How often is a random quantum state k–entangled?

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Determine the “size” of certain convex sets that appear naturally in quantum information theory.

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Some notation

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- Φ is positive if $\Phi(\rho) \geq 0$ for all $\rho \in \mathcal{M}_d$, $\rho \geq 0$, i.e. positive semidifinite

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$\mathcal{P}_k(\mathcal{M}_d)$= set of k-positive maps on $\mathcal{M}_d$

This set is a positive cone.

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f.i. via Jamiolkowski-Choi isomorphism

$$\mathcal{P}_k(\mathcal{M}_d) \longleftrightarrow \mathcal{BP}_k(\mathbb{C}^d \otimes \mathbb{C}^d),$$

the space of k-block positive $d^2 \times d^2$ matrices.

The set of *k- entangled* operators on $\mathbb{C}^d \otimes \mathbb{C}^d$ is

$$Ent_k(\mathbb{C}^d \otimes \mathbb{C}^d) =$$

$$\mathrm{conv}\left(\left\{|\xi\rangle\langle\xi| : \xi = \sum_{j=1}^{k} u_j \otimes v_j, u_j, v_j \in \mathbb{C}^d, j = 1, \ldots, k\right\}\right)$$

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$\xi = \sum_{j=1}^{k} u_j \otimes v_j \in \mathbb{C}^d \otimes \mathbb{C}^d$ is called a k- entangled vector, i.e.

k-entangled states have rank $\leq k$.

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- For all k: $Ent_k(\mathbb{C}^d \otimes \mathbb{C}^d) \subset Ent_{k+1}(\mathbb{C}^d \otimes \mathbb{C}^d)$

Via Jamiolkowski-Choi isomorphism k-entangled states on $\mathbb{C}^d \otimes \mathbb{C}^d = \mathbb{C}^{d^2}$ are in correspondence with maps on $\mathcal{M}_d$

$$\mathcal{SP}_k(\mathcal{M}_d) \longleftrightarrow Ent_k(\mathbb{C}^d \otimes \mathbb{C}^d)$$

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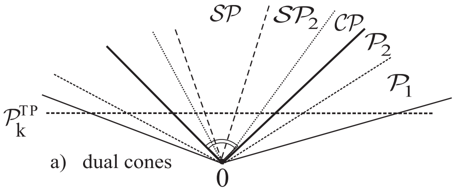
$\mathcal{SP}_k(\mathcal{M}_d)$ is the convex cone of k- superpositive operators Φ on $\mathcal{M}_d$:

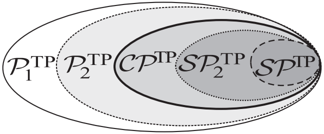
$$\Phi(\rho) = \sum A_i^\dagger \rho A_i$$

such that each A_i has rank $\leq k$.

For $d = 3$



a) dual cones



b) compact, convex sets

Normalizations

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- maps $\Phi : \mathcal{M}_n \to \mathcal{M}_n$ get normalized such that Φ is trace preserving: for all states ρ

  $$Tr(\Phi(\rho)) = Tr(\rho)$$

Here:

$$\mathcal{SP}_k^{TR}(\mathcal{M}_d) =$$
$$\mathcal{SP}_k(\mathcal{M}_d) \cap \{\Phi : \mathcal{M}_d \to \mathcal{M}_d : Tr(\Phi(\rho)) = Tr(\rho), \forall \rho\}$$

Geometrically, this means that we intersect the cones cones with hyperplanes and get convex sets.

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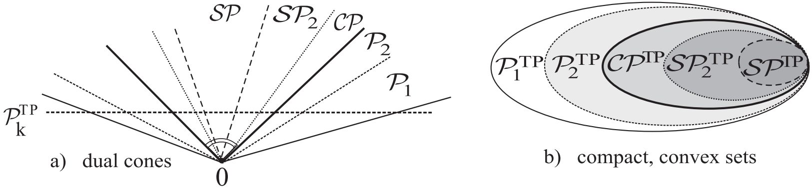

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Measure "size" of a convex body K via **volume radius**

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Urysohn inequality

$$\mathrm{vrad(K)} \leq \frac{1}{2}\mathrm{w(K)}$$

Upper Bound

$$
\begin{aligned}
w\left(Ent_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)\right) &= w\left(ext\left(Ent_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)\right)\right) \\
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&\leq \sqrt{2}w\left(\{\xi : \xi \ k \text{ entangled}, |\xi| = 1\}\right) \\
&= \sqrt{2}w\left(Ent_k^v(\mathbb{C}^d \otimes \mathbb{C}^d)\right)
\end{aligned}
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\begin{aligned}
w(K) &= 2\int_{S^{n-1}} \max_{x\in K}\langle x, u\rangle du \\
&= \gamma_n \int_{\mathbb{R}^n} \max_{x\in K}\langle x, y\rangle d\mu_n(y)
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$N(K,\varepsilon)$ is the smallest N such that there are points $x_1,\dots,x_N$ s.t.

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$$\gamma_n \sim \frac{1}{\sqrt{n}}$$

$$
\begin{aligned}
\mathrm{vrad}\left(\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)\right) &\leq \frac{1}{2}\, w\left(Ent_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)\right) \\
&\leq \frac{1}{\sqrt{2}}\, w\left(Ent_k^v(\mathbb{C}^d \otimes \mathbb{C}^d)\right) \\
&\leq C\gamma_{d^4} \int_0^\infty \sqrt{\log N(Ent_k^v(\mathbb{C}^d \otimes \mathbb{C}^d), \varepsilon))}\; d\varepsilon
\end{aligned}
$$

$$Ent_k^v(\mathbb{C}^d \otimes \mathbb{C}^d) \longleftrightarrow G_{d,k} \times G_{d,k} \times S_{HS}(F,E)$$

$$\tau = \sum_{j=1}^{k} t_j |u_j\rangle\langle v_j| \longleftrightarrow (E, F, T)$$

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$$E = E_\tau = \text{span}\{u_j : 1 \leq j \leq k\}$$

$$F = F_\tau = \text{span}\{v_j : 1 \leq j \leq k\}$$

and $T \in S_{HS}(F, E)$ such that

$$\tau = TP_F$$

Szarek

$$N(G_{d,k},\varepsilon) \le \left(\frac{C}{\varepsilon}\right)^{4k(d-k)}$$

$$N(S_{HS}(F,E),\varepsilon) \le \left(\frac{\tilde{C}}{\varepsilon}\right)^{2k^2}$$

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$$\begin{aligned} N(Ent_k^v(\mathbb{C}^d \otimes \mathbb{C}^d) &\leq \left[\left(\frac{C}{\varepsilon}\right)^{4k(d-k)}\right]^2 \left(\frac{\tilde{C}}{\varepsilon}\right)^{2k^2} \\ &\leq \left(\frac{C'}{\varepsilon}\right)^{8kd} \end{aligned}$$

$$
\begin{aligned}
\mathrm{vrad}\left(\mathrm{Ent}^1_k(\mathbb{C}^d \otimes \mathbb{C}^d)\right) &\leq C\gamma_{d^4} \int_0^1 \sqrt{8kd}\sqrt{\log\left(\frac{C'}{\varepsilon}\right)^{\frac{1}{2}}}\, d\varepsilon \\
&\leq C\frac{k^{\frac{1}{2}}}{d^{\frac{3}{2}}}
\end{aligned}
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$$\begin{aligned} \mathrm{vrad}\left(\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)\right) &\leq C\gamma_{d^4} \int_0^1 \sqrt{8kd}\sqrt{\log\left(\frac{C'}{\varepsilon}\right)^{\frac{1}{2}}}\, d\varepsilon \\ &\leq C\frac{k^{\frac{1}{2}}}{d^{\frac{3}{2}}} \end{aligned}$$

- For $k = 1$ (separable states) and $k = d$ (all states) this gives the right order

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- For $k = 1$ (separable states) and $k = d$ (all states) this gives the right order
- Lower bound: $C\frac{k^{\frac{1}{2}}}{d^{\frac{3}{2}}}$