Controllability of wave and Schrödinger equations with inverse square potential via new energy norm.

Cristian-Mihai, CAZACU

PhD Student

Basque Center for Applied Mathematics (BCAM), Bilbao, Spain

cazacu@bcamath.org

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For any $N \geq 1$, consider $\Omega \subset \mathbb{R}^N$,
Γ=the boundary of Ω,
Γ_0=boundary observable zone with the central of gravity in zero i.e

$$\Gamma_0 := \{x \in \Gamma \mid x \cdot \nu \geq 0\}. \tag{1}$$

The case of interior singularity: $0 \in \Omega$.

Wave-like equation:

$$(W) : \begin{cases} u_{tt} - \Delta u - \frac{\lambda}{|x|^2} u = 0, & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = h(t,x), & (t,x) \in (0,T) \times \Gamma_0, \\ u(t,x) = 0, & (t,x) \in (0,T) \times (\Gamma \setminus \Gamma_0), \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)) & x \in \Omega. \end{cases} \tag{2}$$

Schrödinger-like equation:

$$(S) : \begin{cases} iu_t - \Delta u - \frac{\lambda}{|x|^2} u = 0, & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = h(t,x), & (t,x) \in (0,T) \times \Gamma_0, \\ u(t,x) = 0, & (t,x) \in (0,T) \times (\Gamma \setminus \Gamma_0), \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{3}$$

The problems (W)-(S) are well-posed for $\lambda \leq \lambda_\star := \frac{(N-2)^2}{4}$ because the operator $A_\lambda := -\Delta - \frac{\lambda}{|x|^2}$ is losing the positivity and coercivity for any $\lambda > \lambda_\star$.

Motivation (Hardy inequality): The critical upper-bound $\lambda_\star$ is the sharp constant in the Hardy inequality (Hardy-Littlewood-Polya 1952) i.e. for any $u \in H_0^1(\Omega)$

$$\int_\Omega |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2}, \tag{4}$$

The solutions of (W)-(S) are defined in a weak sense by transposition (Lions '88).

The Hilbert space H_λ

The Hilbert space H_λ induced by the Hardy inequality is defined as the closure of $C_0^\infty(\Omega)$ in the quadratic norm $||\cdot||_{H_\lambda}$,

$$||u||_{H_\lambda}^2 = \int_\Omega \left[|\nabla u|^2 - \lambda \frac{u^2}{|x|^2}\right] dx,$$

for every $\lambda \leq \lambda_\star$.

1. subcritical case $\lambda < \lambda_\star$: $H_0^1(\Omega) = H_\lambda$, due to the equivalence of the norms:
$$\Big(1 - \lambda/\lambda_\star\Big)||u||_{H_0^1(\Omega)} \leq ||u||_{H_\lambda}^2 \leq ||u||_{H_0^1(\Omega)}^2.$$
2. critical case $\lambda = \lambda_\star$: $H_{\lambda_\star}$ is slightly larger than $H_0^1(\Omega)$. Actually, $H_\lambda \subset H^s, s < 1$ (Vazquez-Zuazua, JFA, 2000).

Theorem (Vancostenoble-Zuazua '09)

The system (W) is controllable for any $\lambda \leq \lambda_\star$. More precisely, for any time $T > 2R_\Omega$, $(u_0, u_1) \in L^2(\Omega) \times H_\lambda^{'}$ and $(\overline{u_0}, \overline{u_1}) \in L^2(\Omega) \times H_\lambda^{'}$ there exists $h \in L^2((0,T) \times \Gamma_0)$ such that the solution of (W) satisfies

$$(u_t(T,x), u(T,x)) = (\overline{u_1}(x), \overline{u_0}(x)) \quad \text{for all } x \in \Omega.$$

Theorem (Vancostenoble-Zuazua '09)

The system (S) is controllable for any $\lambda \leq \lambda_\star$. More precisely, for any time $T > 0$, $u_0 \in H_\lambda^{'}$ and $\overline{u_0} \in H_\lambda^{'}$ there exists $h \in L^2((0,T) \times \Gamma_0)$ such that the solution of (S) satisfies

$$u(T,x)) = \overline{u}(x) \quad \text{for all } x \in \Omega.$$

CONTROLLABILITY $\Leftrightarrow$ OBSERVABILITY

$\Downarrow$

J.L. Lions's Hilbert Uniqueness Method [1]: Null-Controllability of systems (W)-(S) suffices to prove the so-called *Observability Inequality* for the adjoint systems

$$(W)_{adj}:\begin{cases} v_{tt}-\Delta v-\lambda\frac{v}{|x|^2}=0, & (t,x)\in Q_T,\\ v(t,x)=0, & (t,x)\in(0,T)\times\Gamma,\\ v(0,x)=v_0(x), & x\in\Omega,\\ v_t(0,x)=v_1(x), & x\in\Omega.\end{cases} \quad (5)$$

respectively

$$(S)_{adj}:\begin{cases} iv_t+\Delta v+\lambda\frac{v}{|x|^2}=0, & (t,x)\in Q_T,\\ v(t,x)=0, & (t,x)\in(0,T)\times\Gamma,\\ v(0,x)=v_0(x), & x\in\Omega,\end{cases} \quad (6)$$

Notations: $Q_T=(0,T)\times\Omega$, $\Sigma_T=(0,T)\times\Gamma$, $\Sigma_0=(0,T)\times\Gamma_0$.

Conservation of energy for $(W)_{adj}$:

$$E_v^\lambda(t):=\frac{1}{2}\Big[||v_t||^2_{L^2(\Omega)}+||v||^2_{H_\lambda}\Big], \quad (7)$$

$$E_v^\lambda(t)=E_v^\lambda(0),\quad \forall t\in[0,T],\forall\lambda\le\lambda(N).$$

Conservation of energy for $(S)_{adj}$:

$$||v(t)||_{L^2(\Omega)}=||v_0||_{L^2(\Omega)},\quad ||v(t)||_{H_\lambda}=||v_0||_{H_\lambda}. \quad (8)$$

The domain of $A_\lambda := -\Delta - \frac{\lambda}{|x|^2}$ is defined as
$D_\lambda = \{u \in H_\lambda \mid -\Delta u - \lambda/|x|^2 \in L^2(\Omega)\}$.
The system (S) is controllable iff there exists $h \in L^2(0,T) \times \Gamma_0)$ such that

$$\int_0^T \int_{\Gamma_0} h \, \frac{\partial v}{\partial \nu} d\sigma dt - < u_t(0), v(0) >_{H'_\lambda, H_\lambda} + < u(0), v_t(0) >_{L^2(\Omega), L^2(\Omega} = 0,$$

where v is the solution of the adjoint system (S_{adj}).
Build the control h as a minimizer of the functional

$$J(v_0, v_1) := \frac{1}{2} \int_0^T \int_{\Gamma_0} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt - (u_1, v_0)_{H'_\lambda, H_\lambda} + (u_0, v_1)_{L^2(\Omega), L^2(\Omega)},$$

The existence of a minimizer of J (the existence of a control h) is assured by the coercivity of J. Coercivity of J is equivalent to the so-called *Observability Inequality*.

Theorem (Observability inequality-Vancostenoble-Zuazua)

For all $\lambda \leq \lambda_\star$, there exist two positive constanta C_1, C_2 such that for all $T \geq 2R_\Omega$, the solution of $(W)_{adj}$ verifies

$$\int_0^T \int_{\Gamma_0} \left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma dt \leq C_2 E_v^\lambda(0). \tag{9}$$

and

$$E_v^\lambda(0) \leq C_1 \int_0^T \int_{\Gamma_0} \left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma dt. \tag{10}$$

Theorem (Observability inequality-Vancostenoble-Zuazua)

For all $\lambda \leq \lambda_\star$, there exist two positive constant C_1, C_2 such that for all $T > 0$, the solution of $(S)_{adj}$ verifies

$$\int_0^T \int_{\Gamma_0} \left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma dt \leq C_2 ||v||^2_{H_\lambda}. \tag{11}$$

and

$$||v||^2_{H_\lambda} \leq C_1 \int_0^T \int_{\Gamma_0} \left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma dt. \tag{12}$$

The Observability inequality relies mainly on the method of multipliers (cf. [1]) and the so called compactness uniqueness argument (cf. [2]), combined with Hardy inequalities above.

First inequalities provide Hidden Regularity for the normal derivative (We do not know a priori that $\partial v/\partial \nu \in L^2$).

Multipliers.

Wave: When integrating

$$(v_{tt} - \Delta v - \frac{\lambda}{|x|^2} v) x \cdot \nabla v = 0$$

formally you get

$$\frac{1}{2} \int_0^T \int_{\Gamma_0} (x \cdot \nu) \Big(\frac{\partial v}{\partial \nu} \Big)^2 d\sigma = T E_v^{\lambda}(0) + \int_\Omega v_t \big(x \cdot v + \frac{N-1}{2} \big) \Big|_0^T dx.$$

Schrödinger: When Integrating

$$(i v_t + \Delta v + \frac{\lambda}{|x|^2} v)(x \cdot \nabla \overline{v}) = 0,$$

and taking the real part, formally you get

$$\frac{1}{2} \int_0^T \int_{\Gamma_0} (x \cdot \nu) \Big| \frac{\partial v}{\partial \nu} \Big|^2 d\sigma = T ||v||_{H_\lambda}^2 + \frac{1}{2} \mathrm{Im} \int_\Omega v x \cdot \nabla \overline{v} dx \Big|_0^T.$$

Warning !

We are not allowed to do integration by parts directly: Even if $(v_0, v_1) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ the solution of (W) does not belong to $H^2(\Omega)$ in space variable ! A smooth initial data yields regularity $D((-\Delta - \lambda/|x|)^k)$ for any natural number k, which, the same, is not in H^2.

Remark

In order to prove the validity of multipliers identity is enough to show the Pohozaev identity for the elliptic operator $-\Delta - \lambda/|x|^2$.

Indeed, when integrating

$$(-\Delta v - \lambda/|x|^2 v) x \cdot \nabla v = 0$$

we obtain formally the Pohozaev identity:

Proposition

$$\int_{\partial\Omega}(x\cdot\nu)\Big(\frac{\partial v}{\partial\nu}\Big)^2 d\sigma = -\int_\Omega \Big(-\Delta v - \lambda\frac{v}{|x|^2}\Big)(x\cdot\nabla v)dx - \frac{N-2}{2}\int_\Omega \Big[|\nabla v|^2 - \lambda\frac{v^2}{|x|^2}\Big]dx, \tag{13}$$

holds true for any $v \in D(-\Delta - \lambda/|x|^2)$.

Replace $-Dv - \lambda/|x|^2 v$ by v_{tt}, integrate in time to obtain multipliers for Waves.

Goal: Prove the validity of Pohozaev identity.

Possibilities to avoid the singularity:

1. regularizing the potential $\lambda/(|x|^2+\epsilon^2)$ and try to pass to the limit as well...
2. use a multiplier of the form $x\cdot\nabla u\theta_\epsilon$, and try to pass to the limit...

All these fail in the critical case $\lambda = \lambda_\star$.

Let us try the following remedy: integrate on $\Omega\setminus B_\varepsilon(0)$ and then try to pass to the limit when ϵ goes to zero... Indeed, we consider $v\in D(-\Delta - \lambda/|x|^2)$ (in particular $v\in H^2(\Omega\setminus B_\epsilon(0))$). Then we are allowed to integrate. We obtain

$$\int_{\Omega_\varepsilon}(-\Delta v-\lambda/|x|^2 v)(x\cdot\nabla v)dx+\frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)(\frac{\partial v}{\partial\nu})^2 d\sigma=\frac{1}{\varepsilon}\int_{S_\varepsilon}|\nabla v\cdot x|^2 d\sigma$$
$$-\frac{\varepsilon}{2}\int_{S_\varepsilon}\Big[|\nabla v|^2-\lambda_\star\frac{v^2}{\varepsilon^2}\Big]d\sigma-$$
$$-\frac{N-2}{2}\int_{\Omega_\varepsilon}\Big[|\nabla v|^2-\lambda_\star\frac{v^2}{|x|^2}\Big]dx. \qquad (14)$$

Surprising:

$$\int_{\Omega_\epsilon}\Big[|\nabla v|^2-\lambda_\star\frac{|u|^2}{|x|^2}\Big]dx\not\to\int_{\Omega}\Big[|\nabla v|^2-\lambda_\star\frac{|u|^2}{|x|^2}\Big]dx, \qquad \varepsilon\to 0, !!!!$$

(Counterexample: $\sin$-like functions, see Vazquez-Zographopoulos 2011).
There exists a Hidden energy at the origin !

New Hardy functional norm: Let us introduce the norm

$$||v||^2_{H_{\lambda_\star,1}}=\int_\Omega\Big|\nabla v+\frac{N-2}{2}\frac{x}{|x|^2}v\Big|^2 dx. \qquad (15)$$

and the corresponding Hilbert space $H_{\lambda_\star,1}$ to be the closure of $C_0^\infty(\Omega)$ in the norm $||\cdot||_{H_{\lambda_\star,1}}$.

Remark:

$$\int_\Omega \Big|\nabla v + \frac{N-2}{2}\frac{x}{|x|^2}v\Big|^2 dx = \int_\Omega \Big[|\nabla v|^2 - \lambda_\star \frac{v^2}{|x|^2}\Big]dx, \qquad \forall v \in C_0^\infty(\Omega).$$

therefore $H_{\lambda_\star} = H_{\lambda_\star,1}$, and $||v||_{H_{\lambda_\star}} = ||v||_{H_{\lambda_\star,1}}$. Then

$$\int_{\Omega_\varepsilon} \Big[|\nabla v|^2 - \lambda_\star \frac{v^2}{|x|^2}\Big]dx = \int_{\Omega_\varepsilon} \Big|\nabla v + \frac{N-2}{2}\frac{x}{|x|^2}v\Big|^2 dx + \frac{N-2}{2\varepsilon}\int_{S_\varepsilon} v^2 d\sigma. \tag{16}$$

and, adding (16) to (14), the following identity holds

$$\int_{\Omega_\varepsilon} (-\Delta - \lambda_\star/|x|^2 v)(x\cdot\nabla v)dx + \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)(\frac{\partial v}{\partial \nu})^2 d\sigma$$
$$= \varepsilon \int_{S_\varepsilon}\Big[\frac{|\nabla v\cdot x|^2}{\varepsilon^2} - \lambda_\star \frac{v^2}{\varepsilon^2}\Big]d\sigma - \frac{\varepsilon}{2}\int_{S_\varepsilon}\Big[|\nabla v|^2 - \lambda_\star \frac{v^2}{\varepsilon^2}\Big]d\sigma$$
$$-(\frac{N-2}{2})\int_{\Omega_\varepsilon}\Big|\nabla v + \frac{N-2}{2}\frac{x}{|x|^2}v\Big|^2 dx. \tag{17}$$

or

$$\int_{\Omega_\varepsilon} (-\Delta - \lambda_\star/|x|^2 v)(x \cdot \nabla v)dx + \frac{1}{2}\int_{\partial\Omega} (x \cdot \nu)(\frac{\partial v}{\partial \nu})^2 d\sigma$$
$$= R[\varepsilon] - \frac{N-2}{2}\int_{\Omega_\varepsilon} \left|\nabla v + \frac{N-2}{2}\frac{x}{|x|^2}v\right|^2 dx. \qquad (18)$$

The most technical difficulty: $v \in D(-\Delta - \lambda_\star/|x|^2)$ implies $R[\varepsilon] \to 0$, as $\varepsilon \to 0$.
With this, we can pass to the limit to obtain

$$\int_{\Omega} (-\Delta - \lambda_\star/|x|^2 v)(x \cdot \nabla v)dx + \frac{1}{2}\int_{\partial\Omega} (x \cdot \nu)(\frac{\partial v}{\partial \nu})^2 d\sigma$$
$$= -\frac{N-2}{2}\int_{\Omega} \left|\nabla v + \frac{N-2}{2}\frac{x}{|x|^2}v\right|^2 dx. \qquad (19)$$

and

$$\int_{\Omega} (-\Delta - \lambda_\star/|x|^2 v)(x \cdot \nabla v)dx + \frac{1}{2}\int_{\partial\Omega} (x \cdot \nu)(\frac{\partial v}{\partial \nu})^2 d\sigma$$
$$= -\frac{N-2}{2}\int_{\Omega} \Big[|\nabla v|^2 - \lambda_\star \frac{|u|^2}{|x|^2}\Big] dx. \qquad (20)$$

is totally correct.

Thank you !

J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, Recherches en Mathématiques Appliquées [Research in Applied Mathematics], vol. 8, Masson, Paris, 1988, Contrôlabilité exacte. [Exact controllability], With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.

Elaine Machtyngier, *Exact controllability for the Schrödinger equation*, SIAM J. Control Optim. **32** (1994), no. 1, 24–34.