T}})$ be a compact real tree, and $\mu^{\mathcal{T}}$ be a Borel measure on $\mathcal{T}$ with full support.

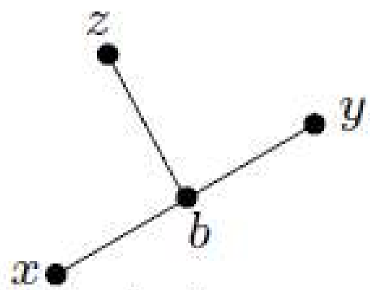
$X^{\mathcal{T}} = (X_t^{\mathcal{T}})_{t\geq 0}$ is a Brownian motion on $(\mathcal{T}, d_{\mathcal{T}}, \mu^{\mathcal{T}})$ if it satisfies:

- Strong Markov diffusion.

- Reversible, invariant measure $\mu^{\mathcal{T}}$.

- For $x, y, z \in \mathcal{T}$,

$$\mathbf{P}_z(\tau_x < \tau_y) = \frac{d_{\mathcal{T}}(b(x,y,z),y)}{d_{\mathcal{T}}(x,y)}.$$



- Mean occupation density when started at x and killed at y,

$$2d_{\mathcal{T}}(b(x,y,z),y)\mu^{\mathcal{T}}(dz).$$

RESISTANCE FORM CONSTRUCTION

"Resistance, $d_{\mathcal{T}} \leftrightarrow$ Electrical energy, $\mathcal{E}_{\mathcal{T}}$"

[Kigami] $\exists$ a symmetric, bilinear form $(\mathcal{E}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}})$ satisfying

$$d_{\mathcal{T}}(x,y)^{-1} = \inf\{\mathcal{E}_{\mathcal{T}}(f,f) : f(x)=1,\ f(y)=0\},$$

for $x \neq y$. Moreover,

$$(\mathcal{E}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}} \cap L^2(\mathcal{T}, \mu^{\mathcal{T}}))$$

is a conservative, irreducible, local, regular Dirichlet form, for any Borel measure $\mu^{\mathcal{T}}$ on $\mathcal{T}$ with full support.

We can subsequently define a corresponding Markov process $X^{\mathcal{T}}$, and it is possible to check that this is Brownian motion on $(\mathcal{T}, d_{\mathcal{T}}, \mu^{\mathcal{T}})$.

FUSING RESISTANCE FORMS

Suppose $\mathcal{M}$ is obtained by gluing together a finite number of pairs of vertices of $\mathcal{T}$, and $\phi : \mathcal{T} \to \mathcal{M}$ is the natural quotient map.

We define a quadratic form on the glued space by setting

$$\mathcal{E}_{\mathcal{M}}(f,f) := \mathcal{E}_{\mathcal{T}}(f \circ \phi, f \circ \phi),$$

for any $f \in \mathcal{F}_{\mathcal{M}}$, where

$$\mathcal{F}_{\mathcal{M}} := \{f : \mathcal{M} \to \mathbb{R} : \ f \circ \phi \in \mathcal{F}_{\mathcal{T}}\}.$$

$(\mathcal{E}_{\mathcal{M}}, \mathcal{F}_{\mathcal{M}})$ is a local, regular Dirichlet form on $L^2(\mathcal{M}, \mu^{\mathcal{M}})$, where $\mu^{\mathcal{M}} := \mu^{\mathcal{T}} \circ \phi^{-1}$. Thus, there is a corresponding Markov diffusion $X^{\mathcal{M}}$, which we call Brownian motion on $\mathcal{M}$.

BROWNIAN MOTION ON $\mathcal{M}_1$

Using the above construction, for almost-every realisation of $\mathcal{M}_1$, the metric space limit of $N^{-1/3}\mathcal{C}_1$, we can define a Brownian motion $X^{\mathcal{M}_1}$, and it is possible to check that

$$\left(N^{-1/3}X^{\mathcal{C}_1}_{\lfloor tN \rfloor}\right)_{t\geq 0} \to \left(X^{\mathcal{M}_1}_t\right)_{t\geq 0},$$

in distribution in both a quenched (for almost-every environment) and annealed (averaged over environments) sense. Here, both $X^{\mathcal{C}_1}$ and $X^{\mathcal{M}_1}$ are started from a distinguished vertex.

The precise topology under which this result is obtained is a generalised Gromov-Hausdorff topology for processes on compact length spaces.

Proof uses restriction to finite line-segment subgraphs.

FROM RANDOM WALK TO MIXING TIME CONVERGENCE

First we check convergence of transition densities:

$$q^N_{\lfloor tN \rfloor}(\rho, x_N) \approx \frac{\mathbf{P}\left(X^{\mathcal{C}_1}_{\lfloor tN \rfloor} \in B(x_N, \varepsilon N^{1/3})\right)}{\pi^{\mathcal{C}_1}(B(x_N, \varepsilon N^{1/3}))} \to \frac{\mathbf{P}\left(X^{\mathcal{M}_1}_t \in B(x, \varepsilon)\right)}{\pi^{\mathcal{M}_1}(B(x, \varepsilon))} \approx q_t(\rho, x).$$

where $N^{-1/3}x_N \to x$ as $N \to \infty$ [C, Hambly]. Then

$$\begin{aligned} t_{\text{mix}}(\mathcal{C}_1, \rho) &:= \inf\left\{m > 0 : \|q^N_m(\rho, \cdot) - 1\|_1 \leq 1/4\right\} \\ &\approx N \inf\{t > 0 : \|q_t(\rho, \cdot) - 1\|_1 \leq 1/4\} \\ &=: N t_{\text{mix}}(\mathcal{M}_1, \rho). \end{aligned}$$

In particular, we can rigourously establish

$$N^{-1} t_{\text{mix}}(\mathcal{C}_1, \rho) \to t_{\text{mix}}(\mathcal{M}_1, \rho).$$

SPECTRAL GROMOV-HAUSDORFF DISTANCE

For compact metric spaces F, F' equipped with Borel probability measures π, π' and jointly continuous heat kernels q, q', define for a compact time interval $I \subset (0, \infty)$,

$$\begin{aligned}
&\Delta_I \Big((F, \pi, q), (F', \pi', q') \Big) \\
&:= \inf_{Z,\phi,\phi',\mathcal{C}} \Bigg\{ d_H^Z(\phi(F), \phi'(F')) + d_P^Z(\pi \circ \phi^{-1}, \pi' \circ \phi'^{-1}) \\
&\quad + \sup_{(x,x'),(y,y') \in \mathcal{C}} \Bigg(d_Z(\phi(x), \phi'(x')) + d_Z(\phi(y), \phi'(y')) \, , \\
&\quad\quad + \sup_{t \in I} \Big| q_t(x,y) - q'_t(x',y') \Big| \Bigg) \Bigg\} .
\end{aligned}$$

This defines a separable metric on (equivalence classes of) triples of the form (F, π, q). cf. work on Riemannian manifolds of [Bérard, Besson, Gallot], [Kasue, Kumura].

GENERAL MIXING TIME CONVERGENCE THEOREM

Suppose that, for any compact interval $I \subset (0, \infty)$,

$$\left(\left(V(G^N), d_{G^N}\right), \pi^N, \left(q^N_{\gamma(N)t}(x,y)\right)_{x,y \in V(G^N), t \in I}\right)$$

converges to

$$\left((F, d_F), \pi, (q_t(x,y))_{x,y \in F, t \in I}\right)$$

in a spectral Gromov-Hausdorff sense, then $t_{\text{mix}}(F) \in (0, \infty)$ and

$$\gamma(N)^{-1} t_{\text{mix}}(G^N) \to t_{\text{mix}}(F).$$

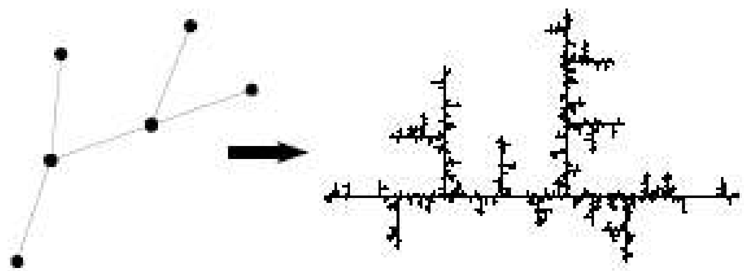
It is also possible to prove the same result when the mixing times are defined in terms of the L^p distance, for any $p \in [1, \infty]$.

EXAMPLE: CRITICAL GALTON-WATSON TREES

For the simple random walk X^N on T^N, a Galton-Watson tree with a critical (mean 1), aperiodic, finite variance offspring distribution, conditioned to have N vertices, started from root ρ^N,

$$\left(N^{-1/2}X^N_{\lfloor tN^{3/2}\rfloor}\right)_{t\geq 0} \to \left(X^{\mathcal{T}}_t\right)_{t\geq 0},$$

where $X^{\mathcal{T}}$ is the Brownian motion on the continuum random tree, started from its root ρ [C].



(Scaling of graphs in [Aldous]. See also [Duquesne, Le Gall].)

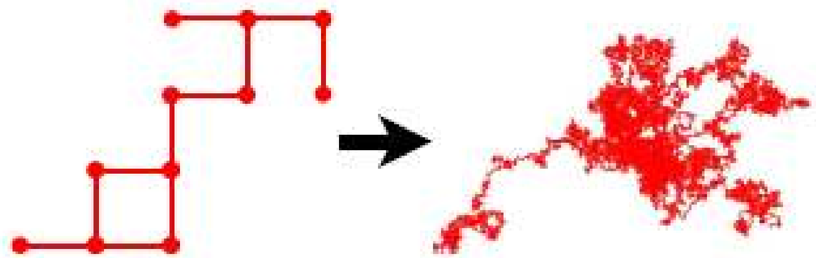
For mixing times: $N^{-3/2}t^p_{\text{mix}}(\rho^N) \to t^p_{\text{mix}}(\rho)$, in distribution.

EXAMPLE: RANDOM WALK TRACE

For the simple random walk X^N on $G^N = S_{[0,N]}$, the trace of the random walk up to time N, in dimensions ≥ 5,

$$\left(N^{-1}X^N_{\lfloor tN^2 \rfloor}\right)_{t\geq 0} \to \left(X^{\mathcal{R}}_{ct}\right)_{t\geq 0},$$

where $X^{\mathcal{R}}$ is the Brownian motion on the range of the Brownian motion up to time 1, $\mathcal{R} := \{B_t : t \in [0,1]\}$.



Result originally proved for entire trace $S_{[0,\infty)}$, see [C].

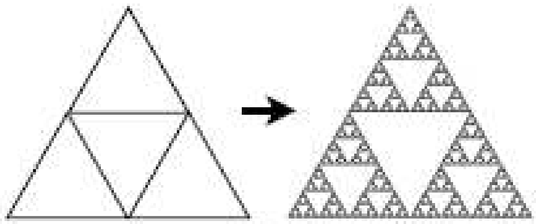
For mixing times: $cN^{-2}t^p_{\text{mix}}(S_{[0,N]}) \to t^p_{\text{mix}}([0,1])$, almost-surely.

EXAMPLE: SELF-SIMILAR FRACTAL GRAPHS

For simple random walk X^N on the pre-nested fractal graph G^N,

$$\left(L^{-N}X^N_{\lfloor t(M\lambda)^N \rfloor}\right)_{t\geq 0} \to \left(X^F_t\right)_{t\geq 0},$$

where L is a length scaling factor, M is a mass scaling factor, and λ is a resistance scaling factor [Lindstrom]. e.g. $L = 2$, $M = 3$, $\lambda = 5/3$ for the S.G.



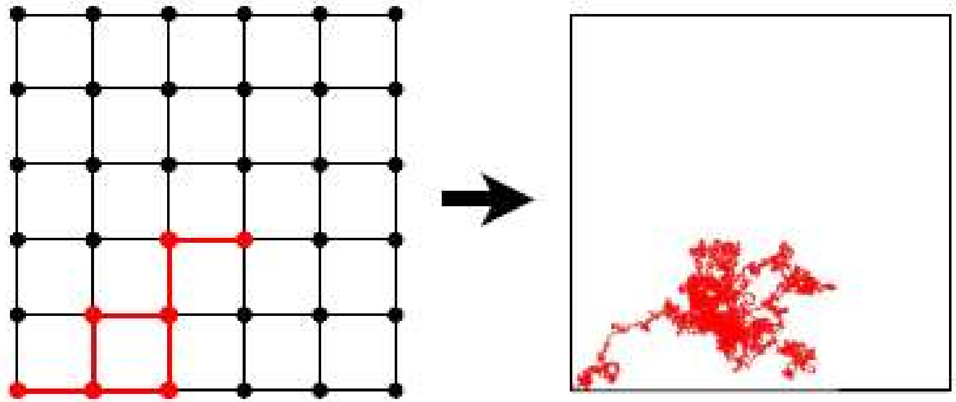
Similarly for p.c.f.s.s. fractal graphs [Kigami] and Sierpinski carpet-type graphs [Barlow, Bass, Kumagai, Teplyaev]. Also random weights in finitely ramified examples [Kumagai, Kusuoka].

For mixing times: $(M\lambda)^{-N} t^p_{\text{mix}}(G^N) \to t^p_{\text{mix}}(F)$, in probability.

EXAMPLE: LATTICE MODELS IN A BOX

For the simple random walk X^N on $\{1,\dots,N\}^d$,

$$\left(N^{-1}X^N_{\lfloor tN^2\rfloor}\right)_{t\geq 0} \to \left(X_t^{[0,1]^d}\right)_{t\geq 0}.$$



For mixing times: $N^{-2}t^p_{\mathrm{mix}}(\{1,\dots,N\}^d) \to t^p_{\mathrm{mix}}([0,1]^d)$.

OPEN PROBLEMS

Lattice homogenisation

Place i.i.d. weights on edges of box $\{1,\ldots,N\}^d$, i.e. random conductor model. Does random walk converge to Brownian motion? Do mixing times converge?

Convergence of spectrum

Do eigenvalues of graphs G^N converge to those of F,

$$-\gamma(N)\ln\lambda_{N,j} \to \lambda_j?$$

In particular, does the spectral gap $\lambda_{N,1}$ converge?